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NEW OSCILLATION CRITERIA FOR GENERALIZED SECOND-ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the authors consider the second-order neutral functional differential equation

$$[p(t)\psi(y(t))(x'(t))^{\gamma}]' + q(t)f(y(\delta(t))) = 0, \quad t \geq t_0,$$

where $x(t) = y(t) + r(t)y(\tau(t))$ and $\gamma > 0$ is a ratio of odd positive integers. They establish some new sufficient conditions for oscillation of all solutions that are substantial improvements to some existing results in the literature. Some examples are included to illustrate the main results.

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1. INTRODUCTION

In this paper, we are concerned with the oscillation of solutions of the second-order nonlinear neutral functional differential equation

$$\left[p(t)\psi(y(t))\left(x'(t)\right)^{\gamma}\right]' + q(t)f(y(\delta(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

where $x(t) = y(t)+r(t)y(\tau(t))$ and $\gamma > 0$ is a ratio of odd positive integers. Throughout this paper, we will assume, without further mention, that the following conditions hold:

(h_1) r, p, q, τ, and δ are real valued continuous positive functions defined for $t \in \mathbb{I} = [t_0, \infty)$ and $0 \leq r(t) < 1$;

(h_2) f, $\psi : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $yf(y) > 0$ for all $y \neq 0$, $\psi(y) > 0$ for $y \neq 0$;

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(h_3) there exist positive constants k, K, and L such that

$$\frac{f(y)}{y^\gamma} \geq k \quad \text{and} \quad \frac{1}{K} \leq \psi(y) \leq \frac{1}{L} \quad \text{for} \quad y \neq 0; \tag{1.2}$$

(h_4) $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, and $\lim_{t\to\infty} \delta(t) = \infty$.

For a given function y, we set

$$x(t) := y(t) + r(t)y(\tau(t)), \quad x^{[1]} := p\psi(y)(x')^\gamma, \quad \text{and} \quad x^{[2]} := \left(x^{[1]}\right)'. \tag{1.3}$$

Let $\sigma_1 = \inf\{\tau(t) : t \in \mathbb{R}\}$, $\sigma_2 = \inf\{\delta(t) : t \in \mathbb{R}\}$, and $\sigma = \min\{\sigma_1, \sigma_2\}$. By a *solution* of (1.1), we mean a nontrivial real-valued function y with $x \in C^1[\sigma, \infty)$, $x^{[1]} \in C^1[\sigma, \infty)$, and such that equation (1.1) is satisfied. Our attention is restricted to those solutions of (1.1) that exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$.

A discussion of the existence and uniqueness of solutions of neutral delay differential equations can be found in [15]. A solution of (1.1) is said to be *oscillatory* if it is defined on some ray $[t_0, \infty)$ and has an unbounded set of zeros; otherwise, it is called *nonoscillatory*. An equation is called oscillatory if all its solutions are oscillatory. In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

In [23], Ye and Xu considered equation (1.1) and studied the oscillation of solutions under the following assumptions:

(A_1) $\gamma > 0$, r, p, q, τ, and δ are real valued continuous positive functions defined for $t \in \mathbb{I} = [t_0, \infty)$;
(A_2) $0 \leq r(t) < 1$, $\delta(t) \leq t$, and $\delta'(t) > 0$ for $t \geq t_0$;
(A_3) f, $\psi : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $yf(y) > 0$ and $\psi(y) > 0$ for $y \neq 0$;
(A_4) there exist positive constants k and L such $f(y) \geq k|y|^{\gamma-1}y$ and $0 < \psi(y) \leq 1/L$ for $y \neq 0$.

They considered the two cases

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}} dt = \infty, \tag{1.4}$$

or

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}} dt < \infty, \tag{1.5}$$

and established several sufficient conditions for oscillation of solutions. Some of the results in [23] are quite interesting, however, there are mistakes in some of the main results. In fact, their conditions are not sharp even when (1.4) does hold. It is known that if (1.4) holds, then we have $x(t)x^{[1]}(t) > 0$ for $t \geq T$ and the other case, namely,

$x(t)x^{[1]}(t) < 0$, can be disregarded. But in case (1.5) holds, we see that $\left(x^{[1]}(t)\right)' < 0$, and there are the two possible cases

$$x(t)x^{[1]}(t) > 0 \quad \text{for} \quad t \geq T, \tag{1.6}$$

and

$$x(t)x^{[1]}(t) < 0 \quad \text{for} \quad t \geq T. \tag{1.7}$$

The mistakes in [23, Theorems 2.3, 2.4, 2.5] are caused by using the inequality

$$y(\delta(t)) \geq (1 - r(\delta(t)))x(\delta(t)), \tag{1.8}$$

to establish oscillation criteria when (1.5) holds. But this inequality is valid only when (1.4) holds, i.e., if (1.6) is satisfied. Also, they proved that the inequality

$$(x^{[1]})' + Q(t)x^{\gamma}(\delta(t)) \leq 0, \text{ for } t \geq T,$$

obtained when (1.4) holds, is the associated inequality when (1.5) holds. This, however, is not true. Hence, the results established in [23] when (1.5) holds are not valid. They can be corrected by finding the appropriate inequality analogous to (1.8) when (1.5) holds, and it certainly would be interesting to see this done. Also, the results in [23] are proved only when the function $\delta(t)$ satisfies $\delta(t) \leq t$ and $\delta'(t) \geq 0$. One question we then ask is: *is it possible to find new oscillation criteria for equation* (1.1) *when* $\delta(t) > t$*?* One of our aims in this paper is to give an affirmative answer to this question.

Oscillation criteria for different types of neutral differential equations can be found in the papers [1, 2, 3, 4, 5, 6, 7, 16, 8, 9, 10, 11, 12, 13, 17, 18, 20, 21, 22, 24, 25, 26, 27] and the reference cited therein. We note that all the results obtained in these papers are given for neutral delay differential equations when (1.4) holds and when $\delta(t) \leq t$.

Our objective in this paper is to use a technique of proof different from that used in [23] and establish some new sufficient conditions for oscillation of (1.1). Some of our oscillation results are of Hille and Nehari types and are essentially new even in the case when (1.4) holds. The results also cover the case $\delta(t) > t$ (which has not been considered before) and do not require that $\delta'(t) \geq 0$ as is the case in [23]. Our results are not only different from those in the above mentioned papers, but also improve some of them. Some examples are given to illustrate the main results.

2. MAIN RESULTS

In this section, we establish some sufficient conditions for the oscillation of all solutions of (1.1) when (1.4) holds. In Subsection 2.1, we consider the case $\delta(t) > t$; the case $\delta(t) \leq t$ will be considered in Subsection 2.2. To prove our main results, we need the following lemmas. They will play important roles in the proofs of our theorems. We will only give proofs for the case where a solution $y(t)$ is positive since if $y(t)$ is negative, then the transformation $y(t) = -z(t)$ transforms the equation into one with the same form as (1.1).

Lemma 2.1. *Assume that* (1.4) *holds, and suppose that equation* (1.1) *has a nonoscillatory solution* y *on* $[t_0, \infty)$. *Then there exists* $T > t_0$ *such that* $x(t)x^{[1]}(t) > 0$ *for* $t \geq T$.

Proof. Let $y(t)$ be a positive solution of (1.1) on $[t_0, \infty)$ and choose $t_1 > t_0$ so that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ on $[t_1, \infty)$. Since y is positive, $q(t) > 0$, and (h_3) holds, we have

$$(x^{[1]}(t))' \leq -kq(t)y^{\gamma}(\delta(t)) < 0, \quad \text{for} \quad t \in [t_1,\ \infty). \tag{2.1}$$

Thus, $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)$ and eventually is of one sign. We claim that $x^{[1]}(t) > 0$ on $[t_2, \infty)$ for some $t_2 > t_1$. If this is not the case, then there is a $t_3 \geq t_1$ such that $x^{[1]}(t_3) = c < 0$. Then, from (2.1), we have $x^{[1]}(t) \leq c$ for $t \geq t_3$, and so (1.2) implies

$$x'(t) \leq \frac{c^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)\psi^{\frac{1}{\gamma}}(y(t))} \leq \frac{K^{\frac{1}{\gamma}}c^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \quad \text{for} \quad t \in [t_3, \infty). \tag{2.2}$$

Integrating the last inequality from t_3 to t implies

$$x(t) = x(t_3) + \int_{t_3}^{t} x'(s)ds \leq x(t_3) + K^{\frac{1}{\gamma}}c^{\frac{1}{\gamma}} \int_{t_3}^{t} \frac{ds}{p^{\frac{1}{\gamma}}(s)} \to -\infty \quad \text{as} \quad t \to \infty, \tag{2.3}$$

by (1.4). Thus, x is eventually negative, and this contradiction completes the proof of the lemma. $\square$

Lemma 2.2. *Assume that* (1.4) *holds and suppose that* (1.1) *has a nonoscillatory solution* y *on* $[t_0, \infty)$. *Then there exists* $T \geq t_0$ *such that*

$$(x^{[1]}(t))' + P(t)x^{\gamma}(\delta(t)) \leq 0 \quad \textit{for} \quad t \geq T, \tag{2.4}$$

where $P(t) = kq(t)(1 - r(\delta(t)))^{\gamma}$.

Proof. Let $y(t)$ be a positive solution of (1.1) on $[t_0, \infty)$ and choose $t_1 > t_0$ so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ on $[t_1,\ \infty)$. Since y is positive and $q(t) > 0$, from Lemma 2.1, we see that

$$x(t) > 0, \quad x'(t) > 0, \quad \text{and} \quad \left(x^{[1]}(t)\right)' < 0 \quad \text{for} \quad t \geq t_2, \tag{2.5}$$

for some $t_2 \geq t_1$. Since $\tau(t) \leq t$ and $r(t) \geq 0$, (2.5) implies

$$x(t) = y(t) + r(t)y(\tau(t)) \leq y(t) + r(t)x(\tau(t)) \leq y(t) + r(t)x(t) \quad \text{for} \quad t \geq t_2.$$

Thus, $y(t) \geq (1 - r(t))x(t)$ for $t \geq t_2$. Then, for $t \geq t_3$, where $t_3 > t_2$ is chosen large enough, we have

$$y(\delta(t)) \geq (1 - r(\delta(t)))x(\delta(t)). \tag{2.6}$$

From (2.1) and the last inequality, we see that (2.4) holds, and this completes the proof. $\square$

2.1. **Oscillation of (1.1) with $\boldsymbol{\delta(t) > t}$.** In this subsection, we establish some sufficient conditions for oscillation of (1.1) when (1.4) holds and $\delta(t) > t$. We start with the following theorem.

Theorem 2.3. *Assume that* (1.4) *holds. Let y be a nonoscillatory solution of* (1.1) *and make the Riccati substitution*

$$u(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}. \tag{2.7}$$

Then $u(t) > 0$ for $t \geq T$, where T is given in Lemma 2.2, and

$$u'(t) + P(t) + \frac{\gamma L^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \left(u(t)\right)^{1+\frac{1}{\gamma}} \leq 0 \quad \textit{for} \quad t \in [T, \infty). \tag{2.8}$$

Proof. Let y be as above, and without loss of generality, assume that there is a $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq t_1$. Then, from Lemma 2.1, there exists $T > t_1$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, and $x^{[2]}(t) < 0$ for $t \geq T$. From the definition of $u(t)$, we have

$$u'(t) = \frac{x^{\gamma}(t)x^{[2]}(t) - \left(x^{\gamma}(t)\right)' x^{[1]}(t)}{x^{\gamma}(t)\left(x(t)\right)^{\gamma}} = \frac{x^{[2]}(t)}{x(\delta(t))^{\gamma}} \frac{\left(x(\delta(t)\right)^{\gamma}}{\left(x(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)' x^{[1]}(t)}{x^{\gamma}(t)\left(x(t)\right)^{\gamma}}.$$

From Lemma 2.2, we see that

$$u'(t) \leq -P(t)\frac{\left(x(\delta(t))\right)^{\gamma}}{\left(x(t)\right)^{\gamma}} - \frac{\left(x^{\gamma}(t)\right)' x^{[1]}(t)}{x^{\gamma}(t)\left(x(t)\right)^{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.9}$$

Now, (1.3) and (1.2) give

$$\frac{x'}{x} = \frac{\left(x^{[1]}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\psi^{\frac{1}{\gamma}}(y)x} \geq \frac{L^{\frac{1}{\gamma}}\left(x^{[1]}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}x} = \frac{L^{\frac{1}{\gamma}}u^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}}.$$

This implies that

$$\frac{\left(x^{\gamma}(t)\right)' x^{[1]}(t)}{x^{\gamma}(t)\left(x(t)\right)^{\gamma}} = \gamma u(t)\frac{x'(t)}{x(t)} \geq \gamma L^{\frac{1}{\gamma}}\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(u(t)\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T.$$

Substituting into (2.9), we have

$$u'(t) \leq -P(t)\frac{\left(x(\delta(t)\right)^{\gamma}}{\left(x(t)\right)^{\gamma}} - \gamma L^{\frac{1}{\gamma}}\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(u(t)\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.10}$$

Next, consider the coefficient of $P(t)$ in (2.10). Since $\delta(t) > t$ and $x(t)$ is increasing, we have

$$x(\delta(t)) > x(t). \tag{2.11}$$

Using (2.11) in (2.10), we obtain inequality (2.8), and this completes the proof of the theorem. $\square$

Theorem 2.4 (Leighton-Wintner type)**.** *Assume that* (1.4) *holds. If*

$$\int_{t_0}^{\infty} P(s)ds = \infty, \tag{2.12}$$

then every solution of (1.1) *oscillates.*

Proof. Suppose, to the contrary, that y is a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$, where T is given in Theorem 2.3. Let u be defined as in Theorem 2.3. Then, $u(t) > 0$ for $t \geq T$ and

$$-u'(t) \geq P(t) + \frac{\gamma L^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} (u(t))^{1+\frac{1}{\gamma}} > P(t) \quad \text{for} \quad t \geq T. \tag{2.13}$$

From the definition of $x^{[1]}(t)$, we see that

$$x'(t) = \left(\frac{x^{[1]}(t)}{p(t)\psi(y(t))}\right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$x(t) = x(T) + \int_T^t \left(\frac{1}{p(s)\psi(y(s))} x^{[1]}(s)\right)^{\frac{1}{\gamma}} ds \quad \text{for} \quad t \geq T.$$

Since $x^{[1]}(t)$ is positive and decreasing, we have

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)\psi(y(s))}\right)^{\frac{1}{\gamma}} ds > L^{\frac{1}{\gamma}} \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} ds,$$

for $t \geq T$. It follows that

$$u(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} < \frac{1}{L} \left(\int_{t_0}^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} ds\right)^{-\gamma} \quad \text{for} \quad t \in [T, \infty),$$

which, in view of (1.4), implies that $\lim_{t\to\infty} u(t) = 0$. Integrating (2.13) from T to ∞, we obtain $u(T) \geq \int_T^{\infty} P(s)ds$, which contradicts (2.12). This completes the proof of the theorem. □

In the following results we consider the case where condition (2.12) may fail, that is, we may have

$$\int_{t_0}^{\infty} P(s)ds < \infty. \tag{2.14}$$

Theorem 2.5. *Assume that* (1.4) *holds. If there exists a positive continuously differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\phi(s)P(s) - \frac{p(s)((\phi'(s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] ds = \infty, \tag{2.15}$$

then every solution of (1.1) *oscillates.*

Proof. Suppose, to the contrary, that y is a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$, where T is given in Theorem 2.3. Let u be defined as in Theorem 2.1. Then, $u(t) > 0$ for $t \geq T$ and (2.8) holds. We then have

$$u'(t) \leq -P(t) - \frac{\gamma L^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} (u(t))^{\frac{\gamma+1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.16}$$

Multiplying (2.16) by $\phi(s)$ and integrating from T to $t \geq T$, we have

$$\int_T^t \phi(s)P(s)ds \leq -\int_T^t \phi(s)u'(s)ds - \int_T^t \frac{\gamma L^{\frac{1}{\gamma}}\phi(s)}{p^{\frac{1}{\gamma}}(s)} (u(s))^{\frac{\gamma+1}{\gamma}} ds.$$

An integration by parts yields

$$\int_T^t \phi(s)P(s)ds \leq u(T)\phi(T) + \int_T^t \phi'(s)u(s)ds - \int_{t_1}^t \frac{\gamma L^{\frac{1}{\gamma}}\phi(s)}{p^{\frac{1}{\gamma}}(s)}(u(s))^{\frac{\gamma+1}{\gamma}} ds.$$

Setting $B = \phi'(s)$, $A = \gamma L^{\frac{1}{\gamma}}\phi(s)p^{-1/\gamma}(s) > 0$, and applying the inequality

$$Bw - Aw^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \tag{2.17}$$

we obtain

$$\int_T^t \left[\phi(s)P(s) - \frac{p(s)(\phi'(s))^{\gamma+1}(s)}{L(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right] ds < \phi(T)u(T),$$

which contradicts condition (2.15). Thus, every solution of (1.1) oscillates, and this completes the proof of the theorem. □

From Theorem 2.5, we can obtain different conditions for the oscillation of (1.1) by making different choices for $\phi(t)$, For instance, if $\phi(t) = t$, we have the following result.

Corollary 2.6. *Assume that* (1.4) *holds. If*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[sP(s) - \frac{p(s)}{L(\gamma+1)^{\gamma+1}s^\gamma}\right] ds = \infty, \tag{2.18}$$

then every solution of (1.1) *oscillates.*

Another method of choosing test functions can be developed by considering functions in the class $\Re$ consisting of kernels of two variables. We say that the function $H \in \Re$ provided H is defined for $t_0 \leq s \leq t$, $H(t,s) \geq 0$, and $H(t,t) = 0$ for $t \geq s \geq t_0$. Important examples of H include $H(t,s) = (t-s)^m$ for $m \geq 1$.

The following theorem gives new oscillation criteria for (1.1) which can be considered as an extension of the Kamenev type oscillation criterion. The proof is similar to those in [16, 21] if we use the inequality (2.8). We omit the details.

Theorem 2.7. *Assume that* (1.4) *holds. Let* $\phi(t)$ *be defined as in Theorem* 2.5 *and let* $H \in \Re$. *If for* $t > s$,

(2.19)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)P(s) - \frac{p(s)((\phi'(s))^{\gamma+1}(\frac{\partial}{\partial s}H(t,s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)} \right] ds = \infty,$$

then every solution of (1.1) *oscillates.*

With appropriate choices of the functions H it is possible to establish a number of oscillation criteria for (1.1). For example, if there exists a function $h(t,s) \in \Re$ such that

$$\frac{\partial}{\partial s} H(t,s) := -h(t,s) H^{\frac{\gamma}{1+\gamma}}(t,s),$$

then Theorem 2.7 yields the following oscillation result.

Corollary 2.8. *Assume that* (1.4) *holds. Let* $\phi(t)$ *be defined as in Theorem* 2.5 *and let* $H \in \Re$. *If for* $t > s$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)P(s) - \frac{p(s)((\phi'(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] ds = \infty,$$

then every solution of equation (1.1) *is oscillatory.*

Choosing $\phi(s) = 1$ and $H(t,s) = (t-s)^m$ for $m \geq 1$, Corollary 2.8 gives the following Kamenev type oscillation criterion.

Corollary 2.9. *Assume that* (1.4) *holds. If for* $m > 1$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m P(s) - \frac{m^{\gamma+1} p(s)((t-s)^{m-1})^{\gamma+1}}{L(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] ds = \infty,$$

then every solution of (1.1) *oscillates.*

As an example of a result of these types, we have the following.

Example 2.10. Consider the second-order neutral equation

$$\left[y(t) + \frac{1}{2} y(\tau(t)) \right]'' + \frac{\lambda}{t^2} y(\delta(t)) = 0 \quad \text{for} \quad t \in [1, \infty). \tag{2.20}$$

Here $\gamma = 1$, $p(t) = 1$, $\psi(y) = 1$, $r(t) = 1/2$, $f(u) = u$, $q(t) = \lambda/t^2$ where $\lambda > 0$ is a constant, $\tau(t) < t$, $\delta(t) > t$, and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$. We have $P(t) = \lambda/2t^2$ and it is easy to see that $(h_1) - (h_3)$, (1.4), and (2.14) are satisfied. To apply Corollary 2.6, we need to examine condition (2.18). Note that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sP(s) - \frac{p(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}} \right] ds = \limsup_{t\to\infty} \int_{1}^{t} \left(\frac{\lambda s}{2s^2} - \frac{1}{4s} \right) ds = \infty,$$

provided that $\lambda > 1/2$. Hence, by Corollary 2.6, every solution of (2.20) oscillates if $\lambda > 1/2$.

For the second order differential equation

$$x''(t) + p(t)x(t) = 0, \tag{2.21}$$

Hille [14] proved that every solution of (2.21) oscillates if

$$\liminf_{t\to\infty} t\int_t^\infty p(s)ds > \frac{1}{4}. \tag{2.22}$$

Nehari [19], by a different approach, proved that if

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t s^2 p(s)ds > \frac{1}{4}, \tag{2.23}$$

then every solution of (2.21) oscillates. In the following, we present some extensions of these results and establish new oscillation criteria of Hille and Nehari types for equation (1.1). We will make use the following notation:

$$a := \liminf_{t\to\infty} \frac{t^\gamma}{p(t)}\int_t^\infty P(s)ds \quad \text{and} \quad b := \liminf_{t\to\infty} \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{p(t)}P(s)ds.$$

Theorem 2.11. *Assume that* (1.4) *holds and that* $p' \geq 0$. *Let* y *be a nonoscillatory solution of* (1.1) *and set*

$$R_* := \liminf_{t\to\infty} \frac{t^\gamma u(t)}{p(t)} \quad \textit{and} \quad R^* := \limsup_{t\to\infty} \frac{t^\gamma u(t)}{p(t)},$$

where u *is defined in* (2.7). *Then,*

$$a \leq R_* - R_*^{1+\frac{1}{\gamma}} L^{\frac{1}{\gamma}}, \tag{2.24}$$

and

$$a + b \leq \frac{1}{L}. \tag{2.25}$$

Proof. Let y be as above and assume that there is a $T > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, $y(\delta(t)) > 0$, and Lemmas 2.1 and 2.2 hold for $t \geq T$. From Theorem 2.3, we have

$$-u'(t) \geq P(t) + \frac{\gamma L^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}(u(t))^{\frac{\gamma+1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.26}$$

First, we prove (2.24). Integrating (2.26) from t to ∞ and using the fact that $\lim_{t\to\infty} u(t) = 0$ (see the proof of Theorem 2.4), we obtain

$$u(t) \geq \int_t^\infty P(s)ds + \gamma L^{\frac{1}{\gamma}} \int_t^\infty \frac{(u(s))^{\frac{\gamma+1}{\gamma}}ds}{p^{\frac{1}{\gamma}}(s)} \quad \text{for} \quad t \geq T. \tag{2.27}$$

It follows from (2.27) that

$$\frac{t^\gamma u(t)}{p(t)} \geq \frac{t^\gamma}{p(t)}\int_t^\infty P(s)ds + \frac{\gamma L^{\frac{1}{\gamma}} t^\gamma}{p(t)} \int_t^\infty \frac{(u(s))^{\frac{\gamma+1}{\gamma}}ds}{p^{\frac{1}{\gamma}}(s)} \quad \text{for} \quad t \geq T. \tag{2.28}$$

Let $\epsilon > 0$ be given; then, by the definition of a and R_* we can choose $T_1 \in [T, \infty)$, sufficiently large, so that

$$\frac{t^\gamma}{p(t)} \int_t^\infty P(s)ds \geq a - \epsilon \quad \text{and} \quad \frac{t^\gamma u(t)}{p(t)} \geq R_* - \epsilon \quad \text{for} \quad t \geq T_1. \tag{2.29}$$

From (2.28) and (2.29) and the fact that $p' \geq 0$, we have

$$\begin{aligned} \frac{t^\gamma u^\gamma(t)}{p(t)} &\geq (a-\epsilon) + \gamma L^{\frac{1}{\gamma}} \frac{t^\gamma}{p(t)} \int_t^\infty \frac{s\,(u(s))^{\frac{1}{\gamma}}\, s^\gamma u(s)}{p^{\frac{1}{\gamma}}(s) s^{\gamma+1}} ds \\ &\geq (a-\epsilon) + (R_* - \epsilon)^{1+\frac{1}{\gamma}} \frac{\gamma L^{\frac{1}{\gamma}} t^\gamma}{p(t)} \int_t^\infty \frac{p(s)}{s^{\gamma+1}} ds \\ &\geq (a-\epsilon) + (R_* - \epsilon)^{1+\frac{1}{\gamma}} \gamma L^{\frac{1}{\gamma}} t^\gamma \int_t^\infty \frac{ds}{s^{\gamma+1}} \quad \text{for} \quad t \geq T_1. \end{aligned} \tag{2.30}$$

Then, from (2.30), we have

$$\frac{t^\gamma u^\gamma(t)}{p(t)} \geq (a-\epsilon) + (R_* - \epsilon)^{1+\frac{1}{\gamma}} L^{\frac{1}{\gamma}}.$$

Taking the lim inf of both sides as $t \to \infty$, we have $R_* \geq a - \epsilon + L^{\frac{1}{\gamma}} (R_* - \epsilon)^{1+\frac{1}{\gamma}}$. Since $\epsilon > 0$ is arbitrary, this implies

$$a \leq R_* - R_*^{1+\frac{1}{\gamma}} L^{\frac{1}{\gamma}}, \tag{2.31}$$

and this proves (2.24).

To prove (2.25), first multiply both sides of (2.26) by $t^{\gamma+1}/p(t)$ and integrate from T to $t \geq T$ to obtain

$$\int_T^t \frac{s^{\gamma+1}}{p(s)} u'(s)ds \leq -\int_T^t \frac{s^{\gamma+1}}{p(s)} P(s)ds - \gamma L^{\frac{1}{\gamma}} \int_T^t \left(\frac{s^\gamma u(s)}{p(s)}\right)^{\frac{\gamma+1}{\gamma}} ds.$$

Integrating by parts, we have

$$\begin{aligned} \frac{t^{\gamma+1} u(t)}{p(t)} \leq \frac{T^{\gamma+1} u(T)}{p(T)} &+ \int_T^t \left(\frac{s^{\gamma+1}}{p(s)}\right)' u(s)ds \\ &- \int_T^t \frac{s^{\gamma+1} P(s)ds}{p(s)} - \gamma L^{\frac{1}{\gamma}} \int_T^t \left(\frac{s^\gamma u(s)}{p(s)}\right)^{\frac{\gamma+1}{\gamma}} ds. \end{aligned}$$

Since $p'(t) \geq 0$,

$$\left(\frac{s^{\gamma+1}}{p(s)}\right)' \leq \frac{(\gamma+1)s^\gamma}{p(s)}, \tag{2.32}$$

so

$$\begin{aligned} \frac{t^{\gamma+1} u(t)}{p(t)} \leq \frac{T^{\gamma+1} u(T)}{p(T)} &+ \int_T^t (\gamma+1) \left(\frac{s^\gamma u(s)}{p(s)}\right) ds \\ &- \int_T^t \frac{s^{\gamma+1}}{p(s)} P(s)ds - \gamma L^{\frac{1}{\gamma}} \int_T^t \left(\frac{s^\gamma u(s)}{p(s)}\right)^{\frac{\gamma+1}{\gamma}} ds. \end{aligned}$$

Hence,

$$\frac{t^{\gamma+1}u(t)}{p(t)} \leq \frac{T^{\gamma+1}u(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}}{p(s)}P(s)ds$$
$$+ \int_T^t \left\{(\gamma+1)\frac{s^\gamma u(s)}{p(s)} - \gamma L^{\frac{1}{\gamma}} \left(\frac{s^\gamma u(s)}{p(s)}\right)^{\frac{\gamma+1}{\gamma}}\right\} ds \quad \text{for} \quad t \geq T.$$

Applying inequality (2.17) with $w = \frac{s^\gamma u}{p}$, $A = \gamma L^{\frac{1}{\gamma}}$, and $B = \gamma + 1$, we obtain

$$\begin{aligned}\frac{t^{\gamma+1}u(t)}{p(t)} &\leq \frac{T^{\gamma+1}u(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}}{p(s)}P(s)ds + \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{(\gamma+1)^{\gamma+1}}{\gamma^\gamma L}ds \\ &= \frac{T^{\gamma+1}u(T)}{p(T)} - \int_T^t \frac{s^{\gamma+1}}{p(s)}P(s)ds + \frac{(t-T)}{L} \quad \text{for} \quad t \geq T.\end{aligned}$$

It follows that

$$\frac{t^\gamma u(t)}{p(t)} \leq \frac{T^{\gamma+1}u(T)}{tp(T)} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}P(s)ds}{p(s)} + \frac{1}{L}(1-\frac{T}{t}) \quad \text{for} \quad t \geq T.$$

Taking the lim sup of both sides as $t \to \infty$, we obtain

$$R^* \leq -b + \frac{1}{L}$$

Together with inequality (2.31), we have

$$a \leq R_* - R_*^{1+\frac{1}{\gamma}} L^{\frac{1}{\gamma}} \leq R_* \leq R^* \leq -b + \frac{1}{L}.$$

Therefore, $a + b \leq \frac{1}{L}$, and this completes the proof of (2.25). □

Using Theorem 2.5, the following result is easy to prove.

Theorem 2.12. *Assume that* (1.4) *holds and that* $p' \geq 0$. *If*

$$\liminf_{t\to\infty} \frac{t^\gamma}{p(t)} \int_t^\infty P(s)ds > \frac{\gamma^\gamma}{L(\gamma+1)^{\gamma+1}}, \tag{2.33}$$

then every solution of (1.1) *oscillates.*

Proof. Suppose that y is a nonoscillatory solution of equation (1.1), say, $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ with T as in Theorem 2.3. With u given in (2.7), Theorem 2.11 gives

$$a \leq R_* - R_*^{\frac{\gamma+1}{\gamma}} L^{\frac{1}{\gamma}}.$$

Applying (2.17) to this last inequality, we have

$$a \leq \frac{\gamma^\gamma}{L(\gamma+1)^{\gamma+1}},$$

which contradicts (2.33). This completes the proof of the theorem. □

Another consequence of Theorem 2.11 is the following oscillation result.

Theorem 2.13. *Assume that* (1.4) *holds and* $p' \geq 0$. *If*

$$\liminf_{t\to\infty} \frac{t^\gamma}{p(t)} \int_t^\infty P(s)ds + \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{p(t)} P(s)ds > \frac{1}{L}, \tag{2.34}$$

then every solution of (1.1) *oscillates.*

2.2. **Oscillation Criteria with $\delta(t) \leq t$.** In this subsection, we establish some sufficient conditions for oscillation of (1.1) with $\delta(t) \leq t$. We will use the following notation:

$$A(t) := P(t)\alpha^\gamma(t) \quad \text{and} \quad \alpha(t) := \frac{L^{\frac{1}{\gamma}} J(\delta(t), T)}{K^{\frac{1}{\gamma}} J(t, T)}, \quad \text{where} \quad J(u, v) := \int_v^u \frac{1}{p^{\frac{1}{\gamma}}(s)} ds.$$

Theorem 2.14. *Assume that* (1.4) *holds. Let* y *be a nonoscillatory solution of* (1.1) *and make the Riccati substitution*

$$w(t) := \frac{x^{[1]}(t)}{x^\gamma(t)}. \tag{2.35}$$

Then $w(t) > 0$ *for* $t \geq T$, *where* T *is given in Lemma* 2.2, *and*

$$w'(t) + A(t) + \gamma L^{\frac{1}{\gamma}} \frac{1}{p^{\frac{1}{\gamma}}(t)} (w(t))^{1+\frac{1}{\gamma}}(t) \leq 0 \quad \textit{for} \quad t \in [T, \infty). \tag{2.36}$$

Proof. Let y be as above and assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$. From the definition of w (see the proof of Theorem 2.3), we have

$$w'(t) \leq -P(t) \frac{(x(\delta(t)))^\gamma}{(x(t))^\gamma} - \gamma L^{\frac{1}{\gamma}} \frac{1}{p^{\frac{1}{\gamma}}(t)} (w(t))^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.37}$$

Now, consider the coefficient of $P(t)$ in (2.37). Since $x^{[1]}(t) = p\psi(y)\,(x')^\gamma(t)$ is decreasing for $t \geq T$, we have

$$\begin{aligned} x(t) - x(\delta(t)) = \int_{\delta(t)}^t \frac{(x^{[1]}(s))^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(s)\psi^{\frac{1}{\gamma}}(y(s))} ds &\leq (x^{[1]}(\delta(t)))^{\frac{1}{\gamma}} \int_{\delta(t)}^t \frac{1}{p^{\frac{1}{\gamma}}(s)\psi^{\frac{1}{\gamma}}(y(s))} ds \\ &\leq K^{\frac{1}{\gamma}} (x^{[1]}(\delta(t)))^{\frac{1}{\gamma}} \int_{\delta(t)}^t \frac{1}{p^{\frac{1}{\gamma}}(s)} ds, \end{aligned}$$

and this implies that

$$\frac{x(t)}{x(\delta(t))} \leq 1 + \frac{K^{\frac{1}{\gamma}} (x^{[1]}(\delta(t)))^{\frac{1}{\gamma}}}{x(\delta(t))} \int_{\delta(t)}^t \frac{1}{p^{\frac{1}{\gamma}}(s)} ds. \tag{2.38}$$

On the other hand, we have

$$x(\delta(t)) > x(\delta(t)) - x(T) = \int_T^{\delta(t)} \frac{(x^{[1]}(s))^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(s)\psi^{\frac{1}{\gamma}}(y(s))} ds \geq L^{\frac{1}{\gamma}} (x^{[1]}(\delta(t)))^{\frac{1}{\gamma}} \int_T^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} ds,$$

which leads to

$$\frac{(x^{[1]}(\delta(t)))^{\frac{1}{\gamma}}}{x(\delta(t))} < \frac{1}{L^{\frac{1}{\gamma}}} \left(\int_T^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)} ds \right)^{-1},$$

This and (2.38) imply

$$\frac{x(t)}{x(\delta(t))} < 1 + \frac{K^{\frac{1}{\gamma}} \int_{\delta(t)}^{t} p^{-\frac{1}{\gamma}}(s)ds}{L^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds} = \frac{L^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds + K^{\frac{1}{\gamma}} \int_{\delta(t)}^{t} p^{-\frac{1}{\gamma}}(s)ds}{L^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds}$$
$$\leq \frac{K^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds + K^{\frac{1}{\gamma}} \int_{\delta(t)}^{t} p^{-\frac{1}{\gamma}}(s)ds}{L^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds} = \frac{K^{\frac{1}{\gamma}} \int_{T}^{t} p^{-\frac{1}{\gamma}}(s)ds}{L^{\frac{1}{\gamma}} \int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)ds} = \frac{1}{\alpha(t)}$$

for $t \geq T$. Hence,

$$x(\delta(t)) \geq \alpha(t)x(t) \quad \text{for} \quad t \geq T. \tag{2.39}$$

This implies that

$$\frac{(x(\delta(t))^{\gamma}}{(x(t))^{\gamma}} \geq (\alpha(t))^{\gamma} \quad \text{for} \quad t \geq T. \tag{2.40}$$

Substituting (2.40) into (2.37) gives inequality (2.36), and this completes the proof of the theorem. □

The following result is the companion to Theorem 2.4 above.

Theorem 2.15 (Leighton-Wintner type)**.** *Assume that* (1.4) *holds. If*

$$\int_{t_0}^{\infty} A(s)ds = \infty, \tag{2.41}$$

then every solution of (1.1) *oscillates.*

Proof. Let y be a nonoscillatory solution of (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ with T given in Theorem 2.14. Let w be defined as in Theorem 2.14; then, $w(t) > 0$ and

$$-w'(t) \geq A(t) + \frac{\gamma L^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} (w(t))^{1+\frac{1}{\gamma}} > A(t) \quad \text{for} \quad t \geq T. \tag{2.42}$$

The reminder of the proof is similar to the proof of Theorem 2.4 and hence is omitted. □

Next, we consider the case where condition (2.41) may fail, that is, we may have

$$\int_{t_0}^{\infty} A(s)ds < \infty. \tag{2.43}$$

Proceeding as in the proofs of Theorems 2.5 and 2.7, we can use inequality (2.36) to obtain the following results.

Theorem 2.16. *Assume that* (1.4) *holds. If there exists a positive continuously differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)A(s) - \frac{p(s)((\phi'(s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] ds = \infty, \tag{2.44}$$

then every solution of (1.1) *oscillates.*

Theorem 2.17. *Assume that* (1.4) *holds. If there exists a positive continuously differentiable function $\phi(t)$ and a function $H \in \Re$ such that for $t > s$,*

(2.45)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi'(s))^{\gamma+1}(\frac{\partial}{\partial s}H(t,s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)} \right] ds = \infty,$$

then every solution of (1.1) *oscillates.*

Analogous to Corollaries 2.8 and 2.9, we have the following results.

Corollary 2.18. *Assume that* (1.4) *holds. Let $\phi(t)$ be defined as in Theorem* 2.17 *and $H \in \Re$. If for $t > s$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi'(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{L(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] ds = \infty, \tag{2.46}$$

then every solution of (1.1) *oscillates.*

Corollary 2.19. *Assume that* (1.4) *holds. If for $m > 1$*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m A(s) - \frac{m^{\gamma+1}p(s)((t-s)^{m-1})^{\gamma+1}}{L(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] ds = \infty, \tag{2.47}$$

then every solution of (1.1) *oscillates.*

Next, we will establish some new oscillation results for (1.1) of the Hille and Nehari types in the delay case. We need the following notation:

$$A := \liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_t^{\infty} A(s)ds \quad \text{and} \quad B := \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{p(t)} A(s)ds.$$

The proof of the following theorem is similar to the proof of Theorem 2.11 by using inequality (2.36) in place of (2.8). We omit the details.

Theorem 2.20. *Assume that* (1.4) *holds and $p' \geq 0$. Let y be a nonoscillatory solution of* (1.1) *and set*

$$Q_* := \liminf_{t\to\infty} t^{\gamma}w(t)/p(t) \quad \textit{and} \quad Q^* := \limsup_{t\to\infty} t^{\gamma}w(t)/p(t)$$

where w is defined in (2.35)*. Then*

$$A \leq Q_* - Q_*^{1+\frac{1}{\gamma}} L^{\frac{1}{\gamma}} \quad \textit{and} \quad A + B \leq \frac{1}{L}.$$

From Theorem 2.20 we can obtain the following results that are analogous to Theorems 2.12 and 2.13 above.

Theorem 2.21. *Assume that* (1.4) *holds and $p' \geq 0$. If*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_t^{\infty} A(s)ds > \frac{\gamma^{\gamma}}{L(\gamma+1)^{\gamma+1}}, \tag{2.48}$$

then every solution of (1.1) *oscillates.*

Theorem 2.22. *Assume that* (1.4) *holds and* $p' \geq 0$. *If*

$$\liminf_{t\to\infty} \frac{t^\gamma}{p(t)} \int_t^\infty A(s)ds + \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{p(t)} A(s)ds > \frac{1}{L}, \tag{2.49}$$

then every solution of (1.1) *oscillates.*

We will now give some examples to illustrate our results.

Example 2.23. Consider the second-order neutral differential equation

$$\left(\frac{1}{t^2}\left(\left(y(t) + \frac{\delta^{-1}(t)-1}{\delta^{-1}(t)} y(\tau(t))\right)'\right)^\gamma\right)' + \frac{\lambda}{\alpha^\gamma(t)t} y^\gamma(\delta(t)) = 0, \quad t \in [1,\infty), \tag{2.50}$$

where $\lambda > 0$, $\gamma > 0$ is a ratio of odd positive integers, $\lim_{t\to\infty}\delta(t) = \lim_{t\to\infty}\tau(t) = \infty$, $\tau(t) \leq t$, and $\delta(t) \leq t$, and assume that $\delta^{-1}(t)$, the inverse function of $\delta(t)$, exists. Here, $\alpha(t) := \frac{J(\delta(t),T)}{J(t,T)}$, where $J(t,T) = \int_T^t (s^2)^{1/\gamma}\, ds$, for any $T \geq 1$. We have $\psi(y) = 1$,

$$p(t) = \frac{1}{t^2}, \quad r(t) = \frac{\delta^{-1}(t)-1}{\delta^{-1}(t)} = 1 - \frac{1}{\delta^{-1}(t)}, \quad \text{and} \quad q(t) = \frac{\lambda}{\alpha^\gamma(t)t}.$$

Noting that $A(t) = P(t)\alpha^\gamma(t) = \alpha^\gamma(t)q(t)(1-r(\delta(t))^\gamma = \lambda/t^{\gamma+1}$, it is easy to see that $(h_1)-(h_4)$, (1.4), and (2.43) hold. Finally, we need to examine condition (2.44). Note that by choosing $\phi(t) = t^\gamma$, we have

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\phi(s)A(s) - \frac{p(s)((\phi'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right] ds$$
$$= \limsup_{t\to\infty} \int_{t_0}^t \left[\frac{\lambda}{s} - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^3}\right] ds = \infty.$$

Therefore, by Theorem 2.16, every solution of (2.50) oscillates.

Example 2.24. Consider the neutral delay differential equation

$$\left(\frac{2+y^2(t)}{1+y^2(t)}\left(\left(y(t) + \frac{1}{t^2} y\left(t - \pi/2\right)\right)'\right)^3\right)' + \frac{(t-1)^6}{t^7} y^3(t-1) = 0, \quad t \geq 2. \tag{2.51}$$

Then, we have

$$p(t) = 1, \quad r(t) = \frac{1}{t^2}, \quad q(t) = \frac{(t-1)^6}{t^7}, \quad \tau(t) = t - \frac{\pi}{2}, \quad \delta(t) = t-1,$$

$$f(u) = u^3, \quad \text{and} \quad 1 \leq \psi(u) = \frac{2+u^2(t)}{1+u^2(t)} \leq 2,$$

so $(h_1)-(h_4)$ and (1.4) hold with $k = 1$ and $\gamma = 3$. Thus,

$$1 - r(\delta(t)) = 1 - \frac{1}{(t-1)^2} = \frac{t\,(t-2)}{(t-1)^2},$$

$$\alpha(t) := \frac{L^{\frac{1}{3}}\,(t-3)}{K^{\frac{1}{3}}(t-2)} = \frac{(t-3)}{2^{\frac{1}{3}}(t-2)},$$

$$P(t) = kq(t)\,(1-r(\delta(t)))^3 = \frac{(t-2)^3}{t^4},$$

and we have

$$A(t) = P(t)\alpha^3(t) = \frac{(t-3)^3}{2t^4}.$$

It is easy to see that

$$\int_{t_0}^{\infty} A(s)ds = \infty.$$

All the assumption of Theorem 2.15 are satisfied, so every solution of (2.51) oscillates.

Example 2.25. Consider the delay differential equation

$$\left(\frac{2+y^2(t)}{1+y^2(t)}\left(\left(y(t)+\frac{1}{t^2}y\left(t-\pi/2\right)\right)'\right)^3\right)' + \frac{\lambda(t-1)^6}{t^{10}}y^3(t-1) = 0, \quad t \geq 2. \tag{2.52}$$

This is the same equation as in Example 2.24 except that

$$q(t) = \frac{\lambda(t-1)^6}{t^{10}}.$$

Here,

$$P(t) = \frac{\lambda(t-2)^3}{t^7} \quad \text{and} \quad A(t) = \frac{\lambda(t-3)^3}{2t^7}.$$

Clearly,

$$\int_{2}^{\infty} A(s)ds < \infty.$$

To apply Theorem 2.21 we need to examine condition (2.48). In this case we see that

$$\begin{aligned}\liminf_{t\to\infty} \frac{t^\gamma}{p(t)} \int_t^\infty A(s)ds &= \liminf_{t\to\infty} t^3 \int_t^\infty A(s)ds \\ &= \liminf_{t\to\infty} t^3 \int_t^\infty \lambda\left(\frac{1}{2t^4} - \frac{9}{2t^5} + \frac{27}{2t^6} - \frac{27}{2t^7}\right) dt \\ &= \frac{\lambda}{6}.\end{aligned}$$

All the hypotheses of Theorem 2.21 are satisfied provided that

$$\lambda > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} = \frac{81}{64},$$

i.e., every solution of (2.52) oscillates if $\lambda > 81/64$.

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