

ON STABILITY OF ABSTRACT MEASURE DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, an existence as well as an existence and uniqueness result are proved for an abstract measure delay integro-differential equation. The extendability and stability of solutions are also discussed. An illustrative example is included.

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1. INTRODUCTION

A functional integro-differential equation with delay is a hereditary system in which the rate of change, or the derivative of the unknown function or set-function, depends upon the past history. A functional integro-differential equation of neutral type is a hereditary system in which the derivative of the unknown function is determined by the values of a state variable as well as the derivative of the state variable over some past interval in the phase space. Although the general theory and the basic results for integro-differential equations have now been thoroughly investigated, the study of functional integro-differential equations is nowhere near complete. In recent years, there has been an increasing interest in such equations among mathematicians in many countries.

The study of abstract measure differential equations was initiated by Sharma [11, 12] and Dhage *et al.* [6], while the study of abstract measure integro-differential equations was initiated and developed at length in a series of papers by Dhage [1, 2, 4] and Dhage and Bellale [5]. However, the study of functional abstract measure integro-differential equations has not appeared in the literature. The study of abstract measure delay differential equations was initiated by Joshi [8], Joshi and Deo [9], and Shendge and Joshi [13], and subsequently developed by Dhage [1]–[4]. Following the

approach in the above mentioned papers, here we prove existence and stability results for an abstract measure delay integro-differential equation. The results in this paper extend the results of Joshi [8] on abstract measure delay differential equations, under weaker conditions, to abstract measure integro-differential equations.

2. PRELIMINARIES

Let \mathbb{R} denote the real line and let \mathbb{R}^n be an Euclidean space with respect to the norm $|\cdot|_n$ defined by

$$(2.1) \quad |x|_n = \max\{|x_1|, \dots, |x_n|\}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Let X be a real Banach space with any convenient norm $\|\cdot\|$. For any two points x, y in X , the *segment* \overline{xy} in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}.$$

Let x_0 and y_0 be two fixed points in X such that $\overline{0y_0} \subset \overline{0x_0}$, where 0 is the zero vector of X . Let z be a point of X , such that $\overline{0x_0} \subset \overline{0z}$. For this z and $x \in \overline{y_0z}$, define the sets S_x and \overline{S}_x by

$$S_x = \{rx : -\infty < r < 1\}$$

and

$$\overline{S}_x = \{rx : -\infty < r \leq 1\}.$$

For $x_1, x_2 \in \overline{y_0z}$, we write $x_1 < x_2$ (or $x_2 > x_1$) if $\overline{y_0x_1} \subset \overline{y_0x_2}$. Let the positive number $\|x_0 - y_0\|$ be denoted by w . For each $x \in \overline{x_0z}$, $z > x_0$, let x_w denote that element of $\overline{y_0z}$ satisfying

$$x_w < x \quad \text{and} \quad \|x - x_w\| = w.$$

Note that x_w and wx are not the same points unless $w = 0$ and $x = 0$.

Let M denote the σ -algebra of all subsets of X so that (X, M) becomes a measurable space. Let $\text{ca}(X, M)$ be the space of all vector measures (signed measures) and define a norm $\|\cdot\|$ on $\text{ca}(X, M)$ by

$$(2.2) \quad \|p\| = |p|_n(X)$$

where $|p|$ is a total variation measure of p and is given by

$$(2.3) \quad |p|_n(X) = \sum_{i=1}^{\infty} |p(E_i)|_n,$$

for all $E_i \subset X$ with $X = \bigcup_{i=1}^{\infty} E_i$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. It is known that $\text{ca}(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (2.2). Let μ be a σ -finite measure on X and let $p \in \text{ca}(X, M)$. We say p is *absolutely continuous with respect*

to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for all $E \in M$. In this case, we write $p \ll \mu$.

For a fixed $x_0 \in X$, let M_0 be the smallest σ -algebra on $\overline{S_{x_0}}$ containing $\{x_0\}$ and the sets $S_x, x \in \overline{y_0x_0}$. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ -algebra of all sets containing M_0 and the sets of the form $\overline{S_x}$ for $x \in \overline{x_0z}$. For a given $H > 0$, we define the set B_H by

$$B_H = \{u \in \mathbb{R}^n \mid |u|_n < H\}.$$

Finally, let $L^1_\mu(S_z, \mathbb{R})$ denote the space of all μ -integrable real-valued functions h on S_z with the norm $\|\cdot\|_{L^1_\mu}$ defined by

$$\|h\|_{L^1_\mu} = \int_{S_z} |h(x)| d|\mu|.$$

3. STATEMENT OF THE PROBLEM

Let μ be a σ -finite real measure on X . Given a $p \in ca(X, M)$ with $p \ll \mu$, consider the abstract measure delay integro-differential equation (in short delay AMIGDE) involving the delay w ,

$$(3.1) \quad \left. \begin{aligned} \frac{dp}{d\mu} &= \int_{\overline{S_{x_w}}} f(t, p(\overline{S_t}), p(\overline{S_{t_w}})) d\mu \text{ a.e. } [\mu] \text{ on } \overline{x_0z}, \\ p(E) &= q(E), \quad E \in M_0, \end{aligned} \right\}$$

where q is a given initial known measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ , and the function $f : S_z \times B_H \times B_H \rightarrow \mathbb{R}^n$ is such that the map $x \mapsto \lambda(p(\overline{S_x})) = \int_{\overline{S_{x_w}}} f(t, p(\overline{S_t}), p(\overline{S_{t_w}})) d\mu$ is μ -integrable for each $p \in ca(S_z, M_z)$.

Details on Radon-Nikodym derivatives along with some of their properties may be found in Rudin [10].

For what follows, we need to define the class of sets C_H by

$$C_H = \{q \in ca(S_{x_0}, M_0) \mid \|q\| + C < H\},$$

where $C > 0$. Here, it is implied that H is large enough so that C_H is not empty.

Definition 3.1. Given an initial real measure $q \in C_H$ on M_0 , a vector $p \in ca(S_z, M_z)$ ($z > x$) is said to be a solution of the delay AMIGDE (3.1) if

- (i) $p(E) = q(E), E \in M_0,$
- (ii) $p \ll \mu$ on $\overline{x_0z},$
- (iii) p satisfies (3.1) a.e. $[\mu]$ on $\overline{x_0z}.$

Remark 3.2. The delay AMIGDE (3.1) is equivalent to the abstract measure integral equation

$$(3.2) \quad p(E) = \begin{cases} \int_E \left(\int_{\overline{S}_{x_w}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu, & \text{if } E \in M_z, \quad E \subset \overline{x_0 z}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

A solution p of the delay AMIGDE (3.1) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

We will employ Schauder's fixed point theorem to prove our main existence result for the delay AMIGDE (3.1). Before stating this result, we give a useful definition.

Definition 3.3. An operator Q on a Banach space X into itself is called compact if for any bounded subset S of X , $Q(S)$ is a relatively compact subset of X . We say that Q is totally bounded if $Q(S)$ is a totally bounded subset of X for each bounded subset S of X . If Q is continuous and totally bounded, then it is said to be completely continuous on X .

Note that every compact operator is totally bounded, but the converse may not be true. However, both the notions coincide on bounded subsets of X . Details on different types of such operators may be found in Granas and Dugundji [7].

Theorem 3.4 (Smart [14, p. 15]). *Let S be a non-empty, closed, convex and bounded subset of the Banach space X and let $Q : S \rightarrow S$ be a continuous and compact operator. Then the operator equation $Qx = x$ has a solution.*

In the next section we prove our main existence theorem for the delay AMIGDE (3.1) under suitable conditions on the function f .

4. EXISTENCE AND UNIQUENESS THEOREMS

We need the following definitions.

Definition 4.1. A function $\beta : S_z \times B_H \times B_H \rightarrow \mathbb{R}^n$ is said to satisfy conditions of Carathéodory type (or simply, is Carathéodory) if

- (i) $x \rightarrow \beta(x, y, z)$ is μ -measurable for each $(y, z) \in B_H \times B_H$, and
- (ii) $(y, z) \rightarrow \beta(x, y, z)$ is continuous for almost every μ and $x \in \overline{x_0 z}$.

Definition 4.2. A Carathéodory function β is called L^1_μ -Carathéodory if

- (iii) for each given real number $\rho > 0$, $\rho < H$, there exists a function $h_\rho \in L^1_\mu(S_z, \mathbb{R})$ such that

$$|\beta(x, y, z)|_n \leq h_\rho(x) \text{ a.e. } [\mu] \text{ and } x \in \overline{x_0 z}$$

for all $y, z \in B_H$ with $|y|_n \leq \rho$ and $|z|_n \leq \rho$.

Definition 4.3. A Carathéodory function $\beta : S_z \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called $L^1_\mu(\mathbb{R}^n)$ -Carathéodory if there exists a function $h \in L^1_\mu(S_z, \mathbb{R})$ such that

$$|\beta(x, y, z)|_n \leq h(x) \text{ a.e. } [\mu] \text{ and } x \in \overline{x_0 z}$$

for all $y, z \in B_H$.

We consider the following set of assumptions.

(A₀) $\mu(\{x_0\}) = 0$.

(A₁) For any $z > x_0$, the σ -algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined by

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2), \quad E_1, E_2 \in M_z.$$

(A₂) The function q is continuous on M_z with respect to the pseudo-metric d defined in (A₁).

(A₃) The function $f(x, y, z)$ is $L^1_\mu(\mathbb{R}^n)$ -Carathéodory.

(A₄) There exist functions $\ell_1, \ell_2 \in L^1_\mu(S_z, \mathbb{R}^+)$ such that

$$|f(x, y_1, z_1) - f(x, y_2, z_2)|_n \leq \ell_1(x)|y_1 - y_2|_n + \ell_2(x)|z_1 - z_2|_n$$

for all $y_1, z_1, y_2, z_2 \in B_H$.

Our first existence result is the following.

Theorem 4.4. *Suppose that assumptions (A₁)–(A₃) hold. Then, for a given initial measure $q \in C_H$, the delay AMIGDE (3.1) admits a solution $p(\overline{S}_{x_0}, q)$ on $\overline{x_0 x_1}$ for some $x_1 \in \overline{x_0 z}$.*

Proof. Let $\{r_n\}$ ($r_n > 1$) be a decreasing sequence of real numbers such that $r_n \rightarrow 1$ as $n \rightarrow \infty$, and

$$S_{r_0 x_0} \supset S_{r_1 x_0} \cdots \supset S_{r_n x_0} \supset \cdots .$$

Then we have

$$\lim_{n \rightarrow \infty} \mu(\{S_{r_n x_0} - S_{x_0}\}) = 0.$$

Therefore, there exists a real number r and a point $x_1 = r x_0$ such that

$$(4.1) \quad \int_{\overline{x_0 x_1}} \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu| < H - \|q\|.$$

This is possible by virtue of (A₀) and the positiveness of $|\mu|$.

Now in the Banach space $\text{ca}(S_{x_1}, M_{x_1})$, we define a subset S by

$$(4.2) \quad S = \{p \in \text{ca}(S_z, M_z) \mid p(E) = q(E) \text{ if } E \in M_0 \text{ and } \|p\| \leq K\},$$

where the constant K is given by

$$(4.3) \quad K = \|q\| + \int_{\overline{x_0 x_1}} \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu|.$$

From (4.1)–(4.3), it follows that

$$\|q\| < H \quad \text{for all } q \in \text{ca}(S_{x_0}, M_{x_0}).$$

Define an operator T from S into $\text{ca}(S_{x_1}, M_{x_1})$ by

$$(4.4) \quad Tp(E) = \begin{cases} \int_E \left(\int_{\overline{S_{x_w}}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu, & \text{if } E \in M_z, \quad E \subset \overline{x_0 z}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

We shall show that the operator T satisfies all the conditions of Theorem 3.4 on S .

Step I: We show that T continuously maps S into itself. First, we show that the operator T maps S into itself. Let $p \in S$ be arbitrary and let $E \in M_{x_1}$. Then, there are sets $F \in M_0$ and $G \in M_{x_1}$, $G \subset \overline{x_0 x_1}$, such that $E = F \cup G$. We then have

$$\begin{aligned} |Tp(E)|_n &\leq |q(F)|_n + \int_G \left(\int_{\overline{S_{x_w}}} |f(t, p(\overline{S}_t), p(\overline{S}_{t_w}))|_n d|\mu| \right) d|\mu| \\ &\leq \|q\| + \int_G \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu| \\ &\leq \|q\| + \int_{\overline{x_0 x_1}} \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu| \\ &= K \end{aligned}$$

for all $E \in M_{x_1}$. From the definition of the norm in $\text{ca}(S_{x_1}, M_{x_1})$, we have

$$\|Tp\| \leq \|q\| + \int_{\overline{x_0 x_1}} \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu| = K.$$

This shows that T maps S into itself.

Next, we show that T is continuous on S . Let $\{p_n\}$ be a sequence of vector measures in S converging to a vector measure p , that is, $\lim_{n \rightarrow \infty} \|p_n - p\| = 0$. Set $m_1 = |\mu|(x_0 x_1)$. Then for $\eta > 0$, by hypothesis (A₃), there exists $\delta > 0$ such that

$$\|p_n - p\| < \delta \quad \text{implies} \quad |f(x, p_n(\overline{S}_x), p_n(\overline{S}_{x_w})) - f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))|_n < \frac{\eta}{m_1^2}.$$

Therefore, for any $E \in M_{x_1}$,

$$\begin{aligned} |Tp_n(E) - Tp(E)|_n &\leq \int_E \left(\int_{\overline{S_{x_w}}} |f(t, p_n(\overline{S}_t), p_n(\overline{S}_{t_w})) - f(t, p(\overline{S}_t), p(\overline{S}_{t_w}))|_n d|\mu| \right) d|\mu| \\ &\leq \int_E \left(\int_{\overline{S_{x_w}}} \frac{\eta}{m_1^2} d|\mu| \right) d|\mu| \\ &< \eta \end{aligned}$$

provided $\|p_n - p\| < \delta$. This shows that T is a continuous operator on S .

Step II: Next, we show that $T(S)$ is a uniformly bounded and equi-continuous set in $\text{ca}(S_{x_1}, M_{x_1})$. Now, as in Step I, we see that $T(S)$ is a subset of S and hence it

is a uniformly bounded set in $ca(S_{x_1}, M_{x_1})$. From the definition of the map T ,

$$Tp(E) = \begin{cases} \int_E \left(\int_{\overline{S_{x_w}}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0x_1}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

To show that $T(S)$ is an equi-continuous set in $ca(S_{x_1}, M_{x_1})$, let $E_1, E_2 \in M_z$. Then there are sets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{x_1}$ with $G_1, G_2 \subset \overline{x_0x_1}$ and

$$F_i \cap G_i = \emptyset, \quad i = 1, 2.$$

Recalling the set-identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1) \quad \text{and} \quad G_2 = (G_2 - G_1) \cup (G_2 \cap G_1),$$

we have

$$\begin{aligned} Tp(E_1) - Tp(E_2) &= q(F_1) - q(F_2) \\ &+ \int_{G_1 - G_2} \left(\int_{\overline{S_{x_w}}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu \\ &- \int_{G_2 - G_1} \left(\int_{\overline{S_{x_w}}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu. \end{aligned}$$

Since $f(x, y, z)$ is $L^1_\mu(\mathbb{R}^n)$ -Carathéodory, we have that

$$\begin{aligned} |Tp(E_1) - Tp(E_2)|_n &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \Delta G_2} \left(\int_{\overline{S_{x_w}}} |f(t, p(\overline{S}_t), p_n(\overline{S}_{t_w}))|_n d|\mu| \right) d|\mu| \\ &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \Delta G_2} \left(\int_{\overline{S_{x_w}}} h(t) d|\mu| \right) d|\mu| \\ &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \Delta G_2} \|h\|_{L^1_\mu} d|\mu|. \end{aligned}$$

Assuming that $d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \rightarrow 0$, we have $E_1 \rightarrow E_2$, and consequently $F_1 \rightarrow F_2$ and $|\mu|(G_1 \Delta G_2) \rightarrow 0$. From the continuity of q on M_0 , it follows that

$$\begin{aligned} |Tp(E_1) - Tp(E_2)|_n &\leq |q(F_1) - q(F_2)|_n + \int_{G_1 \Delta G_2} \|h\|_{L^1_\mu} d|\mu| \\ &\rightarrow 0 \quad \text{as } E_1 \rightarrow E_2 \end{aligned}$$

uniformly for all $p \in S$. This shows that $T(S)$ is an equi-continuous set in $ca(S_{x_1}, M_{x_1})$. Thus, $T(S)$ is a uniformly bounded and equi-continuous set in $ca(S_z, M_z)$, so it is compact in the norm topology on $ca(S_{x_1}, M_{x_1})$. Now an application of the Arzelá-Ascoli Theorem yields that $T(S)$ is a compact subset of $ca(S_{x_1}, M_{x_1})$. As a result, T is a continuous and compact operator on S . Theorem 3.4 then yields that the operator equation $p = Tp$ has a solution in S . As a result, the delay AMIGDE (3.1) has a solution on $\overline{x_0x_1}$. This completes the proof of the theorem. \square

Our next existence result is proved by an application of the Banach fixed point theorem.

Theorem 4.5. *Suppose that assumptions (A₀)–(A₂) and (A₄) hold. If there exists a point $x_1 \in \overline{x_0 z}$ such that the inequality (4.1) holds and*

$$(4.5) \quad m_1 \left(\|\ell_1\|_{L^1_\mu} + \|\ell_2\|_{L^1_\mu} \right) < 1,$$

where $m_1 = |\mu|(\overline{x_0 x_1})$, then for a given initial measure $q \in C_H$, the delay AMIGDE (3.1) has a unique solution $p(\overline{S}_{x_0}, q)$ on $\overline{x_0 x_1}$.

Proof. Proceeding with the arguments as in the proof of Theorem 4.4 shows that the operator T defined by (4.3) maps a subset S of $\text{ca}(S_{x_1}, M_{x_1})$ into itself, where S is defined by (4.2). Condition (4.5) can then be used to show that T is continuous and is a contraction operator on S . So by an application of Banach's fixed point theorem, there is a unique solution to the operator equation $Tp = p$. This corresponds to a unique solution $p(\overline{S}_{x_0}, q)$ to the delay AMIGDE (3.1) existing on $\overline{x_0 x_1}$, and completes the proof of the theorem. \square

5. EXTENSION AND STABILITY

A solution of the delay AMIGDE (3.1) so obtained can be extended to the larger segment whenever $\mu(\{x_1\}) = 0$. An existence result in this situation is the following.

Theorem 5.1. *Under the hypotheses of Theorem 4.4, let $p(\overline{S}_{x_0}, q)$ be a solution of the delay AMIGDE (3.1) on $\overline{x_0 x_1}$. Then the solution p can be extended to a larger segment if $\mu(\{x_1\}) = 0$.*

Proof. If $\mu(\{x_1\}) = 0$, then following the proof of Theorem 4.4, we may choose a point $x_2 \in \overline{x_1 z}$ such that (4.1) holds. Then repeating the arguments, it can be proved that p is a solution of the delay AMIGDE (3.1) that is defined on $\overline{x_1 x_2}$. \square

Next, we shall obtain a result concerning the stability of solutions for the delay AMIGDE (3.1) in the sense of following definition.

Definition 5.2. Let $q \in C_H$. If for each $\varepsilon > 0$, $\varepsilon < H$, there exists a number $\eta = \eta(\varepsilon)$ and a solution $p(\overline{S}_{x_0}, q)$ to the AMIGDE (3.1) such that $\|q\| < \eta$ implies $\|p\| < \varepsilon$, then we say p is locally stable with respect to the initial measure q .

We will make use of the following conditions to prove our main result in this section.

(B₁) The function $f : S_z \times B_H \times B_H \rightarrow \mathbb{R}^n$ is μ -integrable and $f(x, 0, 0) = 0$ for all $x \in S_z$ for some $z > x_0$.

(B₂) Given $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|f(x, y_1, z_1) - f(x, y_2, z_2)|_n \leq \delta [|y_1 - y_2|_n + |z_1 - z_2|_n]$$

for all $y_1, y_2, z_1, z_2 \in B_H$ with $|y_1|_n \leq \varepsilon, |y_2|_n \leq \varepsilon, |z_1|_n \leq \varepsilon,$ and $|z_2|_n \leq \varepsilon.$

Theorem 5.3. *Let the assumptions (B₁)–(B₂) hold. Then for each $\varepsilon > 0$ and a fixed number $b \in (0, 1)$, there exists a unique solution $p(\overline{S}_{x_0}, q)$ of the delay AMIGDE (3.1) satisfying $\|p\| \leq \varepsilon$ whenever $\|q\| \leq b\varepsilon.$*

Proof. Let $\delta = \frac{1-b}{2m^2},$ where $m = \mu(\overline{x_0z}).$ Then corresponding to this $\delta,$ there is a number $\varepsilon > 0$ such that

$$(5.1) \quad |f(x, y_1, z_1) - f(x, y_2, z_2)|_n \leq \delta [|y_1 - y_2|_n + |z_1 - z_2|_n]$$

for all $y_1, y_2, z_1, z_2 \in \mathbb{R}$ with $|y_1|_n \leq \varepsilon, |y_2|_n \leq \varepsilon, |z_1|_n \leq \varepsilon,$ and $|z_2|_n \leq \varepsilon.$

Define a subset $\overline{S}(\varepsilon)$ of $\text{ca}(S_z, M_z)$ by

$$(5.2) \quad \overline{S}(\varepsilon) = \{p \in \text{ca}(S_z, M_z) \mid \|p\| \leq \varepsilon\}.$$

Let $p \in \overline{S}(\varepsilon).$ Using (B₁) and (B₂), we obtain

$$(5.3) \quad |f(x, p(\overline{S}_x), p(\overline{S}_{x_w}))|_n \leq 2\delta \varepsilon.$$

Define an operator T from $\overline{S}(\varepsilon)$ into $\text{ca}(S_z, M_z)$ by

$$(5.4) \quad Tp(E) = \begin{cases} \int_E \left(\int_{\overline{S}_{x_w}} f(t, p(\overline{S}_t), p(\overline{S}_{t_w})) d\mu \right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0z}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

Now, if $E \in M_z,$ then there are two disjoint sets F and G in M_z such that

$$E = F \cup G, \quad F \in M_0, \quad G \subset \overline{x_0z}.$$

Hence, for $E \in M_z,$ from (5.2) and (5.3), it follows that

$$\begin{aligned} |Tp(E)|_n &\leq |q|_n(E) + \int_G \left(\int_{\overline{S}_{x_w}} |f(t, p(\overline{S}_t), p(\overline{S}_{t_w}))|_n d|\mu| \right) d|\mu| \\ &\leq \|q\| + 2 \int_G \left(\int_{\overline{S}_{x_w}} \delta \varepsilon d|\mu| \right) d|\mu| \\ &\leq \|q\| + 2m^2\delta \varepsilon \\ &\leq b\varepsilon + (1 - b)\varepsilon \\ &= \varepsilon. \end{aligned}$$

for all $p \in \overline{S}(\varepsilon).$ Therefore, $\|Tp\| \leq \varepsilon$ for all $p \in \overline{S}(\varepsilon).$

This shows that T maps $\overline{S}(\varepsilon)$ into itself. Next, we show that T is a contraction operator on $\overline{S}(\varepsilon)$. Let $p_1, p_2 \in \overline{S}(\varepsilon)$. Then, by (B₂),

$$\begin{aligned}
 |Tp_1(E) - Tp_2(E)|_n &\leq \left| \int_E \left(\int_{\overline{S}_{x_w}} f(t, p_1(\overline{S}_t), p_1(\overline{S}_{t_w})) d\mu \right) d|\mu| \right. \\
 &\quad \left. - \int_E \left(\int_{\overline{S}_{x_w}} f(t, p_2(\overline{S}_t), p_2(\overline{S}_{t_w})) d|\mu| \right) d|\mu| \right| \\
 &\leq \int_E \left(\int_{\overline{S}_{x_w}} |f(t, p_1(\overline{S}_t), p_1(\overline{S}_{t_w})) - f(t, p_2(\overline{S}_t), p_2(\overline{S}_{t_w}))|_n d|\mu| \right) d|\mu| \\
 &\leq \int_E \left(\int_{\overline{S}_{x_w}} \delta [|p_1(\overline{S}_t) - p_2(\overline{S}_t)|_n + |p_1(\overline{S}_{t_w}) - p_2(\overline{S}_{t_w})|_n] d|\mu| \right) d|\mu| \\
 &\leq 2 \int_E \left(\int_{\overline{S}_{x_w}} \delta \|p_1 - p_2\| d|\mu| \right) d|\mu| \\
 &\leq 2m^2\delta \|p_1 - p_2\| \\
 (5.5) \qquad &= (1 - b)\|p_1 - p_2\|
 \end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. This further implies that

$$\|Tp_1 - Tp_2\| \leq \alpha \|p_1 - p_2\|$$

where, $\alpha = (1 - b) < 1$. This shows that T is a contraction operator on $\overline{S}(\varepsilon)$ with the contraction constant α . Therefore, by an application of the contraction mapping principle, there is a unique solution $p(\overline{S}_{x_0}, q)$ of the delay AMIGDE (3.1) satisfying $\|p\| \leq \varepsilon$ whenever $\|q\| \leq b\varepsilon$. This completes the proof. \square

Example 5.4. Let $X = \mathbb{R}$, μ be the Lebesgue measure on \mathbb{R} , $\overline{S}_x = [0, x]$, $x > 0$, and $q(E) = \mu(E)$, $E \subset [0, 1]$. Consider the delay AMIGDE

$$(5.6) \qquad \frac{dp}{d\mu} = 6 \int_{\overline{S}_{x-1/2}} p(\overline{S}_{t-1/2}) d\mu.$$

and

$$(5.7) \qquad p(E) = q(E), \quad E \subset [0, 1].$$

Here, $w = 1/2$. For $0 \leq x \leq 1$, we observe that

$$p(\overline{S}_x) = p([0, x]) = q([0, x]) = x.$$

If $x \in [1, 2]$, then we have

$$\begin{aligned}
 p(\overline{S}_x) &= q(\overline{S}_1) + \int_{[1,x]} \left(\int_{[1,t-1/2]} 6p(\overline{S}_{s-1/2}) ds \right) dt \\
 &= 1 + 6 \int_1^x \left(\int_1^{t-1/2} (s - 1/2) ds \right) dt \\
 &= 1 + 3 \int_1^x \left[(t - 1)^2 - \frac{1}{4} \right] dt \\
 &= 1 + 3 \int_1^x (t - 1)^2 dt - 3 \int_1^x \frac{1}{4} dt \\
 &= (x - 1)^3 - \frac{3}{4}(x - 1) + 1.
 \end{aligned}$$

Again, if $2 \leq x \leq 3$, then we obtain

$$p(\overline{S}_x) = x^3 - 3x^2 - \frac{15}{4}x + \frac{51}{4},$$

and so on. In this way, the solution p for the linear delay AMIGDE (5.6) can be found recursively on $[0, \infty)$.

Remark 5.5. The above example suggests a method to compute the solution of an AMIGDE, in the particular case when $f(x, y, z)$ is linear in y and z .

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