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SHADOWING PROPERTY FOR INDUCED SET-VALUED DYNAMICAL SYSTEMS OF SOME EXPANSIVE MAPS

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ABSTRACT. In this paper, we study the shadowing property for induced set-valued dynamical systems of some expansive maps. We show that if f is a positively expansive open map, then the induced map F has shadowing property. We introduce the notion of ball expansive maps, and show that such maps have shadowing property.

Keywords: pseudo-orbit; shadowing property; induced set-valued map; expansive maps; ball expansive maps

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1. INTRODUCTION

The notion of pseudo-orbit goes back at least to Birkhoff [2], and plays an important role in the investigation of properties of discrete dynamical systems. In the study of dynamical systems, people often make computer simulations in which there are always no real trajectories of dynamical systems. Then it arises naturally that what is the relationship between the computer output and the underlying dynamics? Bowen [3] and Conley [5] independently discovered that pseudo-orbit could be used as a conceptual tool for discussing this relationship. Can the numerically obtained pseudo-orbits reflect the behavior of the real ones? So it is important to find out in which cases a pseudo-orbit can be shadowed (traced) by a real trajectory. This problem has been well studied in the last several decades, for example, the shadowing near a hyperbolic set of a homeomorphism (see [1, 10, 17]) and the shadowing in structurally stable systems (see [14, 13]). In [8], Gedeon and Kuchta find a necessary and sufficient condition under which continuous maps of type 2^n can possess shadowing property.

Another important topic is the induced set-valued map. Given a continuous map f on the metric space X, the set-valued map induced by f, denoted by F, is the natural extension of f to $\mathscr{K}(X)$, the space of all non-empty compact subsets of X.

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As is well known, the dynamical behavior of the points of X is important and has caught the attention of many scholars. However, in many fields such as computer simulation, biological species, demography, etc, it is not sufficient to know how the points of X move, but it is necessary to understand the dynamical behavior of the subsets of X. So it makes sense to study the set-valued dynamical system $(F, \mathscr{K}(X))$ associated to the system (f, X). In recent years, the connection between dynamical properties of the base map f and the induced map F has attracted many researchers' attention, see for instance [11, 12, 9].

The concept of positively expansive map was introduced by Williams [18] and Eisenberg [6]. Among the dynamical properties of expanding maps is the shadowing of the pseudo-orbits which Bowen called "the most important dynamical property of Axiom A diffeomorphisms" [4]. In [15], Sakai investigated various shadowing properties for a positively expansive map on a compact metrizable space. In the present paper, we will focus on shadowing property of the induced set-valued map of some expansive map. Inspired by Sakai's work [15], we show that if f is a positively expansive open map, then the induced map F has shadowing property (Theorem 3.2). Moreover, we propose a new class of expansive maps – ball expansive maps (see Definition 3.6), and prove that if f is ball expansive, then the induced map F has shadowing property Theorem 3.8.

2. PRELIMINARIES

Let X be a compact metrizable space, and $f : X \to X$ be a continuous map. Fix any metric d for X, which is compatible with the topology of X. Recall that:

- a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called *an orbit* for f, if $x_{i+1} = f(x_i)$ for all $i \geq 0$;
- A sequence $\{x_i\}_{i=0}^{\infty}$ in X is called a *δ-pseudo-orbit* ($\delta > 0$) for f, if $d(f(x_i), x_{i+1}) < \delta$, for all $i \geq 0$.
- Let $\varepsilon, \delta > 0$. We say that a δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ for f is *ε-shadowed* by an orbit $\{f^i(y)\}_{i=0}^{\infty}$, if

$$d(f^i(y), x_i) < \varepsilon \quad \text{for all } i \geq 0.$$

  Here, f^i is the i-th iteration of f with itself.

Definition 2.1. Let $f : X \to X$ be a continuous map. We say that f has the *shadowing property* (or *pseudo-orbit tracing property*) on X, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit for f is ε-shadowed by some orbit for f.

Let $\mathscr{K}(X)$ denote the collection of all nonempty compact subsets of X, i.e.,

$$\mathscr{K}(X) = \{A \subset X : \ A \text{ is nonempty and compact}\}.$$

$\mathscr{K}(X)$ will be referred to as an hyperspace of X. The Vietoris topology $\mathbb{V}$ on $\mathscr{K}(X)$ is the topology generated by the basis $\mathbb{B}$ consisting of all sets of the form

$$\mathscr{B}(U_1, U_2, \dots, U_n) := \left\{ A \in \mathscr{K}(X) : A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset,\ 1 \leq i \leq n \right\},$$

where $U_1, U_2, \dots, U_n$ are non-empty open subsets of X.

Let $A, B \subset X$ be nonempty subsets. The distance from a point x to A is defined by $d(x, A) = \inf\{d(x, a) : a \in A\}$. We put

$$e_d(A, B) := \sup\{d(x, B) : x \in A\}.$$

The *Hausdorff metric* H_d on $\mathscr{K}(X)$ is defined by

$$H_d(A, B) := \max\{e_d(A, B), e_d(B, A)\}.$$

Endowed with the Housdorff metric, $\mathscr{K}(X)$ becomes a complete separable metric space. It is well known that the topology induced by the Hausdorff metric H_d on $\mathscr{K}(X)$ coincides with the Vietoris topology $\mathbb{V}$ on $\mathscr{K}(X)$(see[7]).

For $a \in X, A \in \mathscr{K}(X)$ and $\varepsilon > 0$, we define the ε-balls in (X, d) and $(\mathscr{K}(X), H_d)$ by

$$B_d(a, \varepsilon) = \{x \in X : d(x, a) < \varepsilon\},$$
$$\mathbb{B}_d(A, \varepsilon) = \{K \in \mathscr{K}(X) : H_d(A, K) < \varepsilon\},$$

respectively.

Set $\mathscr{P}(X) = \{A \subset X :$ nonempty, finite$\}$. Since every compact set in X can be approximated by a finite subset of X under the Hausdorff metric, we have the following simple fact.

Proposition 2.2. For a compact metrizable space X, $\overline{\mathscr{P}(X)} = \mathscr{K}(X)$. More precisely, for every $\varepsilon > 0$, there exists an $n_\varepsilon \in N$ such that

$$\forall\ K \in \mathscr{K}(X),\ \exists\ P = \{x_1, x_2, \dots, x_{n_\varepsilon}\} \in \mathscr{P}(X) \quad \text{satisfying } H_d(K, P) < \varepsilon.$$

Definition 2.3. Let f be a continuous map on topological space X. We define $F : \mathscr{K}(X) \to \mathscr{K}(X)$ as

$$F(A) = \{f(a) : a \in A\}, \quad \forall\ A \in \mathscr{K}(X),$$

and F is called the *natural extension* of f to $\mathscr{K}(X)$.

One can easily prove the following proposition.

Proposition 2.4. If $f : X \to X$ is continuous under d, then $F : \mathscr{K}(X) \to \mathscr{K}(X)$ is continuous under H_d.

Proposition 2.5. If the continuous map $f : X \to X$ is open under d, then $F : \mathscr{K}(X) \to \mathscr{K}(X)$ is open under H_d.

Proof. It is sufficient to prove that $F(\mathscr{B}(U_1, U_2, \ldots, U_n))$ is an open set of $\mathscr{K}(X)$ for any basis element $\mathscr{B}(U_1, U_2, \ldots, U_n)$ in $\mathbb{B}$.

Since f is open, $f(U_i)(i = 1, 2, \ldots, n)$ are open sets of X. Therefore, $\mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n))$ is an open set of $\mathscr{K}(X)$. Now it suffices to prove that

$$F(\mathscr{B}(U_1, U_2, \ldots, U_n)) = \mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n)).$$

If $K \in F(\mathscr{B}(U_1, U_2, \ldots, U_n))$, there exists a $K_0 \in \mathscr{B}(U_1, U_2, \ldots, U_n)$ such that $F(K_0) = K$. As

$$K_0 \subset \cup_{i=1}^n U_i, \quad K_0 \cap U_i \neq \emptyset, \quad i = 1, 2, \ldots, n,$$

we deduce that

$$F(K_0) \subset f(\cup_{i=1}^n U_i) = \cup_{i=1}^n f(U_i) \quad \text{and} \quad F(K_0) \cap f(U_i) \neq \emptyset, \quad i = 1, 2, \ldots, n,$$

so that $K = F(K_0) \in \mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n))$. Therefore,

$$F(\mathscr{B}(U_1, U_2, \ldots, U_n)) \subset \mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n)).$$

On the other hand, if $K \in \mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n))$, then $K_1 := F^{-1}(K)$ is compact. According to $K \subset \cup_{i=1}^n f(U_i) = f(\cup_{i=1}^n U_i)$, we have that $K_1 \subset \cup_{i=1}^n U_i$. Moreover, $K \cap f(U_i) \neq \emptyset$ means that $K_1 \cap U_i \neq \emptyset$, $i = 1, 2, \ldots, n$. Therefore,

$$K = F(K_1) \in F(\mathscr{B}(U_1, U_2, \ldots, U_n)).$$

That is,

$$\mathscr{B}(f(U_1), f(U_2), \ldots, f(U_n)) \subset F(\mathscr{B}(U_1, U_2, \ldots, U_n)).$$

The proof is complete. □

3. MAIN RESULTS

We begin by recalling the concept of positive expansive maps on X. A map $f : X \to X$ is *positively expansive*, if there exist a metric d on X and a constant $c > 0$ such that for any two points x, y, $x \neq y$, the inequality $d(f^n(x), f^n(y)) > c$ holds for some n. Such a number $c > 0$ is called an *expansive constant*. This property does not depend on the choice of metric on X, though may depend on c. It is well known that every expansive differentiable map on a C^∞-closed manifold are positively expansive (see[16]).

The following result comes from Sakai [15, Theorem 1] and will play an important role in the proof of the first main result of this paper (Theorem 3.2).

Theorem 3.1 (Sakai)**.** *Let $f : X \to X$ be a positively expansive map on a compact metrizable space X. Then the following conditions are equivalent:*

(1) *f is an open map;*
(2) *f has the shadowing property.*

The first main result is the following.

Theorem 3.2. *Let $f : X \to X$ be a continuous map on compact metrizable space X. If f is positively expansive open map, then $F : \mathscr{K}(X) \to \mathscr{K}(X)$ has shadowing property.*

Proof. It follows from Theorem 3.1 that, for all $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that every δ_1-pseudo orbit for f is $\frac{\varepsilon}{2}$-shadowed by some orbit for f.

Take $\delta = \frac{1}{3}\delta_1$ and let $\{K_0, K_1, K_2, \dots\}$ be a given δ-pseudo orbit of $(\mathscr{K}(X), F)$; that is, $H_d(F(K_i), K_{i+1}) < \delta$, for all $i \geq 0$. We will construct a finite set W whose trajectory ε-shadows the pseudo-orbit $\{K_0, K_1, K_2, \dots\}$; that is,

$$H_d(K_n, F^n(W)) \leq \varepsilon, \quad \forall n \in N. \tag{3.1}$$

By the uniform continuity of f, there exists $0 < \varepsilon_1 \leq \varepsilon$ such that

$$d(x, y) < \varepsilon_1 \Rightarrow d(f(x), f(y)) < \delta. \tag{3.2}$$

Take $\varepsilon_0 = \min\{\frac{\varepsilon}{2}, \varepsilon_1, \delta\}$. Since X is compact, we can find a finite subset $A = \{a_1, a_2, \dots, a_l\} \in \mathscr{P}(X)$ satisfying $\bigcup_{a_i \in A} B_d(a_i, \varepsilon_0) = X$. Similarly, there exists an $A_i = \{x_{ij}\}_{j=1}^{l_i} \subset A$, $(l_i \leq l)$ such that

$$K_i \subset \bigcup_{j=1}^{l_i} B_d(x_{ij}, \varepsilon_0) \quad \text{and} \quad K_i \bigcap B_d(x_{ij}, \varepsilon_0) \neq \emptyset, \quad j = 1, \dots, l_i. \tag{3.3}$$

It follows from (3.3) that, for all $x_{0j_0} \in A_0$ $(j_0 = 1, 2, \dots, l_0)$, there exists an $\tilde{x} \in K_0$ such that $d(x_{0j_0}, \tilde{x}) < \varepsilon_0$. Combining (3.2) with this, we have $d(f(x_{0j_0}), f(\tilde{x})) < \delta$. Since $H_d(F(K_0), K_1) < \delta$, we can find an $z_{1j_1} \in K_1$ satisfying $d(f(\tilde{x}), z_{1j_1}) < \delta$. Using (3.3) again, there exists $x_{1j_1} \in A_1$ such that $z_{1j_1} \in B_d(x_{1j_1}, \varepsilon_0)$. Therefore,

$$\begin{aligned} d(f(x_{0j_0}), x_{1j_1}) &\leq d(f(x_{0j_0}), f(\tilde{x})) + d(f(\tilde{x}), z_{1j_1}) + d(z_{1j_1}, x_{1j_1}) \\ &< \delta + \delta + \varepsilon_0 < 3\delta = \delta_1. \end{aligned}$$

Repeating the process, for each $j_0 \in \{1, 2, \dots, l_0\}$, we can find a sequence $\{x_{0j_0}, x_{1j_1}, \dots, x_{nj_n}, \dots\}$ such that

$$x_{nj_n} \in A_n \subset A \quad \text{and} \quad d(f(x_{nj_n}), x_{n+1,j_{n+1}}) < \delta_1.$$

This means that $\{x_{0j_0}, x_{1j_1}, \dots, x_{nj_n}, \dots\}$ is a δ_1-pseudo orbit of f, so we can find an $y_{j_0} \in X$ satisfying

$$d(f^n(y_{j_0}), x_{n,j_n}) < \frac{\varepsilon}{2}. \tag{3.4}$$

Let $W = \{y_{j_0} : j_0 = 1, 2, \dots, l_0\}$. Appealing again to (3.3), we find, for any $x \in K_n$, some $x_{nj_n} \in A_n$ with $x \in B_d(x_{nj_n}, \varepsilon_0)$. Together with (3.4), we get

$$d(x, f^n(y_{j_0})) \leq d(x, x_{nj_n}) + d(x_{nj_n}, f^n(y_{j_0})) \leq \frac{\varepsilon}{2} + \varepsilon_0 < \varepsilon.$$

Thus, we obtain

$$e_d(K_n, F^n(W)) = \max\{d(x, F^n(W)) : x \in K_n\} \le \max\{d(x, f^n(y_{j_0}) : x \in K_n\} \le \varepsilon. \tag{3.5}$$

On the other hand, for any $y_{j_0} \in W$, by the similar argument as above, we can easily find $z_{nj_n} \in B_d(x_{nj_n}, \varepsilon_0) \cap K_n$ satisfying $d(f^n(y_{j_0}), x_{nj_n}) < \varepsilon$. Thus, we get

$$e_d(F^n(W), K_n) = \max\{d(f^n(y_{j_0}), K_n) : y_{j_0} \in W\} \le \max\{d(f^n(y_{j_0}), x_{nj_n}) : p_{j_0} \in T\} \le \varepsilon. \tag{3.6}$$

Finally, according to (3.5) and (3.6), it is true that

$$H_d(K_n, F^n(W)) \le \varepsilon, \quad \forall\, n \in N.$$

We complete the proof. □

Remark 3.3. Recalling Theorem 3.1, one may try to prove the shadowing property of F by verifying that F is open and positive expansive. Indeed, Proposition 2.5 tells us that if f is open, then F also is open. However, positive expansivity of F can not be deduced from positive expansivity of f. The following example illustrates this.

Example 3.4. Let $S = \{0, 1\}$. Set $S^{Z+} = \prod_0^{+\infty} S = \{x = \{x_i\}_0^{+\infty} : x_i \in S\}$. The metric on S^{Z+} is defined as

$$d(x, y) = \sum_{i=0}^{+\infty} \frac{|x_i - y_i|}{2^i} \qquad \forall\, x, y \in S^{Z+}. \tag{3.7}$$

We define the shift mapping σ on S^{Z+} as

$$\sigma : \{x_0, x_1, \ldots, x_i, \ldots\} \mapsto \{x_1, x_2, \ldots, x_{i+1}, \ldots\}.$$

Then one can easily verify that $\sigma : S^{Z+} \to S^{Z+}$ is positively expansive with expansive constant $c = \frac{1}{2}$. But the natural extension of σ, denoted by Σ, on $\mathscr{K}(S^{Z+})$ is not positively expansive. We give an explanation for this. For any $\varepsilon > 0$, we choose $k \in N$ such that $2^{-k+1} < \varepsilon$.

Let $\theta = \{\theta_i\}_0^{+\infty}$, where $\theta_i = 0$, for all $i = 0, 1, 2, \ldots,$

$$\alpha = \{\alpha_i\}_0^{+\infty}, \quad \text{where } \alpha_i = \begin{cases} 1, & \text{if } i = (2m-1)k \text{ for } m = 1, 2, \ldots \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta = \{\beta_i\}_0^{+\infty}, \quad \text{where } \beta_i = \begin{cases} 1, & \text{if } i = (2m)k \text{ for } m = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Now we take $A = \{\theta, \alpha, \beta\}$, $B = \{\alpha, \beta\}$. obviously it is true that $A, B \in \mathscr{K}(S^{Z+})$. We easily obtain that

$$e_d(\Sigma^n(B), \Sigma^n(A)) = 0,$$

and for each $n \in N$, that

$$e_d(\Sigma^n(A), \Sigma^n(B)) = \min\{d(\sigma^n(\theta), \sigma^n(\alpha)), d(\sigma^n(\theta), \sigma^n(\alpha))\} < 2^{-k+1} < \varepsilon.$$

So for any positive integers n, we get that $H_d(\Sigma^n(A), \Sigma^n(B)) < \varepsilon$, which implies that Σ is not positively expansive.

Remark 3.5. The shift mapping σ on symbolic space S^{Z+} maps open balls to open balls. In fact, for all $x \in S^{Z+}$ and $r > 0$, we have that

$$\sigma(B(x,r)) = \begin{cases} B(\sigma(x), 2r), & \text{for } r \leq 1, \\ B(\sigma(x), 2r-2), & \text{for } 1 < r \leq 2, \\ S^{Z+}, & \text{for } 2 < r. \end{cases}$$

So, σ is an open map. By the above example we know that σ is positively expansive. Therefore, the induced map $\Sigma : \mathscr{K}(S^{Z+}) \to \mathscr{K}(S^{Z+})$ has shadowing property, from Theorem 3.2.

Next, we turn to present the second main result of this paper. We first introduce the definition of ball expansive map.

Definition 3.6. A continuous map f on a compact metric space X is said to be *ball expansive*, if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\overline{B}_d(f(x), \varepsilon + \delta) \subset f(\overline{B}_d(x, \varepsilon)), \quad \forall\, x \in X.$$

Example 3.7. It is easy to see that the map $f(z) = z^2$ on the unit circle $\mathbb{S}^1$, as well as the tent map $T : [0,1] \to [0,1]$ defined by $Tx = 2x$ when $0 \leq x \leq \frac{1}{2}$ and $Tx = 2(1-x)$ when $\frac{1}{2} < x \leq 1$, are both ball expansive.

Theorem 3.8. *Let f be a continuous map on compact metrizable space X. If $f : X \to X$ is ball expansive, then $F : \mathscr{K}(X) \to \mathscr{K}(X)$ has shadowing property.*

Proof. For all $\varepsilon > 0$, take $\varepsilon_1 = \frac{1}{2}\varepsilon$. There exists $\delta_1 > 0$ such that $\overline{B}_d(f(x), \varepsilon_1 + \delta_1) \subset f(\overline{B}_d(x, \varepsilon_1))$. Let $\delta_2 = \frac{1}{3}\delta_1$. By uniform continuity of f, there exists $0 < \varepsilon_2 \leq \varepsilon$ such that

$$d(x,y) < \varepsilon_2 \Rightarrow d(f(x), f(y)) < \delta_2. \tag{3.8}$$

Take $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \delta_2\}$. Since X is compact, there exists a finite subset $A = \{a_1, a_2, \ldots, a_l\} \in \mathscr{P}(X)$, such that $\bigcup_{a_i \in A} B_d(a_i, \varepsilon_0) = X$. Let $\{K_0, K_1, K_2, \ldots\}$ be a given δ_2-pseudo-orbit of $(\mathscr{K}(X), F)$, that is,

$H_d(F(K_i), K_{i+1}) < \delta_2$ for all $i \geq 0$¿ We will construct a finite set T, whose trajectory ε-shadows the pseudo-orbit $\{K_0, K_1, K_2, \dots\}$, i.e.

$$H_d(K_n, F^n(T)) \leq \varepsilon, \quad \forall\, n \in N. \tag{3.9}$$

Since each K_i is compact, there exists $A_i = \{x_{ij}\}_{j=1}^{l_i} \subset A(l_i \leq l)$ such that

$$K_i \subset \bigcup_{j=1}^{l_i} B_d(x_{ij}, \varepsilon_0) \quad \text{and} \quad K_i \bigcap B_d(x_{ij}, \varepsilon_0) \neq \emptyset, \quad j = 1, \dots, l_i. \tag{3.10}$$

By (3.10), for all $x_{0j_0} \in A_0$ $(j = 1, 2, \dots, l_0)$ there exists $\tilde{x} \in K_0$ such that $d(x_{0j_0}, \tilde{x}) < \varepsilon_0$. Combining (3.8), we have $d(f(x_{0j_0}),\ f(\tilde{x})) < \delta_2$. Since $H_d(F(K_0), K_1) < \delta_2$, we can find $z_{1j_1} \in K_1$ satisfying $d(f(\tilde{x}), z_{1j_1}) < \delta_2$. Using (3.10) again, there exists $x_{1j_1} \in A_1$ such that $z_{1j_1} \in B_d(x_{1j_1}, \varepsilon_0)$. Therefore,

$$\begin{aligned} d(f(x_{0j_0}), x_{1j_1}) &\leq d(f(x_{0j_0}), f(\tilde{x})) + d(f(\tilde{x}), z_{1j_1}) + d(z_{1j_1}, x_{1j_1}) \\ &< \delta_2 + \delta_2 + \varepsilon_0 < 3\delta_2 = \delta_1. \end{aligned}$$

Repeating the process, for each $j_0 \in \{1, 2, \dots, l_0\}$, we can find a sequence $\{x_{0j_0}, x_{1j_1}, \dots, x_{nj_n}, \dots\}$ such that $x_{nj_n} \in A_n$ and $d(f(x_{nj_n}), x_{n+1,j_{n+1}}) < \delta_1$.

For each $j_0 \in \{1, 2, \dots, l_0\}$, define $P_0^{j_0}, P_1^{j_0}, P_2^{j_0} \cdots$ as follows:

$$P_0^{j_0} = \overline{B}_d(x_{0j_0}, \varepsilon_1) \quad \text{and} \quad P_n^{j_0} = P_{n-1}^{j_0} \cap f^{-n}(\overline{B}_d(x_{nj_n}, \varepsilon_1)), \quad n \geq 1.$$

Combining the condition $\overline{B}_d(f(x), \varepsilon_1 + \delta_1) \subset f(\overline{B}_d(x, \varepsilon_1))$, we know that

$$f^n(P_n^{j_0}) = \overline{B}_d(x_{nj_n}, \varepsilon_1), \quad n = 0, 1, 2, \dots. \tag{3.11}$$

This also means that $\bigcap\limits_{n=0}^{\infty} P_n^{j_0}$ is not empty for each $j_0 \in \{1, 2, \dots, l_0\}$.

Choose $p_{j_0} \in \bigcap\limits_{n=0}^{\infty} P_n^{j_0}$ $(j_0 = 1, 2, \dots, l_0)$, and let $T = \{p_{j_0} : j = 1, 2, \dots, l_0\}$.

For any $x \in K_n$, by (3.10), there exists $x_{nj_n} \in A_n$ with $x \in B_d(x_{nj_n}, \varepsilon_0)$. Together with (3.11), we get

$$d(x, f^n(p_{j_0})) \leq d(x, x_{nj_n}) + d(x_{nj_n}, f^n(p_{j_0})) \leq \varepsilon_1 + \varepsilon_0 < \varepsilon.$$

Thus, we obtain

$$\begin{aligned} e_d(K_n, F^n(T)) &= \max\{d(x, F^n(T)) : x \in K_n\} \\ &\leq \max\{d(x, f^n(p_{j_0}) : x \in K_n\} \\ &\leq \varepsilon. \end{aligned} \tag{3.12}$$

On the other hand, for any $p_{j_0} \in T$, using the similar argument as above, we can easily find $y_{nj_n} \in B_d(x_{nj_n}, \varepsilon_0) \cap K_n$ satisfying $d(f^n(p_{j_0}), y_{nj_n}) < \varepsilon$. Thus, we get

$$\begin{aligned} e_d(F^n(T), K_n) &= \max\{d(F^n(p_{j_0}), K_n) : p_{j_0} \in T\} \\ &\leq \max\{d(f^n(p_{j_0}), y_{nj_n}) : p_{j_0} \in T\} \\ &\leq \varepsilon. \end{aligned} \tag{3.13}$$

Finally, according to (3.12) and (3.13), it is true that

$$H_d(K_n, F^n(T)) \leq \varepsilon, \quad \forall\, n \in N.$$

We complete the proof. □

Remark 3.9. We want to point out that the positively expansive map and ball expansive map are qualitatively different. In fact, any one of them can not imply the other. The following two examples are devoted to illustrate this.

Example 3.10. Let $X = [0, 1]$ and consider a function $f : X \to X$ defined as

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & \text{for } 0 \leq x \leq \frac{1}{4}, \\ \frac{3}{2} - 2x, & \text{for } \frac{1}{4} < x \leq \frac{3}{8}, \\ \frac{6}{5}(1 - x), & \text{for } \frac{3}{8} < x \leq 1. \end{cases}$$

By simple calculation, we know that for all $\varepsilon > 0$, $\delta = \frac{\varepsilon}{5}$ meets the definition of ball expansive map. On the other hand, f is not positively expansive. Indeed, let $x = 0$ and $y = \frac{7}{12}$, then $f^n(x) = f^n(y)$, for all $n \geq 1$.

Example 3.11. Let $X = \mathbb{S}^1 \times I$, where $\mathbb{S}^1$ and I denote the unit circle and interval $[0, 1]$, respectively. For any two points $x_k = (e^{\theta_k i}, t_k) \in X, k = 1, 2$, the distance between x_1 and x_2 is defined to be

$$d(x_1, x_2) = \max\{(\theta_1 - \theta_2) \bmod 2\pi, |t_1 - t_2|\}.$$

Consider a continuous map $f : X \to X$ defined as

$$f(e^{\theta i}, t) = (e^{(t+2)\theta i}, t), \quad \forall\, (e^{\theta i}, t) \in X.$$

For all $n \in N$, we have

$$d(f^n(x_1), f^n(x_2)) = \max\{((t_1 + 2)^n\theta_1 - (t_2 + 2)^n\theta_2) \bmod 2\pi, |t_1 - t_2|\}.$$

From this one easily deduce that f is positively expansive with some sufficiently small expansive constant. However, f is not ball expansive. In fact, take $\varepsilon = \frac{1}{2}$, $x_0 = (e^{\pi i}, 0)$, then $f(x_0) = (e^{2\pi i}, 0)$. For any $\delta > 0$, set $\delta_0 = \min\{1, \frac{1+\delta}{2}\}$, and $y_0 = (e^{2\pi i}, \delta_0)$. One can see that $y_0 \in \overline{B}_d(f(x), \varepsilon + \delta)$, but $y_0 \notin f(\overline{B}_d(x, \varepsilon))$.

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