

## EXISTENCE OF SOLUTION TO AN INTEGRODYNAMIC EQUATION

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**ABSTRACT.** We prove existence of solution to an integrodynamic equation on time scales under some suitable conditions on the functions involved. In some particular cases, uniqueness is also demonstrated. Some applications of our results are provided.

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### 1. INTRODUCTION

In this paper we shall be concerned with the existence of solutions of the following integrodynamic equation

$$(1.1) \quad x^\Delta(t) = F \left( t, x(t), \int_a^t K[t, \tau, x(\tau)] \Delta\tau \right), \quad x(a) = A, \quad t \in [a, b]_{\mathbb{T}}^\kappa,$$

where  $a, b \in \mathbb{T}$ ,  $K : [a, b]_{\mathbb{T}}^\kappa \times [a, b]_{\mathbb{T}}^{\kappa^2} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $F : [a, b]_{\mathbb{T}}^\kappa \times \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuous functions (see Section 2 for the notation used in this paper). The variable  $t$  in (1.1) is defined in a so called *time scale* which is nothing more than a nonempty closed subset of the real numbers. Stefan Hilger [11] introduced the theory of time scales in an attempt to unify ideas from continuous and discrete calculus and succeeded. It is worth mentioning that unification of continuous and discrete calculus is not all that the time scales setting accomplishes; for example, the Cantor Set and  $\bigcup_{k \in \mathbb{Z}} [k, k + \frac{1}{2}]$  (this is used to model population dynamics [11]) are time scales and therefore it also defines a calculus on them.

As is well known integrodifferential equations and their discrete analogues find many applications in various mathematical problems. Moreover, it appears to be advantageous to model certain processes by employing a suitable combination of both differential equations and difference equations at different stages in the process under consideration (see [17] and references therein for more details).

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Some results concerning existence of a solution to some particular cases of the integrodynamic equation in (1.1) were obtained in [12, 18]. Here we will make use of the well known in the literature *Topological Transversality Theorem* to prove the existence of a solution to the above mentioned equation.

This paper is organized as follows: in Section 2 we give a brief introduction to some topological and time scales concepts, present some results and a proof of an integral inequality needed throughout. In Section 3 we state and prove the two main results within the paper (cf. Theorem 3.1 and Theorem 3.2) and discuss a particular case of Theorem 3.2. Finally, in Section 4 are presented some applications of our results.

## 2. PRELIMINARIES

In this section we provide some concepts and results about time scales calculus and fixed point theory. We also give a proof of an integral inequality (cf. Lemma 2.3) needed in the next section.

A *Time Scale* is an arbitrary nonempty closed subset of  $\mathbb{R}$  and is denoted by  $\mathbb{T}$ . For  $a, b \in \mathbb{T}$  with  $a < b$ , we define the time scales interval by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

The *forward jump operator*  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \quad \text{for all } t \in \mathbb{T},$$

while the *backward jump operator*  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  is defined by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \quad \text{for all } t \in \mathbb{T},$$

with  $\inf \emptyset = \sup \mathbb{T}$  (i.e.,  $\sigma(M) = M$  if  $\mathbb{T}$  has a maximum  $M$ ) and  $\sup \emptyset = \inf \mathbb{T}$  (i.e.,  $\rho(m) = m$  if  $\mathbb{T}$  has a minimum  $m$ ).

A point  $t \in \mathbb{T}$  is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if  $\sigma(t) = t$ ,  $\sigma(t) > t$ ,  $\rho(t) = t$  or  $\rho(t) < t$ , respectively. The *graininess function*  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by  $\mu(t) = \sigma(t) - t$ .

The set  $\mathbb{T}^{\kappa}$  is derived from  $\mathbb{T}$  as follows: if  $\mathbb{T}$  has a left-scattered maximum  $M$  then  $\mathbb{T}^{\kappa} = \mathbb{T} - \{M\}$ ; otherwise,  $\mathbb{T}^{\kappa} = \mathbb{T}$ . Also  $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ .

Assume  $f : \mathbb{T} \rightarrow \mathbb{R}$  is a function and let  $t \in \mathbb{T}^{\kappa}$ . Then we define  $f^{\Delta}(t)$  to be the number (provided it exists) given by

$$f^{\Delta}(t) = \lim_{s \rightarrow t, s \neq \sigma(t)} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

We call  $f^{\Delta}(t)$  the *delta derivative* (or  $\Delta$ -derivative) of  $f$  at  $t$ . A function is said to be *delta differentiable* if it has a delta derivative at each  $t \in \mathbb{T}^{\kappa}$ . We note that, if  $\mathbb{T} = \mathbb{R}$ , then  $f^{\Delta}(t) = f'(t)$  while, if  $\mathbb{T} = \mathbb{Z}$ , then  $f^{\Delta}(t) = f(t+1) - f(t)$ .

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. It is worth mentioning that every continuous function is also rd-continuous. We will denote the set of all real valued rd-continuous functions defined on  $\mathbb{T}$  by  $C_{rd}(\mathbb{T}, \mathbb{R})$ .

It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function  $F$  with  $F^\Delta = f$ , and in this case a  $\Delta$ -integral is defined by  $\int_a^b f(t)\Delta t = F(b) - F(a)$ . If  $\mathbb{T} = \mathbb{R}$ , then  $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$  while, if  $\mathbb{T} = \mathbb{Z}$ , then  $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$ .

Define by  $C([a, b]_{\mathbb{T}}, \mathbb{R})$  [sometimes we will only write  $C[a, b]_{\mathbb{T}}$ ] the set of all real valued continuous functions on  $[a, b]_{\mathbb{T}}$  and by  $C^1([a, b]_{\mathbb{T}}, \mathbb{R})$  the set of all delta differentiable functions whose derivative is continuous on  $[a, b]_{\mathbb{T}}^\kappa$ , and equip the spaces  $C([a, b]_{\mathbb{T}}, \mathbb{R})$ ,  $C^1([a, b]_{\mathbb{T}}, \mathbb{R})$  with the norms

$$\|u\|_0 = \sup_{t \in [a, b]_{\mathbb{T}}} |u(t)|, \quad \|u\|_1 = \sup_{t \in [a, b]_{\mathbb{T}}} |u(t)| + \sup_{t \in [a, b]_{\mathbb{T}}^\kappa} |u^\Delta(t)|,$$

respectively.

We now state some useful results about time scales.

**Lemma 2.1** ([4, Theorem 1.117]). *Let  $t_0 \in \mathbb{T}^k$  and assume  $k : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$  is continuous at  $(t, t)$ , where  $t \in \mathbb{T}^k$  with  $t > t_0$ . In addition, assume that  $k(t, \cdot)$  is rd-continuous on  $[t_0, \sigma(t)]$ . Suppose that for each  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $t$ , independent of  $\tau \in [t_0, \sigma(t)]$ , such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta_1}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where  $k^{\Delta_1}$  denotes the delta derivative of  $k$  with respect to the first variable. Then,

$$g(t) := \int_{t_0}^t k(t, \tau)\Delta\tau \text{ implies } g^\Delta(t) = \int_{t_0}^t k^{\Delta_1}(t, \tau)\Delta\tau + k(\sigma(t), t).$$

**Lemma 2.2** (see [9]). *Consider a delta differentiable function  $r : [a, b]_{\mathbb{T}} \rightarrow (0, \infty)$  with  $r^\Delta(t) \geq 0$  on  $[a, b]_{\mathbb{T}}^\kappa$ . Define*

$$G(x) = \int_1^x \frac{ds}{g(s)}, \quad x > 0,$$

where  $g \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$  is positive and nondecreasing on  $(0, \infty)$ . Then, for each  $t \in [a, b]_{\mathbb{T}}$  we have

$$G(r(t)) \leq G(r(a)) + \int_a^t \frac{r^\Delta(\tau)}{g(r(\tau))}\Delta\tau.$$

To make use of the Topological Transversality Theorem it is essential to obtain a priori bounds on the possible solutions of the equation in study. The next lemma is indispensable in the sequel. The notation  $\mathbb{R}_0^+ = [0, \infty)$  is used throughout.

**Lemma 2.3.** Let  $f, g \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$  and  $u, p \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$ , with  $p(t)$  positive and nondecreasing on  $[a, b]_{\mathbb{T}}$ ,  $k \in C([a, b]_{\mathbb{T}}^{\kappa} \times [a, b]_{\mathbb{T}}^{\kappa^2}, \mathbb{R}_0^+)$ , and  $c \in \mathbb{R}_0^+$ . Moreover, let  $w_1, w_2, w_3 \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$  be nondecreasing functions with  $\{w_1, w_2, w_3\}(x) > 0$  for  $x > 0$ . Let  $M' = \max_{(t,s) \in [a, b]_{\mathbb{T}}^{\kappa} \times [a, b]_{\mathbb{T}}^{\kappa^2}} k(t, s)$  and  $M = (b - a)M'$ . Define two functions as follows,

$$w(x) = \max\{w_1(x), w_2(Mw_3(x))\}, \quad x \in \mathbb{R}_0^+,$$

$$G(x) = \int_1^x \frac{1}{w(s)} ds, \quad x > 0,$$

and assume that  $\lim_{x \rightarrow \infty} G(x) = \infty$ . If, for all  $t \in [a, b]_{\mathbb{T}}$ , the following inequality holds

$$u(t) \leq p(t) + \int_a^t \left\{ f(s)w_1(u(s) + c) + g(s)w_2 \left( \int_a^s k(s, \tau)w_3(u(\tau) + c)\Delta\tau \right) \right\} \Delta s,$$

then,

$$u(t) \leq G^{-1} \left( G(p(t) + c) + \int_a^t [f(s) + g(s)]\Delta s \right) - c, \quad t \in [a, b]_{\mathbb{T}},$$

where, as usual,  $G^{-1}$  denotes de inverse.

*Proof.* We start by noting that the result trivially holds for  $t = a$ . Consider an arbitrary number  $t_0 \in (a, b]_{\mathbb{T}}$  and define a positive and nondecreasing function  $z(t)$  on  $[a, t_0]_{\mathbb{T}}$  by

$$z(t) = p(t_0) + \int_a^t \left\{ f(s)w_1(u(s) + c) + g(s)w_2 \left( \int_a^s k(s, \tau)w_3(u(\tau) + c)\Delta\tau \right) \right\} \Delta s.$$

Then,  $z(a) = p(t_0)$ , for all  $t \in [a, t_0]_{\mathbb{T}}$  we have  $u(t) \leq z(t)$ , and

$$\begin{aligned} z^\Delta(t) &= f(t)w_1(u(t) + c) + g(t)w_2 \left( \int_a^t k(t, \tau)w_3(u(\tau) + c)\Delta\tau \right) \\ &\leq f(t)w_1(z(t) + c) + g(t)w_2 \left( \int_a^t k(t, \tau)w_3(z(\tau) + c)\Delta\tau \right) \\ &\leq f(t)w_1(z(t) + c) + g(t)w_2(Mw_3(z(t) + c)) \\ &\leq w(z(t) + c) [f(t) + g(t)], \end{aligned}$$

for all  $t \in [a, t_0]_{\mathbb{T}}^{\kappa}$ . Hence,

$$\frac{z^\Delta(t)}{w(z(t) + c)} \leq f(t) + g(t),$$

and, after integrating from  $a$  to  $t$  and with the help of Lemma 2.2, we get

$$G(z(t) + c) \leq G(z(a) + c) + \int_a^t [f(s) + g(s)]\Delta s, \quad t \in [a, t_0]_{\mathbb{T}}.$$

In view of the hypothesis on function  $G$ , we may write

$$z(t) \leq G^{-1} \left( G(z(a) + c) + \int_a^t [f(s) + g(s)]\Delta s \right) - c.$$

Therefore,

$$u(t) \leq G^{-1} \left( G(p(t_0) + c) + \int_a^t [f(s) + g(s)] \Delta s \right) - c,$$

for all  $t \in [a, t_0]_{\mathbb{T}}$ . Setting  $t = t_0$  in the above inequality and having in mind that  $t_0$  is arbitrary, we conclude the proof.  $\square$

To get more insight on time scales calculus we refer the reader to the monographs [4, 13]. Next, we turn to fixed point theory.

Let  $\mathcal{B}$  be a Banach space and  $C \subset \mathcal{B}$  be convex. By a *pair*  $(X, A)$  in  $C$  is meant an arbitrary subset  $X$  of  $C$  and an  $A \subset X$  closed in  $X$ . We call a homotopy  $H : X \times [0, 1] \rightarrow Y$  *compact* if it is a compact map. If  $X \subset Y$ , the homotopy  $H$  is called *fixed point free* on  $A \subset X$  if for each  $\lambda \in [0, 1]$ , the map  $H|_{A \times \{\lambda\}} : A \rightarrow Y$  has no fixed point. We denote by  $\mathcal{C}_A(X, C)$  the set of all compact maps  $F : X \rightarrow C$  such that the restriction  $F|_A : A \rightarrow C$  is fixed point free.

Two maps  $F, G \in \mathcal{C}_A(X, C)$  are called *homotopic*, written  $F \simeq G$  in  $\mathcal{C}_A(X, C)$ , provided there is a compact homotopy  $H_\lambda : X \rightarrow C$  ( $\lambda \in [0, 1]$ ) that is fixed point free on  $A$  and such that  $H_0 = F$  and  $H_1 = G$ .

**Definition 2.4.** Let  $(X, A)$  be a pair in a convex  $C \subset \mathcal{B}$ . A map  $F \in \mathcal{C}_A(X, C)$  is called *essential* provided every  $G \in \mathcal{C}_A(X, C)$  such that  $F|_A = G|_A$  has a fixed point.

**Theorem 2.5** (Topological Transversality [10]). *Let  $(X, A)$  be a pair in a convex  $C \subset \mathcal{B}$ , and let  $F, G$  be maps in  $\mathcal{C}_A(X, C)$  such that  $F \simeq G$  in  $\mathcal{C}_A(X, C)$ . Then,  $F$  is essential if and only if  $G$  is essential.*

The next theorem is very useful to the application of the Topological Transversality Theorem. Its proof can be found in [10].

**Theorem 2.6.** *Let  $U$  be an open subset of a convex set  $C \subset \mathcal{B}$ , and let  $(\bar{U}, \partial U)$  be the pair consisting of the closure of  $U$  in  $C$  and the boundary of  $U$  in  $C$ . Then, for any  $u_0 \in U$ , the constant map  $F|_{\bar{U}} = u_0$  is essential in  $\mathcal{C}_{\partial U}(\bar{U}, C)$ .*

If more is needed on this topic the reader can consult the books [5, 10].

### 3. MAIN RESULTS

We start by noting that if a function is delta differentiable then it is continuous. Hence, there is an inclusion of  $C^1[a, b]_{\mathbb{T}}$  into  $C[a, b]_{\mathbb{T}}$ .

**Theorem 3.1.** *The embedding  $j : C^1[a, b]_{\mathbb{T}} \rightarrow C[a, b]_{\mathbb{T}}$  is completely continuous.*

*Proof.* Let  $B$  be a bounded set in  $C^1[a, b]_{\mathbb{T}}$ . Then, there exists  $M > 0$  such that  $\|x\|_1 \leq M$  for all  $x \in B$ . By the definition of the norm  $\|\cdot\|_1$ , we have that  $\sup_{t \in [a, b]_{\mathbb{T}}} |x(t)| \leq M$ , hence  $\|x\|_0 \leq M$ , i.e.,  $B$  is bounded in  $C[a, b]_{\mathbb{T}}$ . Let now  $\varepsilon > 0$  be given and put

$\delta = \frac{\varepsilon}{M}$ . Then, it is easily seen that, for  $x \in B$  we have  $|x^\Delta(t)| \leq M$  for all  $t \in [a, b]_{\mathbb{T}}^{\kappa}$ . For arbitrary  $t_1, t_2 \in [a, b]_{\mathbb{T}}$  with  $t_1 \neq t_2$ , we use the Mean Value Theorem on time scales [3, Theorem 1.14] to get

$$\frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|} \leq M,$$

i.e., for  $|t_1 - t_2| < \delta$  we have  $|x(t_1) - x(t_2)| < \varepsilon$  and this proves that  $B$  is equicontinuous on  $C[a, b]_{\mathbb{T}}$  (the case  $t_1 = t_2$  is obvious). By the Arzela–Ascoli Theorem,  $B$  is relatively compact and therefore  $j$  is completely continuous.  $\square$

**Theorem 3.2.** *Assume that the functions  $F$  and  $K$  introduced in Section 1 satisfy*

$$(3.1) \quad |F(t, x, y)| \leq f(t)w_1(|x|) + g(t)w_2(|y|) + h(t),$$

$$(3.2) \quad |K(t, s, x)| \leq k(t, s)w_3(|x|),$$

where  $f, g, h, k, w_1, w_2, w_3$  are as in Lemma 2.3 (more specifically,  $h$  is in the same space of  $f$  and  $g$ ).

Then, the integrodynamic equation in (1.1) has a solution  $x \in C^1([a, b]_{\mathbb{T}}, \mathbb{R})$ .

*Proof.* We start considering the following auxiliary problem,

$$(3.3) \quad y^\Delta(t) = F\left(t, y(t) + A, \int_a^t K[t, s, y(s) + A]\Delta s\right), \quad y(a) = 0, \quad t \in [a, b]_{\mathbb{T}}^{\kappa},$$

and showing that it has a solution. To do this, we first establish a priori bounds to the (possible) solutions of the family of problems

$$(3.4) \quad y^\Delta(t) = \lambda F\left(t, y(t) + A, \int_a^t K[t, s, y(s) + A]\Delta s\right), \quad y(a) = 0, \quad t \in [a, b]_{\mathbb{T}}^{\kappa},$$

independently of  $\lambda \in [0, 1]$ .

Integrating both sides of the equation in (3.4) on  $[a, t]_{\mathbb{T}}$ , we obtain,

$$y(t) = \lambda \int_a^t F\left(s, y(s) + A, \int_a^s K[s, \tau, y(\tau) + A]\Delta \tau\right) \Delta s,$$

for all  $t \in [a, b]_{\mathbb{T}}$ . Applying the modulus function to both sides of the last equality we obtain, using the triangle inequality and inequalities (3.1) and (3.2),

$$(3.5) \quad |y(t)| \leq p(t) + \int_a^t \left\{ f(s)w_1(|y(s) + A|) + g(s)w_2\left(\int_a^s k(s, \tau)w_3(|y(\tau) + A|)\Delta \tau\right) \right\} \Delta s,$$

with  $p(t) = \int_a^t h(s)\Delta s$ . Since  $p(t)$  might be zero, we give another bound to  $|y(t)|$  in (3.5), namely, (note also that  $|y(\cdot) + A| \leq |y(\cdot)| + |A|$ )

$$|y(t)| \leq p(t) + 1 + \int_a^t \left\{ f(s)w_1(|y(s)| + |A|) + g(s)w_2 \left( \int_a^s k(s, \tau)w_3(|y(\tau)| + |A|)\Delta\tau \right) \right\} \Delta s.$$

By Lemma 2.3 (with  $u(t) = |y(t)|$ ,  $p(t) = p(t) + 1$ ,  $c = |A|$ ), we deduce that

$$(3.6) \quad |y(t)| \leq G^{-1} \left( G(p(t) + |A|) + \int_a^t [f(s) + g(s)]\Delta s \right) - |A|,$$

for all  $t \in [a, b]_{\mathbb{T}}$ . Denote the right hand side of (3.6) by  $R(t)$ .

Let  $\mathcal{B} = C^1[a, b]_{\mathbb{T}}$  be the Banach space equipped with the norm  $\|\cdot\|_1$  and define a set  $C = \{u \in C^1[a, b]_{\mathbb{T}} : u(a) = 0\}$ . Observe that  $C$  is a convex subset of  $\mathcal{B}$ .

Define the linear operator  $L : C \rightarrow C[a, b]_{\mathbb{T}}$  by  $Lu = u^\Delta$ . It is clearly bijective with a continuous inverse  $L^{-1} : C[a, b]_{\mathbb{T}} \rightarrow C$ . Let  $M_0$  and  $M_1$  be defined by

$$M_0 = R(b),$$

$$M_1 = \sup_{t \in [a, b]_{\mathbb{T}}, |y| \leq M_0} \left| F \left( t, y, \int_a^t K[t, s, y]\Delta s \right) \right|.$$

Consider the family of maps  $\mathcal{F}_\lambda : C[a, b]_{\mathbb{T}} \rightarrow C[a, b]_{\mathbb{T}}$ ,  $0 \leq \lambda \leq 1$ , defined by

$$(\mathcal{F}_\lambda y)(t) = \lambda F \left( t, y(t) + A, \int_a^t K[t, s, y(s) + A]\Delta s \right),$$

and the completely continuous embedding  $j : C \rightarrow C[a, b]_{\mathbb{T}}$  (it is easily seen that the restriction here used of  $j$  of Theorem 3.1 is also completely continuous). Let us consider the set  $Y = \{y \in C : \|y\|_1 \leq r\}$  with  $r = 1 + M_0 + M_1$ . Then we can define a homotopy  $H_\lambda : Y \rightarrow C$  by  $H_\lambda = L^{-1}\mathcal{F}_\lambda j$ . Since  $L^{-1}$  and  $\mathcal{F}_\lambda$  are continuous and  $j$  is completely continuous,  $H$  is a compact homotopy. Moreover, it is fixed point free on the boundary of  $Y$ . Since  $H_0$  is the zero map, it is essential. Because  $H_0 \simeq H_1$ , Theorem 2.5 implies that  $H_1$  is also essential. In particular,  $H_1$  has a fixed point which is a solution of (3.3). To finish the proof, let  $y \in C^1[a, b]_{\mathbb{T}}$  be a solution of (3.3) and define the function  $x(t) = y(t) + A$ ,  $t \in [a, b]_{\mathbb{T}}$ . Then, it is easily seen that  $x(a) = A$  and

$$x^\Delta(t) = F \left( t, x(t), \int_a^t K[t, s, x(s)]\Delta s \right), \quad t \in [a, b]_{\mathbb{T}}^\kappa,$$

i.e.,  $x \in C^1[a, b]_{\mathbb{T}}$  is a solution of (1.1). □

We show next that the above theorem provides a nontrivial generalization to the one presented in [6], i.e., Theorem 3.2 seems to be new even if we let the time scale to be  $\mathbb{T} = \mathbb{R}$ .

Let

$$F(t, x, y) = |y| + \begin{cases} (x+1)\ln(x+1) & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

We will show that, for all  $f, h \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$ ,  $|F(t, x, y)| > f(t)|x| + |y| + h(t)$  for some  $(t, x, y) \in [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2$  [in [6] the assumption on  $F$  was that  $|F(t, x, y)| \leq f(t)|x| + |y| + h(t)$ ]. For that, suppose the contrary, i.e., assume that there exist  $f, h \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$  such that  $|F(t, x, y)| \leq f(t)|x| + |y| + h(t)$  for all  $(t, x, y) \in [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2$ . In particular, for arbitrary  $x > 0$ , we have that,

$$|y| + (x+1)\ln(x+1) = |F(t, x, y)| \leq f(t)x + |y| + h(t),$$

which is equivalent to

$$(x+1)\ln(x+1) \leq f(t)x + h(t),$$

or

$$(3.7) \quad 1 \leq f(t) \frac{x}{(x+1)\ln(x+1)} + h(t) \frac{1}{(x+1)\ln(x+1)}.$$

Now, we fix  $t \in [a, b]_{\mathbb{T}}^{\kappa}$  and let  $x \rightarrow \infty$  in both sides of (3.7). Then,

$$1 \leq \lim_{x \rightarrow \infty} \left\{ f(t) \frac{x}{(x+1)\ln(x+1)} + h(t) \frac{1}{(x+1)\ln(x+1)} \right\}.$$

By the L'Hôpital Theorem we have that

$$\lim_{x \rightarrow \infty} \frac{x}{(x+1)\ln(x+1)} = 0,$$

hence we get the contradiction  $1 \leq 0$ .

Note that the function  $w(x) = (x+1)\ln(x+1)$ ,  $x \geq 0$  is nondecreasing and is such that  $\int_1^{\infty} \frac{1}{w(s)} = \infty$ , which shows that we can apply our result to such a function  $F$  defined as above.

Let, as usual,  $\mathcal{R}$  denote the set of regressive functions and  $\mathcal{R}^+$  the set of positively regressive functions, i.e.,  $p \in \mathcal{R}$  (resp.,  $\mathcal{R}^+$ ) if  $1 + \mu(t)p(t) \neq 0$  (resp.,  $1 + \mu(t)p(t) > 0$ ) for all  $t \in \mathbb{T}^{\kappa}$ .

**Corollary 3.3.** *Assume that  $p \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ . Then, the initial value problem*

$$(3.8) \quad x^{\Delta}(t) = p(t)x(t), \quad x(a) = A,$$

*has a unique solution  $x \in C^1([a, b]_{\mathbb{T}}, \mathbb{R})$ .*

*Proof.* Define  $F(t, x, y) = p(t)x$ , for  $(t, x) \in [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}$ . Then,  $|F(t, x, y)| = |p(t)||x|$  and the existence part follows by Theorem 3.2.



Suppose now that  $x, y \in C^1([a, b]_{\mathbb{T}}, \mathbb{R})$  satisfy (3.8). An integration on  $[a, t]_{\mathbb{T}}$  yields,

$$\begin{aligned} x(t) &= A + \int_a^t p(s)x(s)\Delta s, \\ y(t) &= A + \int_a^t p(s)y(s)\Delta s. \end{aligned}$$

Hence,

$$|x(t) - y(t)| \leq \int_a^t |p(s)||x(s) - y(s)|\Delta s.$$

Using Gronwall's inequality [2] (note that  $|p(s)| \in \mathcal{R}^+$ ) it follows that  $|x(t) - y(t)| \leq 0$  and finally that  $x(t) = y(t)$  for all  $t \in [a, b]_{\mathbb{T}}$ . Hence, the solution is unique.  $\square$

**Remark 3.4.** S. Hilger in his seminal work [11] proved (not only but also) the existence of a unique solution to equation (3.8) with  $p \in C_{rd}([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$  being regressive. In this paper we are requiring  $p(t)$  to be continuous but not a regressive function.

In view of the previous remark it is interesting to think of what happens when the function  $p$  is not regressive. This is shown in the following result.

**Proposition 3.5.** *Suppose that  $p \in C([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$  is not regressive. Then, there exists  $t_0 \in [a, b]_{\mathbb{T}}^{\kappa}$  such that the solution of (3.8) satisfies  $x(t) = 0$  for all  $t \in [\sigma(t_0), b]_{\mathbb{T}}$ .*

*Proof.* Since  $p$  is not regressive, there exists a point  $t_0 \in [a, b]_{\mathbb{T}}^{\kappa}$  such that  $1 + \mu(t_0)p(t_0) = 0$ . This immediately implies that  $\mu(t_0) \neq 0$  and  $p(t_0) \neq 0$ . Let  $x \in C^1([a, b]_{\mathbb{T}}, \mathbb{R})$  be the unique solution of (3.8). Using the well known formula  $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$  it is easy to derive from (3.8) that  $x^{\Delta}(t)[1 + \mu(t)p(t)] = p(t)x(\sigma(t))$  and in particular that  $x^{\Delta}(t_0)[1 + \mu(t_0)p(t_0)] = p(t_0)x(\sigma(t_0))$ . It follows that  $x(\sigma(t_0)) = 0$ . Now note that, since  $t_0$  is right-scattered, we must have  $\sigma(t_0) \neq a$ . Moreover,  $\tilde{x}(t) = 0$  satisfies (3.8) for all  $t \in [\sigma(t_0), b]_{\mathbb{T}}$ . The uniqueness of the solution completes the proof.  $\square$

#### 4. APPLICATIONS

We now use Theorem 3.2 to prove existence of solution to an integral equation describing some physical phenomena. We emphasize that, for each time scale we get an equation, i.e., we can construct various models in order to (hopefully) better describe the phenomena in question.

Let  $0, b \in \mathbb{T}$  with  $b > 0$  and  $L \in C^1([0, b]_{\mathbb{T}}, \mathbb{R}_0^+)$ . Moreover, assume that a function  $M(t, s) \in C([0, b]_{\mathbb{T}} \times [0, b]_{\mathbb{T}}^{\kappa}, \mathbb{R}_0^+)$  has continuous partial delta derivative with respect to its first variable [denote it by  $M^{\Delta_t}(t, s)$ ] and  $M(\sigma(t), t)$  is continuous for all  $t \in [0, b]_{\mathbb{T}}^{\kappa}$  (we remind the reader that the jump operator  $\sigma$  is not necessarily continuous [4,

Example 1.55]). Suppose that  $w \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$  is nondecreasing, such that  $w(x) > 0$  for all  $x > 0$  and  $\int_1^\infty \frac{1}{w(s)} ds = \infty$ .

Under the above assumptions, we will prove the following result.

**Theorem 4.1.** *The integral equation,*

$$(4.1) \quad x(t) = L(t) + \int_0^t M(t, s)w(x(s))\Delta s, \quad t \in [0, b]_{\mathbb{T}},$$

has a solution  $x \in C^1([0, b]_{\mathbb{T}}, \mathbb{R}_0^+)$ .

*Proof.* Let us define

$$\begin{aligned} F(t, x, y) &= L^\Delta(t) + M(\sigma(t), t)w(|x|) + y, \quad (t, x, y) \in [0, b]_{\mathbb{T}}^\kappa \times \mathbb{R}^2 \\ K(t, s, x) &= M^{\Delta t}(t, s)w(|x|), \quad (t, s, x) \in [0, b]_{\mathbb{T}}^\kappa \times [0, b]_{\mathbb{T}}^{\kappa^2} \times \mathbb{R}. \end{aligned}$$

Then, we have

$$\begin{aligned} |F(t, x, y)| &\leq |L^\Delta(t)| + M(\sigma(t), t)w(|x|) + |y|, \\ |K(t, s, x)| &\leq |M^{\Delta t}(t, s)|w(|x|). \end{aligned}$$

Using Theorem 3.2 with  $f(t) = M(\sigma(t), t)$ ,  $g(t) = 1$ ,  $h(t) = |L^\Delta(t)|$ ,  $k(t, s) = |M^{\Delta t}(t, s)|$ ,  $w_1(x) = w_3(x) = w(x)$  and  $w_2(x) = x$  we conclude that the equation

$$x^\Delta(t) = L^\Delta(t) + M(\sigma(t), t)w(|x(t)|) + \int_0^t M^{\Delta t}(t, s)w(|x(s)|)\Delta s,$$

with initial value  $x(0) = L(0)$  has a solution  $x \in C^1([0, b]_{\mathbb{T}}, \mathbb{R})$ . Now, an integration on  $[0, t]_{\mathbb{T}}$  gives, using Lemma 2.1,

$$x(t) = L(t) + \int_0^t M(t, s)w(|x(s)|)\Delta s, \quad t \in [0, b]_{\mathbb{T}}.$$

By the assumptions on functions  $L$ ,  $M$  and  $w$  we conclude that  $x(t) \geq 0$  for all  $t \in [0, b]_{\mathbb{T}}$ , hence  $x$  is a nonnegative solution of (4.1).  $\square$

**Remark 4.2.** From the point of view of applications it is usual to search for nonnegative solutions of equations of the type given in (4.1).

**Corollary 4.3.** *If in equation (4.1),  $L(t) > 0$  on  $[0, b]_{\mathbb{T}}$  and  $w$  is continuously differentiable on  $(0, \infty)$ , then the solution obtained by Theorem 4.1 is unique.*

*Proof.* Assume that there exist two positive solutions  $x, y$  on  $[0, b]_{\mathbb{T}}$  to (4.1). Then,

$$x(t) - y(t) = \int_0^t M(t, s)[w(x(s)) - w(y(s))]\Delta s, \quad t \in [0, b]_{\mathbb{T}},$$

hence,

$$|x(t) - y(t)| \leq \int_0^t M(t, s)|w(x(s)) - w(y(s))|\Delta s, \quad t \in [0, b]_{\mathbb{T}}.$$

Let us now define the following numbers:

$$\begin{aligned}\gamma_1 &= \min_{s \in [0, b]_{\mathbb{T}}} \{x(s), y(s)\}, \\ \gamma_2 &= \max_{s \in [0, b]_{\mathbb{T}}} \{x(s), y(s)\}, \\ \nu &= \max_{x \in [\gamma_1, \gamma_2]} w'(x), \\ \mu &= \max_{(t, s) \in [0, b]_{\mathbb{T}} \times [0, b]_{\mathbb{T}}^{\neq}} M(t, s).\end{aligned}$$

The mean value theorem guarantees that

$$|w(x(t)) - w(y(t))| \leq \nu |x(t) - y(t)|, \quad t \in [0, b]_{\mathbb{T}},$$

therefore

$$|x(t) - y(t)| \leq \int_0^t \mu \nu |x(s) - y(s)| \Delta s, \quad t \in [0, b]_{\mathbb{T}}.$$

Using the Gronwall inequality (see, e.g., [2]), we conclude that  $|x(t) - y(t)| \leq 0$  on  $[0, b]_{\mathbb{T}}$  and this implies that  $x(t) = y(t)$  on  $[0, b]_{\mathbb{T}}$ .  $\square$

Now we want to mention some particular cases of equation (4.1). Define on  $\mathbb{R}_0^+$  the function  $w(x) = x^r$ ,  $0 \leq r \leq 1$ . Then, the equation

$$(4.2) \quad x(t) = L(t) + \int_0^t M(t, s) [x(s)]^r \Delta s, \quad t \in [0, b]_{\mathbb{T}},$$

has a unique positive solution if  $L(t) > 0$  for all  $t \in [0, b]_{\mathbb{T}}$ . This type of equation appears in many applications such as nonlinear diffusion, cellular mass change dynamics, or studies concerning the shape of liquid drops [14]. Some results concerning existence and uniqueness of solutions of (4.2) [with  $\mathbb{T} = \mathbb{R}$ ] were previously obtained (see [14] and references therein).

If we define  $u(t) = [x(t)]^r$ , it follows from (4.2) that,

$$(4.3) \quad [u(t)]^{\frac{1}{r}} = L(t) + \int_0^t M(t, s) u(s) \Delta s, \quad t \in [0, b]_{\mathbb{T}}.$$

It is easily seen that  $u(t)$  is the unique solution of (4.3) and, if we let  $r = \frac{1}{2}$  and  $M(t, s) = K(t - s)$  for  $K \in C^1(\mathbb{R}_0^+, \mathbb{R}_0^+)$ , it follows that

$$(4.4) \quad [u(t)]^2 = L(t) + \int_0^t K(t - s) u(s) \Delta s, \quad t \in [0, b]_{\mathbb{T}}.$$

This equation appears in the mathematical theory of the infiltration of a fluid from a cylindrical reservoir into an isotropic homogeneous porous medium [15, 16]. Some existence and uniqueness theorems regarding solutions of (4.4) were obtained in [6, 7, 8, 15, 16].

**Remark 4.4.** Corollary 4.3 proves uniqueness of a solution to the integral equation (4.4) under different assumptions on the function  $L$  than those in [6, 7, 8, 15, 16] (obviously considering  $\mathbb{T} = \mathbb{R}$ ). For example, (to prove uniqueness) in [7], the author

considered  $L \in C^1(\mathbb{R}_0^+, \mathbb{R}_0^+)$  such that  $L'(0) \neq 0$ . Therefore, if we, e.g., let  $L(t) = t^2 + 1$ ,  $t \in [0, b]$ , we see that  $L'(0) = 0$ , hence we cannot use the results obtained in [7]. If we let  $L(t) = t$ , Corollary 4.3 cannot be applied.

We end this paper giving an example of what the integrodynamic equation in (1.1) could like on a particular time scale different from  $\mathbb{R}$ , namely, in  $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$  for some  $h > 0$ . Let  $a = A = 0$  and  $b = hm$  for some  $m \in \mathbb{N}$ . Then, on this time scale, (1.1) becomes

$$\frac{x(t+h) - x(t)}{h} = F \left( t, x(t), \sum_{k=0}^{m-1} hK[t, kh, x(kh)] \right), \quad x(0) = 0, \quad t \in [a, b]_{h\mathbb{Z}}^{\kappa}.$$

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