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FUZZY STOCHASTIC INTEGRAL EQUATIONS

MAREK T. MALINOWSKI AND MARIUSZ MICHTA

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland,
m.malinowski@wmie.uz.zgora.pl, m.michta@wmie.uz.zgora.pl

ABSTRACT. In this paper we propose a new approach to fuzzy stochastic integrals of Itô and Aumann type. Then a fuzzy equation with fuzzy stochastic integrals is investigated. The existence and uniqueness of solution is proven. Some typical properties of the solution are also obtained. Similar results to set-valued stochastic integral equations are stated.

Keywords and phrases. Fuzzy set, fuzzy stochastic process, fuzzy stochastic integral, fuzzy stochastic differential equation, fuzzy stochastic integral equation, set-valued stochastic integral equation

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1. INTRODUCTION

The theory of fuzzy differential equations has focused much attention in the last decades since it provides good models for dynamic systems under uncertainty. In [18] this theory has been started with a concept of H-differentiability for fuzzy mappings introduced in [42]. Currently the literature on this topic is very rich. For a significant collection of the results and further references we refer the reader to the monographs [9, 27] and to the research articles e.g. [1, 2, 7, 8, 15, 26, 28, 37, 38, 39, 40].

Recently some results have been published concerning random fuzzy differential equations (see [11, 12, 31]). The random approach can be adequate in modelling of the dynamics of real phenomena which are subjected to two kinds of uncertainty: randomness and fuzziness, simultaneously. Here fuzzy random variables and fuzzy stochastic processes play a crucial role.

The next natural step in modelling of dynamic systems under uncertainty should be the theory of fuzzy stochastic differential equations (understood in their integral form). Here, a main problem is the notion of stochastic fuzzy Itô integral. A first background for such a kind of research has been made in [19, 17, 30]. Although these papers differ from each other in the considered settings, the main idea is always the same: to define a stochastic set-valued Itô integral (as a measurable set-valued mapping) and then, by using the Stacking Theorem, to introduce a concept of stochastic

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fuzzy integral (as a fuzzy stochastic process). The authors of [19, 30] emphasize their motivations to such studies as the beginning of fundations to the theory of fuzzy stochastic differential equations and inclusions.

As far as we know there are two papers concerning this new area, i.e. [20] and [41]. However the approaches presented there are different. In [20] all the considerations are made in the setup of fuzzy sets space of a real line, and the main result on the existence and uniqueness of the solution is obtained under very particular conditions imposed on the structure of integrated fuzzy stochastic processes such that a maximal inequality for fuzzy stochastic Itô integrals holds. Unfortunately the paper [20] contains the gaps. In view of [44] we find out that the intersection property of a set-valued Itô integral (defined as a measurable set-valued mapping) may not hold true in general. Thus a definition of fuzzy stochastic integral, which is used in [20], seems to be incorrect. On the other hand, in [41] a proposed approach does not contain a notion of a fuzzy stochastic Itô integral. The method presented there is based on sets of appropriately chosen selections. In [32] we proposed a third approach to stochastic fuzzy differential equations. We gave a result of existence and uniqueness of the solution to stochastic fuzzy differential equation where the diffusion term (appropriate fuzzy stochastic Itô integral) was of some special form, i.e. it was the embedding of real d-dimensional Itô integral into fuzzy numbers space.

In this paper we propose completely different approach to the notions of fuzzy stochastic integrals. We employ here the notion of set-valued stochastic integral which was widely used in the theory of stochastic inclusions and their applications (see e.g. [3, 4, 5, 6, 22, 23, 24, 25, 33, 34, 35] and references therein). As distinct from the approaches in [19, 29, 30, 32, 41] we treat the fuzzy stochastic integrals as the fuzzy sets of the space of square-integrable random vectors. Then we consider the fuzzy integral equations in which these fuzzy stochastic integrals appear.

The paper is organized as follows. In Section 2 we recall some facts from the set-valued stochastic analysis. We give also the definitions and some properties of set-valued trajectory stochastic integrals of Itô and Aumann type. The Section 3 is started with a short résumé about fuzzy sets and fuzzy stochastic processes. Next the results of preceding section are used to establish the notions of fuzzy trajectory stochastic integrals of Itô and Aumann type. Some properties of these fuzzy stochastic integrals are also stated. Finally a fuzzy stochastic integral equation, with fuzzy trajectory stochastic Itô integral and fuzzy trajectory stochastic Aumann integral, is investigated. We present the results on existence and uniqueness of the solution as well as on boundedness of the solution, on continuous dependence on initial conditions and on the stability of the solution. In Section 4 we formulate the parallel results which are established for set-valued stochastic integral equations.

2. PRELIMINARIES

Let $\mathcal{X}$ be a separable Banach space, $\mathcal{K}^b(\mathcal{X})$ the family of all nonempty closed and bounded subsets of $\mathcal{X}$. Similarly by $\mathcal{K}^b_c(\mathcal{X})$ we denote the family of all nonempty closed, bounded and convex subsets of $\mathcal{X}$. The Hausdorff metric $H_{\mathcal{X}}$ in $\mathcal{K}^b(\mathcal{X})$ is defined by

$$H_{\mathcal{X}}(A,B) = \max\left\{\sup_{a\in A}\operatorname{dist}_{\mathcal{X}}(a,B), \sup_{b\in B}\operatorname{dist}_{\mathcal{X}}(b,A)\right\},$$

where $\operatorname{dist}_{\mathcal{X}}(a,B) = \inf_{b\in B}\|a-b\|_{\mathcal{X}}$ and $||\cdot||_{\mathcal{X}}$ denotes a norm in $\mathcal{X}$.

It is known that $(\mathcal{K}^b(\mathcal{X}), H_{\mathcal{X}})$ is a complete metric space, $\mathcal{K}^b_c(\mathcal{X})$ is its closed subspace. For nonempty subsets A_1, A_2, B_1, B_2 of $\mathcal{X}$ it holds (see [14])

$$H_{\mathcal{X}}(A_1 + A_2, B_1 + B_2) \le H_{\mathcal{X}}(A_1, B_1) + H_{\mathcal{X}}(A_2, B_2),$$

where $A_1 + A_2$ denotes the Minkowski sum of A_1 and A_2.

Let $(U, \mathcal{U}, \mu)$ be a measure space. Recall that a set-valued mapping $F\colon U \to \mathcal{K}^b(\mathcal{X})$ is said to be measurable if it satisfies:

$$\{u \in U : F(u) \cap C \neq \emptyset\} \in \mathcal{U} \text{ for every closed set } C \subset \mathcal{X}.$$

A measurable multifunction F is said to be L^p-integrably bounded ($p \ge 1$), if there exists $h \in L^p(U, \mathcal{U}, \mu; \mathbb{R}_+)$ such that the inequality $|||F||| \le h$ holds μ-a.e., where

$$|||A||| = H_{\mathcal{X}}(A, \{0\}) = \sup_{a\in A}||a||_{\mathcal{X}} \text{ for } A \in \mathcal{K}^b(\mathcal{X}),$$

and $\mathbb{R}_+ = [0, \infty)$. Consequently, it is known (see [13]) that F is L^p-integrably bounded if and only if $|||F||| \in L^p(U, \mathcal{U}, \mu; \mathbb{R}_+)$.

Let $\mathcal{M}$ be a set of $\mathcal{U}$-measurable mappings $f\colon U \to \mathcal{X}$. The set $\mathcal{M}$ is said to be decomposable if for every $f_1, f_2 \in \mathcal{M}$ and every $A \in \mathcal{U}$ it holds $f_1\mathbf{1}_A + f_2\mathbf{1}_{U\setminus A} \in \mathcal{M}$.

Denote $I = [0, T]$, where $T < \infty$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all P-null sets.

Let $\{B(t)\}_{t\in I}$ be an $\{\mathcal{A}_t\}$-adapted Wiener process. We put $U = I \times \Omega$, $\mathcal{U} = \mathcal{N}$, where $\mathcal{N}$ denotes the σ-algebra of the nonanticipating elements in $I \times \Omega$, i.e.

$$\mathcal{N} = \{A \in \beta_I \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\},$$

where β_I is the Borel σ-algebra of subsets of I and $A^t = \{\omega : (t, \omega) \in A\}$ for $t \in I$. Finally we set $\mu = \lambda \times P$ as a measure, where λ is the Lebesgue measure on (I, β_I).

A d-dimensional stochastic process $f\colon I \times \Omega \to \mathbb{R}^d$ is called nonanticipating if f is $\mathcal{N}$-measurable. Clearly $\mathcal{N} \subset \beta_I \otimes \mathcal{A}$. It is known that f is $\mathcal{N}$-measurable if and only if f is $\beta_I \otimes \mathcal{A}$-measurable and $\{\mathcal{A}_t\}$-adapted.

Consider the space

$$L^2_{\mathcal{N}}(\lambda \times P) := L^2(I \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d).$$

Then for every $f \in L^2_{\mathcal{N}}(\lambda \times P)$ and $\tau, t \in I$, $\tau < t$ the Itô stochastic integral $\int_\tau^t f(s)dB(s)$ exists and one has $\int_\tau^t f(s)dB(s) \in L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$. Moreover

$$\mathbb{E}\Big|\int_\tau^t f(s)dB(s)\Big|^2 = \mathbb{E}\left(\int_\tau^t |f(s)|^2 ds\right) = \int\limits_{[\tau,t]\times\Omega} |f|^2 ds \times dP,$$

where $|\cdot|$ denotes usual Euclidean norm in $\mathbb{R}^d$ (see [16] for details).

Let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a set-valued stochastic process, i.e. a family $\{F(t)\}_{t\in I}$ of $\mathcal{A}$-measurable set-valued mappings $F(t)\colon \Omega \to \mathcal{K}^b(\mathbb{R}^d)$, $t \in I$. We call F nonanticipating if it is $\mathcal{N}$-measurable. Let us define the set

$$S^2_{\mathcal{N}}(F, \lambda \times P) := \{f \in L^2_{\mathcal{N}}(\lambda \times P) : f \in F,\ \ \lambda \times P\text{-a.e.}\}.$$

If F is $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded, then by Kuratowski and Ryll-Nardzewski Selection Theorem (see e.g. [21]) it follows that $S^2_{\mathcal{N}}(F, \lambda \times P) \neq \emptyset$. Hence for every $\tau, t \in I$, $\tau < t$ we can define the set-valued trajectory Itô stochastic integral

$$J_\tau^t(F) := \left\{\int_\tau^t f(s)dB(s) : f \in S^2_{\mathcal{N}}(F, \lambda \times P)\right\}.$$

Remark 2.1. By the above definition we have $J_\tau^t(F) \subset L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

In the rest of the paper, for the sake of convenience, we will write L^2 instead of $L^2(\Omega, \mathcal{A}, P; \mathbb{R}^d)$.

Proposition 2.2. *Let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process. Then*

- (i) *$S^2_{\mathcal{N}}(F, \lambda \times P)$ is nonempty, bounded, closed, weakly compact and decomposable subset of $L^2_{\mathcal{N}}(\lambda \times P)$,*
- (ii) *$J_\tau^t(F)$ is nonempty, bounded, closed and weakly compact subset of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ for every $\tau, t \in I$, $\tau < t$.*

Proof. The proof of part (i) follows immediately from the assumptions imposed on F. The decomposability of $S^2_{\mathcal{N}}(F, \lambda \times P)$ follows by Theorem 3.1 in [13].

For the proof of the part (ii) let us take $u \in J_\tau^t(F)$. Then there exists $f \in S^2_{\mathcal{N}}(F, \lambda \times P)$ such that $u = \int_\tau^t f(s)dB(s)$. Thus by Doob's maximal inequality we have

$$\begin{aligned}\mathbb{E}|u|^2 &\leq \mathbb{E}\left(\sup_{v\in[\tau,t]}\Big|\int_\tau^v f(s)dB(s)\Big|^2\right)\\ &\leq 2\mathbb{E}\Big|\int_\tau^t f(s)dB(s)\Big|^2 = \int\limits_{[\tau,t]\times\Omega} |f|^2 ds \times dP\end{aligned}$$

$$\leq \int_{I\times\Omega} |||F|||^2 ds \times dP.$$

Thus we have that $J_\tau^t(F)$ is a bounded subset of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ for $\tau, t \in I$, $\tau < t$. Under the assumptions, the set $S^2_{\mathcal{N}}(F, \lambda \times P)$ is also norm-closed. Moreover, from the above estimations it follows that the sets $S^2_{\mathcal{N}}(F, \lambda \times P)$ and $J_\tau^t(F)$ are bounded in reflexive spaces $L^2_{\mathcal{N}}(\lambda \times P)$ and $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$, respectively. Thus the set $J_\tau^t(F)$ is conditionally weakly compact and $S^2_{\mathcal{N}}(F, \lambda \times P)$ is weakly compact (see [10]). Let $\{u_n\} \subset J_\tau^t(F)$ and let us suppose $u_n \to u$ in $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$. Then there exist a sequence $\{f_n\} \subset S^2_{\mathcal{N}}(F, \lambda \times P)$ such that $u_n = \int_\tau^t f_n(s)dB(s)$ for $n \in \mathbb{N}$. Hence, by Itô isometry for Itô integral, we infer that $\{f_n\}$ is a Cauchy sequence in $L^2_{\mathcal{N}}(\lambda \times P)$. Thus it has to be convergent to some element $f \in L^2_{\mathcal{N}}(\lambda \times P)$. Since the set $S^2_{\mathcal{N}}(F, \lambda \times P)$ is closed, we conclude that $u = \int_\tau^t f(s)dB(s)$. This proves the closedness of the set $J_\tau^t(F)$ and consequently its weak compactness. □

Remark 2.3. If $F\colon I \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ is a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process, then $S^2_{\mathcal{N}}(F, \lambda \times P)$ and $J_\tau^t(F)$ are convex sets.

Theorem 2.4. *For each $n \in \mathbb{N}$, let $F_n\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating set-valued stochastic processes such that F_1 is $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded and*

$$F_1 \supset F_2 \supset \ldots \supset F \quad \lambda \times P\text{-a.e.},$$

where $F := \bigcap_{n=1}^{\infty} F_n \quad \lambda \times P$-a.e. Then for every $\tau, t \in I$, $\tau < t$ it holds

$$J_\tau^t(F) = \bigcap_{n=1}^{\infty} J_\tau^t(F_n).$$

Proof. Firstly, let us note that by Theorem 3.3 in [21] a set-valued mapping F is nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded. Since

$$S^2_{\mathcal{N}}(F_1, \lambda \times P) \supset S^2_{\mathcal{N}}(F_2, \lambda \times P) \supset \ldots \supset S^2_{\mathcal{N}}(F, \lambda \times P),$$

we infer that $S^2_{\mathcal{N}}(F, \lambda \times P) \subset \bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F_n, \lambda \times P)$. Let us suppose that there exists $f \in \bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F_n, \lambda \times P) \setminus S^2_{\mathcal{N}}(F, \lambda \times P)$. Then for every $n \in \mathbb{N}$ it holds $f \in F_n$ $\lambda \times P$-a.e., and consequently $f \in F \;\; \lambda \times P$-a.e. Since $f \in L^2_{\mathcal{N}}(\lambda \times P)$, it has to belong to the set $S^2_{\mathcal{N}}(F, \lambda \times P)$ too. This leads to the contradiction. Hence $S^2_{\mathcal{N}}(F, \lambda \times P) = \bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F_n, \lambda \times P)$. Thus for every fixed $\tau, t \in I$, $\tau < t$ we have

$$\left\{ \int_\tau^t f(s)dB(s) : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\} =$$
$$\bigcap_{n=1}^{\infty} \left\{ \int_\tau^t f(s)dB(s) : f \in S^2_{\mathcal{N}}(F_n, \lambda \times P) \right\},$$

what completes the proof. $\square$

Theorem 2.5. *Let $F_1, F_2\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be the nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic processes. Then for every $\tau, t \in I$, $\tau < t$ it holds*

$$H^2_{L^2}\left(J^t_\tau(F_1), J^t_\tau(F_2)\right) \leq \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP.$$

Proof. Let us fix $\tau, t \in I$ such that $\tau < t$. Then, for a fixed selection $f_1 \in S^2_{\mathcal{N}}(F_1, \lambda \times P)$ let us define a function $\varphi\colon I \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ by

$$\varphi(t, \omega, x) := |f_1(t, \omega) - x|^2.$$

Then $\varphi(\cdot, \cdot, x)$ is $\mathcal{N}$-measurable for every fixed $x \in \mathbb{R}^d$ and the mapping $\varphi(t, \omega, \cdot)$ is continuous for every fixed $(t, \omega) \in I \times \Omega$. Hence by Theorem 2.2 in [13] we obtain

$$\begin{aligned}
&\operatorname{dist}^2_{L^2}\left(\int_\tau^t f_1(s) dB(s), J^t_\tau(F_2)\right) \\
&\quad = \inf_{f_2 \in S^2_{\mathcal{N}}(F_2, \lambda\times P)} \left\| \int_\tau^t f_1(s) dB(s) - \int_\tau^t f_2(s) dB(s) \right\|^2_{L^2} \\
&\quad = \inf_{f_2 \in S^2_{\mathcal{N}}(F_2, \lambda\times P)} \int\limits_{[\tau,t]\times\Omega} \varphi(s, \omega, f_2(s, \omega)) ds \times dP \\
&\quad = \int\limits_{[\tau,t]\times\Omega} \inf_{x \in F_2(s,\omega)} \varphi(s, \omega, x) ds \times dP \\
&\quad \leq \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP.
\end{aligned}$$

Therefore we claim that

$$\sup_{j_1 \in J^t_\tau(F_1)} \operatorname{dist}^2_{L^2}\left(j_1, J^t_\tau(F_2)\right) \leq \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP.$$

In a similar way one can show that

$$\sup_{j_2 \in J^t_\tau(F_2)} \operatorname{dist}^2_{L^2}\left(j_2, J^t_\tau(F_1)\right) \leq \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP.$$

$\square$

Lemma 2.6. *Let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process. Then for every $\tau, a, t \in I$ such that $\tau \leq a \leq t$ it holds*

$$J^t_\tau(F) = J^a_\tau(F) + J^t_a(F). \tag{2.1}$$

Proof. It is clear that $J_\tau^t(F) \subset J_\tau^a(F) + J_a^t(F)$. We want to show that the opposite inclusion holds. Let us fix the random variables j_1, j_2 that belong to $J_\tau^a(F)$ and $J_a^t(F)$, respectively. Then there exist $f_1, f_2 \in S^2_{\mathcal{N}}(F, \lambda \times P)$ such that $j_1 = \int_\tau^a f_1(s)dB(s)$, $j_2 = \int_a^t f_2(s)dB(s)$. Let us take any $f_3 \in S^2_{\mathcal{N}}(F, \lambda \times P)$. Since $S^2_{\mathcal{N}}(F, \lambda \times P)$ is decomposable with respect to σ-algebra $\mathcal{N}$, we infer that

$$\begin{aligned} f(s,\omega) = {} & f_1(s,\omega)\mathbf{1}_{[\tau,a]\times\Omega}(s,\omega) + f_2(s,\omega)\mathbf{1}_{(a,t]\times\Omega}(s,\omega) \\ & + \ f_3(s,\omega)\mathbf{1}_{(t,T]\times\Omega}(s,\omega) \end{aligned}$$

belongs to $S^2_{\mathcal{N}}(F, \lambda \times P)$, too. Observe that $j_1 + j_2 = \int_\tau^t f(s)dB(s)$. This completes the proof. □

Application of (2.1) together with Theorem 2.5 yields the following result.

Theorem 2.7. *Let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process. Then the mapping*

$$[\tau, T] \ni t \mapsto J_\tau^t(F) \in \mathcal{K}^b(L^2)$$

is continuous with respect to the metric H_{L^2}.

Now we consider the set-valued trajectory Aumann stochastic integral. Similarly as in the preceding considerations, let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process. Then for $\tau, t \in I$, $\tau < t$ we define set-valued trajectory Aumann stochastic integral $L_\tau^t(F)$ as a subset of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ and described by

$$L_\tau^t(F) := \left\{ \int_\tau^t f(s)ds : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\}.$$

Then for every $f \in S^2_{\mathcal{N}}(F, \lambda \times P)$ one has

$$\mathbb{E}\left| \int_\tau^t f(s)ds \right|^2 \le (t - \tau) \int\limits_{[\tau,t]\times\Omega} |f|^2 ds \times dP.$$

Proposition 2.8. *Let $F\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process. Then set-valued trajectory stochastic integral $L_\tau^t(F)$ is nonempty, bounded, closed and weakly compact subset of $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$.*

Proof. The boundedness property follows by the inequality above. The conditional weak compactness follows as in the proof of Proposition 2.2. Let $\{u_n\} \subset L_\tau^t(F)$ and let $u_n \to u$ in $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$. Then there exists a sequence $\{f_n\} \subset S^2_{\mathcal{N}}(F, \lambda \times dP)$ such that $u_n = \int_\tau^t f_n(s)ds$. Since the set $S^2_{\mathcal{N}}(F, \lambda \times P)$ is weakly compact, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \rightharpoonup f$ in $L^2_{\mathcal{N}}(\lambda \times P)$, where $\rightharpoonup$ denotes weak convergence. But $S^2_{\mathcal{N}}(F, \lambda \times P)$ is weakly closed, thus $f \in S^2_{\mathcal{N}}(F, \lambda \times P)$. For a fixed

$\tau, t \in I$, $\tau < t$, let us consider a linear operator $T_\tau^t\colon L^2_{\mathcal{N}}(\lambda \times P) \to L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$ defined by $T_\tau^t(g) := \int_\tau^t g(s)ds$. By the inequality above, it follows that T_τ^t is norm-to-norm continuous. Hence by Dunford–Schwartz Theorem (see [10]) it is equivalent to the fact that T_τ^t is continuous with respect to weak topologies. Therefore $u_{n_k} = T_\tau^t(f_{n_k}) \rightharpoonup T_\tau^t(f)$ in $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$, and hence $u = T_\tau^t(f)$. This proves the closedness of the set $L_\tau^t(F)$ in the norm topology of the space $L^2(\Omega, \mathcal{A}_t, P; \mathbb{R}^d)$, and consequently also its weak compactness. □

Remark 2.9. If $F\colon I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ is a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic process, then $L_\tau^t(F)$ is a convex set.

Clearly we have also that for every $\tau, a, t \in I$, $\tau \le a \le t$ it holds

$$L_\tau^t(F) = L_\tau^a(F) + L_a^t(F).$$

Moreover, if $F_1, F_2\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ are the nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic processes, then for every $\tau, t \in I$, $\tau < t$, similarly as in the proof of Theorem 2.5, one can derive the following:
for every $f_1 \in S^2_{\mathcal{N}}(F_1, \lambda \times P)$ we have

$$\begin{aligned}
&\operatorname{dist}^2_{L^2}\left(\int_\tau^t f_1(s)ds, L_\tau^t(F_2)\right) \\
&= \inf_{f_2 \in S^2_{\mathcal{N}}(F_2, \lambda\times P)} \mathbb{E}\left|\int_\tau^t (f_1(s) - f_2(s))ds\right|^2 \\
&\le (t-\tau) \inf_{f_2 \in S^2_{\mathcal{N}}(F_2, \lambda\times P)} \int\limits_{[\tau,t]\times\Omega} |f_1(s) - f_2(s)|^2 ds \times dP \\
&= (t-\tau) \int\limits_{[\tau,t]\times\Omega} \inf_{x \in F_2(s,\omega)} \varphi(s,\omega,x) ds \times dP \\
&\le (t-\tau) \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP,
\end{aligned}$$

where $\varphi\colon I \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is defined as $\varphi(s,\omega,x) := |f_1(s,\omega) - x|^2$. Hence we have the following result.

Theorem 2.10. *Let $F_1, F_2\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be the nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded set-valued stochastic processes. Then for every $\tau, t \in I$, $\tau < t$ it holds*

$$H^2_{L^2}\left(L_\tau^t(F_1), L_\tau^t(F_2)\right) \le (t-\tau) \int\limits_{[\tau,t]\times\Omega} H^2_{\mathbb{R}^d}(F_1, F_2) ds \times dP.$$

Remark 2.11. Similarly as in Theorem 2.7 one obtains that the mapping

$$[\tau, T] \ni t \mapsto L_\tau^t(F) \in \mathcal{K}^b(L^2)$$

is continuous with respect to H_{L^2}.

Finally, using the same arguments as in the proof of Theorem 2.4 one can show the following result.

Theorem 2.12. *For each $n \in \mathbb{N}$, let $F_n\colon I \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a nonanticipating set-valued stochastic processes such that F_1 is $L^2_{\mathcal{N}}(\lambda \times P)$-integrably bounded and*

$$F_1 \supset F_2 \supset \cdots \supset F \quad \lambda \times P\text{-a.e.},$$

where $F := \bigcap_{n=1}^{\infty} F_n \quad \lambda \times P$-a.e. Then for every $\tau, t \in I$, $\tau < t$ it holds

$$L^t_\tau(F) = \bigcap_{n=1}^{\infty} L^t_\tau(F_n).$$

3. FUZZY TRAJECTORY STOCHASTIC INTEGRALS AND FUZZY STOCHASTIC INTEGRAL EQUATION

As it was announced in the Introduction, in this section we recall some fundamental facts concerning fuzzy sets and fuzzy stochastic processes. Then we formulate the results of existence of fuzzy trajectory stochastic integrals of Itô and Aumann type. Finally we consider the fuzzy stochastic integral equations with the fuzzy trajectory stochastic integrals.

Let $\mathcal{X}$ be a given separable, reflexive Banach space. By a fuzzy set u of the space $\mathcal{X}$ we mean a function $u\colon \mathcal{X} \to [0,1]$. We denote this fact as $u \in \mathcal{F}(\mathcal{X})$. For $\alpha \in (0,1]$ denote $[u]^\alpha := \{x \in \mathcal{X} : u(x) \geq \alpha\}$ and let $[u]^0 := \mathrm{cl}_{\mathcal{X}}\{x \in \mathcal{X} : u(x) > 0\}$, where $\mathrm{cl}_{\mathcal{X}}$ denotes the closure in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. The sets $[u]^\alpha$ are called the α-level sets of fuzzy set u, and 0-level set is called the support of u.

We will use the following Representation Theorem of Negoita–Ralescu [36].

Theorem 3.1. *Let $M \subset \mathcal{X}$ be a set and let $\{C_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that*

(i) *$C_0 = M$,*
(ii) *$C_0 \supset C_\alpha \supset C_\beta$ for $0 \leq \alpha \leq \beta$,*
(iii) *if $\alpha_n \nearrow \alpha$ then $C_\alpha = \bigcap_{n=1}^{\infty} C_{\alpha_n}$.*

Then there exists $u \in \mathcal{F}(\mathcal{X})$ such that $[u]^\alpha = C_\alpha$ for every $\alpha \in [0,1]$. Moreover

$$u(x) = \begin{cases} \sup\{\alpha : x \in C_\alpha\}, & \text{if } x \in M, \\ 0, & \text{if } x \notin M. \end{cases}$$

In the sequel we will deal with fuzzy sets which have some additional properties. Therefore we introduce the notation

$$\mathcal{F}^b(\mathcal{X}) = \{u \in \mathcal{F}(\mathcal{X}) : [u]^\alpha \in \mathcal{K}^b(\mathcal{X}) \text{ for every } \alpha \in [0,1]\},$$

$$\mathcal{F}^b_c(\mathcal{X}) = \{u \in \mathcal{F}^b(\mathcal{X}) : [u]^\alpha \in \mathcal{K}^b_c(\mathcal{X}) \text{ for every } \alpha \in [0,1]\}.$$

The following metric $D_{\mathcal{X}}$ in $\mathcal{F}^b(\mathcal{X})$ is often used:

$$D_{\mathcal{X}}(u,v) := \sup_{\alpha\in[0,1]} H_{\mathcal{X}}([u]^\alpha, [v]^\alpha) \quad \text{for } u,v \in \mathcal{F}^b(\mathcal{X}).$$

This metric generalizes the Hausdorff metric $H_{\mathcal{X}}$, and has the property

$$D_{\mathcal{X}}(u_1+u_2, v_1+v_2) \le D_{\mathcal{X}}(u_1,v_1) + D_{\mathcal{X}}(u_2,v_2) \text{ for } u_1,u_2,v_1,v_2 \in \mathcal{F}^b_c(\mathcal{X}),$$

where the addition of fuzzy sets is defined levelwise, i.e.

$$[u_1+u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha \text{ for } \alpha \in [0,1].$$

It is known (see [43]) that $(\mathcal{F}^b(\mathcal{X}), D_{\mathcal{X}})$ is a complete metric space. It is also easy to see that $\mathcal{F}^b_c(\mathcal{X})$ is a closed subset of $\mathcal{F}^b(\mathcal{X})$.

For our aims we will consider two cases of $\mathcal{X}$. Namely we will take $\mathcal{X} = \mathbb{R}^d$ or $\mathcal{X} = L^2$, where we assume (from now on) that σ-algebra $\mathcal{A}$ is separable with respect to probability measure P.

By a fuzzy random variable we mean a function $u\colon \Omega \to \mathcal{F}^b(\mathcal{X})$ such that $[u(\cdot)]^\alpha\colon \Omega \to \mathcal{K}^b(\mathcal{X})$ is an $\mathcal{A}$-measurable set-valued mapping for every $\alpha \in (0,1]$.

A fuzzy set-valued mapping $f\colon I\times\Omega \to \mathcal{F}^b(\mathcal{X})$ is called a fuzzy stochastic process if $f(t,\cdot)\colon \Omega \to \mathcal{F}^b(\mathcal{X})$ is a fuzzy random variable for every $t \in I$.

The fuzzy stochastic process $f\colon I\times\Omega \to \mathcal{F}^b(\mathcal{X})$ is said to be nonanticipating if the set-valued mapping $[f]^\alpha\colon I\times\Omega \to \mathcal{K}^b(\mathcal{X})$ is $\mathcal{N}$-measurable for every $\alpha \in (0,1]$.

A fuzzy stochastic process f is called $\{\mathcal{A}_t\}$-adapted, if $f(t,\cdot)$ is an $\mathcal{A}_t$-measurable fuzzy random variable for every $t \in I$.

Let $f\colon I\times\Omega \to \mathcal{F}^b(\mathcal{X})$ be a nonanticipating fuzzy stochastic process. The process f is said to be $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded, if $\|\|[f]^0\|\| \in L^2_{\mathcal{N}}(\lambda\times P)$.

Consider now $f\colon I\times\Omega \to \mathcal{F}^b(\mathbb{R}^d)$ and assume that it is nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded fuzzy stochastic process. For α-levels ($\alpha \in [0,1]$) of such a fuzzy stochastic process f one can consider set-valued trajectory stochastic integrals $J^t_\tau([f]^\alpha)$ and $L^t_\tau([f]^\alpha)$ for every $\tau,t \in I$, $\tau < t$.

Then by Theorem 2.4, Theorem 2.12 and Theorem 3.1 for every $\tau,t \in I$, $\tau < t$ there exist fuzzy sets in $\mathcal{F}^b(L^2)$ denoted by

$$(\mathcal{F})\int_\tau^t f(s)dB(s) \quad \text{and} \quad (\mathcal{F})\int_\tau^t f(s)ds$$

such that for every $\alpha \in [0,1]$

$$\left[(\mathcal{F})\int_\tau^t f(s)dB(s)\right]^\alpha = J^t_\tau([f]^\alpha), \quad \left[(\mathcal{F})\int_\tau^t f(s)ds\right]^\alpha = L^t_\tau([f]^\alpha).$$

Definition 3.2. The fuzzy sets

$$(\mathcal{F})\int_\tau^t f(s)dB(s) \in \mathcal{F}^b(L^2), \quad (\mathcal{F})\int_\tau^t f(s)ds \in \mathcal{F}^b(L^2)$$

described above are called the fuzzy trajectory stochastic Itô integral and fuzzy trajectory stochastic Aumann integral (respectively) of nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded fuzzy stochastic process $f\colon I\times\Omega\to\mathcal{F}^b(\mathbb{R}^d)$.

Note that the sum of these integrals not necessarily is the fuzzy set from $\mathcal{F}^b(L^2)$.

Remark 3.3. If we assume additionally that f has convex α-level sets, i.e. $f\colon I\times\Omega\to\mathcal{F}^b_c(\mathbb{R}^d)$ then

$$(\mathcal{F})\int_\tau^t f(s)dB(s) \in \mathcal{F}^b_c(L^2), \quad (\mathcal{F})\int_\tau^t f(s)ds \in \mathcal{F}^b_c(L^2),$$

and as a consequence

$$(\mathcal{F})\int_\tau^t f(s)ds + (\mathcal{F})\int_\tau^t f(s)dB(s) \in \mathcal{F}^b_c(L^2).$$

This fact we will use later in (3.3).

One can show that for every $\tau, a, t\in I$, $\tau\le a\le t$ it holds

$$(\mathcal{F})\int_\tau^t f(s)dB(s) = (\mathcal{F})\int_\tau^a f(s)dB(s) + (\mathcal{F})\int_a^t f(s)dB(s), \tag{3.1}$$

$$(\mathcal{F})\int_\tau^t f(s)ds = (\mathcal{F})\int_\tau^a f(s)ds + (\mathcal{F})\int_a^t f(s)ds. \tag{3.2}$$

Directly by Theorem 2.5 and Theorem 2.10 we have the following result.

Corollary 3.4. *Let $f,g\colon I\times\Omega\to\mathcal{F}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded fuzzy stochastic processes. Then for every $\tau,t\in I$, $\tau<t$*

$$D^2_{L^2}\left((\mathcal{F})\int_\tau^t f(s)dB(s), (\mathcal{F})\int_\tau^t g(s)dB(s)\right) \le \int_{[\tau,t]\times\Omega} D^2_{\mathbb{R}^d}(f,g)ds\times dP,$$

and

$$D^2_{L^2}\left((\mathcal{F})\int_\tau^t f(s)ds, (\mathcal{F})\int_\tau^t g(s)ds\right) \le (t-\tau)\int_{[\tau,t]\times\Omega} D^2_{\mathbb{R}^d}(f,g)ds\times dP.$$

Applying (3.1) and (3.2) we obtain:

Corollary 3.5. *Let $f\colon I\times\Omega\to\mathcal{F}^b(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrably bounded fuzzy stochastic process. Then the mappings*

$$[\tau,T]\ni t\mapsto (\mathcal{F})\int_\tau^t f(s)dB(s)\in\mathcal{F}^b(L^2),$$

$$[\tau, T] \ni t \mapsto (\mathcal{F}) \int_\tau^t f(s)ds \in \mathcal{F}^b(L^2),$$

are continuous with respect to the metric D_{L^2}.

In the sequel we want to consider some fuzzy equation in which the fuzzy trajectory stochastic integrals appear.

Denote $L_0^2 = L^2(\Omega, \mathcal{A}_0, P; \mathbb{R}^d)$. Let $f, g\colon I \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ and let $X_0 \in \mathcal{F}_c^b(L_0^2)$. By a fuzzy stochastic integral equation we mean the following relation in the space $\mathcal{F}_c^b(L^2)$:

$$X(t) = X_0 + (\mathcal{F}) \int_0^t f(s, X(s))ds + (\mathcal{F}) \int_0^t g(s, X(s))dB(s) \ \text{ for } \ t \in I. \tag{3.3}$$

Definition 3.6. By a solution to (3.3) we mean a continuous mapping $X\colon I \to \mathcal{F}_c^b(L^2)$ that satisfies (3.3). A solution $X\colon I \to \mathcal{F}_c^b(L^2)$ to (3.3) is unique if

$$X(t) = Y(t) \ \text{ for every } \ \ t \in I,$$

where $Y\colon I \to \mathcal{F}_c^b(L^2)$ is any solution of (3.3).

Below we write down the detailed conditions which will be imposed on the coefficients of the equation.

Assume that $f, g\colon I \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ satisfy:

(f1) for every $u \in \mathcal{F}_c^b(L^2)$ the mappings

$$f(\cdot, \cdot, u), g(\cdot, \cdot, u)\colon I \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$$

are the nonanticipating fuzzy stochastic processes,

(f2) there exists a constant $K > 0$ such that

$$D_{\mathbb{R}^d}\big(f(t,u), f(t,v)\big) + D_{\mathbb{R}^d}\big(g(t,u), g(t,v)\big) \le K D_{L^2}(u,v)$$

for every $(t, \omega) \in I \times \Omega$, and every $u, v \in \mathcal{F}_c^b(L^2)$,

(f3) there exists a constant $C > 0$ such that

$$D_{\mathbb{R}^d}\big(f(t,u), \hat{\theta}\big) + D_{\mathbb{R}^d}\big(g(t,u), \hat{\theta}\big) \le C\big(1 + D_{L^2}(u, \hat{\Theta})\big),$$

for every $(t, \omega) \in I \times \Omega$, and every $u \in \mathcal{F}_c^b(L^2)$.

To describe the symbols $\hat{\theta}, \hat{\Theta}$ let θ, Θ denote the zero elements in $\mathbb{R}^d$ and L^2, respectively. Then the symbols $\hat{\theta}, \hat{\Theta}$ are their fuzzy counterparts, i.e. $[\hat{\theta}]^\alpha = \{\theta\}$ and $[\hat{\Theta}]^\alpha = \{\Theta\}$ for every $\alpha \in [0, 1]$.

Theorem 3.7. *Let* $X_0 \in \mathcal{F}_c^b(L_0^2)$, *and* $f, g\colon I \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ *satisfy the conditions* (f1)-(f3). *Then the equation* (3.3) *has a unique solution.*

Proof. Let us define a sequence $X_n\colon I \to \mathcal{F}_c^b(L^2)$, $n = 0, 1, \dots$ of successive approximations as follows:

$$X_0(t) = X_0, \ \text{ for every } \ t \in I,$$

and for $n = 1, 2, \dots$

$$X_n(t) = X_0 + (\mathcal{F}) \int_0^t f(s, X_{n-1}(s)) ds + (\mathcal{F}) \int_0^t g(s, X_{n-1}(s)) dB(s)$$

for every $t \in I$.

Due to Corollary 3.5, the mappings X_n are continuous.

By Corollary 3.4 and assumptions (f1), (f3) one gets

$$\begin{aligned} D_{L^2}^2\Big((\mathcal{F}) \int_0^t f(s, X_0) ds, \hat{\Theta}\Big) &= D_{L^2}^2\Big((\mathcal{F}) \int_0^t f(s, X_0) ds, (\mathcal{F}) \int_0^t \hat{\theta} ds\Big) \\ &\le T \int\limits_{I \times \Omega} D_{\mathbb{R}^d}^2\big(f(s, X_0), \hat{\theta}\big) ds \times dP \le \eta T^2, \end{aligned}$$

where $\eta = 2C^2(1 + D_{L^2}^2(X_0, \hat{\Theta}))$. Similarly we have

$$\begin{aligned} D_{L^2}^2\Big((\mathcal{F}) \int_0^t g(s, X_0) dB(s), \hat{\Theta}\Big) &= D_{L^2}^2\Big((\mathcal{F}) \int_0^t g(s, X_0) dB(s), (\mathcal{F}) \int_0^t \hat{\theta} ds\Big) \\ &\le \int\limits_{I \times \Omega} D_{\mathbb{R}^d}^2\big(g(s, X_0), \hat{\theta}\big) ds \times dP \le \eta T. \end{aligned}$$

Therefore for every $t \in I$

$$\begin{aligned} D_{L^2}^2\big(X_1(t), X_0(t)\big) &\le 2 D_{L^2}^2\Big((\mathcal{F}) \int_0^t f(s, X_0) ds, \hat{\Theta}\Big) \\ &\quad + 2 D_{L^2}^2\Big((\mathcal{F}) \int_0^t g(s, X_0) dB(s), \hat{\Theta}\Big) \\ &\le 2\eta T(T + 1). \end{aligned}$$

Observe further that for $n = 2, 3, \dots$, using Corollary 3.4 and assumptions (f1)-(f2), one has

$$\begin{aligned} &D_{L^2}^2\big(X_n(t), X_{n-1}(t)\big) \\ &\quad \le 2 D_{L^2}^2\Big((\mathcal{F}) \int_0^t f(s, X_{n-1}(s)) ds, (\mathcal{F}) \int_0^t f(s, X_{n-2}(s)) ds\Big) \\ &\qquad + \ 2 D_{L^2}^2\Big((\mathcal{F}) \int_0^t g(s, X_{n-1}(s)) dB(s), (\mathcal{F}) \int_0^t g(s, X_{n-2}(s)) dB(s)\Big) \\ &\quad \le 2T \int\limits_{I \times \Omega} D_{\mathbb{R}^d}^2\big(f(s, X_{n-1}(s)), f(s, X_{n-2}(s))\big) ds \times dP \\ &\qquad + 2 \int\limits_{I \times \Omega} D_{\mathbb{R}^d}^2\big(g(s, X_{n-1}(s)), g(s, X_{n-2}(s))\big) ds \times dP \end{aligned}$$

$$\le 2K^2(T+1)\int_0^T D^2_{L^2}\big(X_{n-1}(s),X_{n-2}(s)\big)ds.$$

Thus we obtain

$$\sup_{t\in I} D^2_{L^2}\big(X_n(t),X_{n-1}(t)\big)\le \eta K^{-2}\frac{[2K^2T(T+1)]^n}{n!}.$$

Let us consider the space $C(I,\mathcal{F}^b_c(L^2))$ of continuous mappings $X\colon I\to\mathcal{F}^b_c(L^2)$ with a distance ρ defined by

$$\rho(X,Y)=\sup_{t\in I} D_{L^2}(X(t),Y(t))\quad\text{for }\ X,Y\in C(I,\mathcal{F}^b_c(L^2)).$$

It is clear that $\big(C(I,\mathcal{F}^b_c(L^2)),\rho\big)$ is a complete metric space. Since $\{X_n\}\subset C(I,\mathcal{F}^b_c(L^2))$ and for $m<n$

$$\rho(X_n,X_m\big)=\sup_{t\in I} D_{L^2}(X_n(t),X_m(t))\le\sqrt{\eta}K^{-1}\sum_{k=m+1}^{n}\sqrt{\frac{(2K^2T(T+1))^k}{k!}},$$

we infer that $\{X_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\big(C(I,\mathcal{F}^b_c(L^2)),\rho\big)$. Thus there is $X\in C(I,\mathcal{F}^b_c(L^2))$ such that $\rho(X_n,X)\to 0$, as $n\to\infty$.

We shall show that X is a solution to (3.3). Let $t\in I$ be fixed. We have

$$\begin{aligned}
&D^2_{L^2}\Big(X(t),X_0+(\mathcal{F})\int_0^t f(s,X(s))ds+(\mathcal{F})\int_0^t g(s,X(s))dB(s)\Big)\\
&\quad\le 3D^2_{L^2}\big(X_n(t),X(t)\big)\\
&\qquad+3D^2_{L^2}\Big(X_n(t),X_0+(\mathcal{F})\int_0^t f(s,X_{n-1}(s))ds\\
&\qquad+(\mathcal{F})\int_0^t g(s,X_{n-1}(s))dB(s)\Big)\\
&\qquad+6D^2_{L^2}\Big((\mathcal{F})\int_0^t f(s,X_{n-1}(s))ds,(\mathcal{F})\int_0^t f(s,X(s))ds\Big)\\
&\qquad+\ 6D^2_{L^2}\Big((\mathcal{F})\int_0^t g(s,X_{n-1}(s))dB(s),(\mathcal{F})\int_0^t g(s,X(s))dB(s)\Big),
\end{aligned}$$

The first term on the right-hand side of the inequality converges to zero, whereas the second is equal to zero. Since

$$\begin{aligned}
&D^2_{L^2}\Big((\mathcal{F})\int_0^t f(s,X_{n-1}(s))ds,(\mathcal{F})\int_0^t f(s,X(s))ds\Big)\\
&\quad\le TK^2\int_0^T D^2_{L^2}\big(X_{n-1}(s),X(s)\big)ds,\\
&\quad\le T^2K^2\sup_{t\in I} D^2_{L^2}\big(X_{n-1}(t),X(t)\big)\to 0,\ \text{ as }\ n\to\infty,
\end{aligned}$$

and

$$D^2_{L^2}\Big((\mathcal{F})\int_0^t g(s,X_{n-1}(s))dB(s),(\mathcal{F})\int_0^t g(s,X(s))dB(s)\Big)$$

$$\leq K^2 \int_0^T D^2_{L^2}\big(X_{n-1}(s), X(s)\big)ds,$$
$$\leq TK^2 \sup_{t\in I} D^2_{L^2}\big(X_{n-1}(t), X(t)\big) \to 0, \text{ as } n \to \infty,$$

we infer that

$$D_{L^2}\Big(X(t), X_0 + (\mathcal{F})\int_0^t f(s, X(s))ds + (\mathcal{F})\int_0^t g(s, X(s))dB(s)\Big) = 0$$

for every $t \in I$.

For the uniqueness assume that $X\colon I \to \mathcal{F}^b_c(L^2)$ and $Y\colon I \to \mathcal{F}^b_c(L^2)$ are two solutions to (3.3). Then let us notice that

$$D^2_{L^2}\big(X(t), Y(t)\big) \leq 2K^2(T+1)\int_0^t D^2_{L^2}\big(X(s), Y(s)\big)ds.$$

Thus, by Gronwall's lemma, we obtain

$$D^2_{L^2}\big(X(t), Y(t)\big) \leq 0 \quad \text{for every} \quad t \in I.$$

Therefore the uniqueness of the solution follows. □

The next result presents some estimation for the solution to (3.3).

Theorem 3.8. *Let $X_0 \in \mathcal{F}^b_c(L^2_0)$ and $f, g\colon I \times \Omega \times \mathcal{F}^b_c(L^2) \to \mathcal{F}^b_c(\mathbb{R}^d)$ satisfy the assumptions of Theorem* 3.7. *Then the solution X of* (3.3) *satisfies*

$$\sup_{t\in I} D^2_{L^2}\big(X(t), \hat{\Theta}\big) \leq \big(3D^2_{L^2}\big(X_0, \hat{\Theta}\big) + 6C^2T(T+1)\big)e^{6C^2T(T+1)}.$$

Proof. Let us fix $t \in I$. Using Corollary 3.4 and assumption (f3) we obtain

$$\begin{aligned}
D^2_{L^2}\big(X(t), \hat{\Theta}\big) &\leq 3D^2_{L^2}(X_0, \hat{\Theta}) + 3D^2_{L^2}\Big((\mathcal{F})\int_0^t f(s, X(s))ds, \hat{\Theta}\Big) \\
&\quad + 3D^2_{L^2}\Big((\mathcal{F})\int_0^t g(s, X(s))ds, \hat{\Theta}\Big) \\
&\leq 3D^2_{L^2}(X_0, \hat{\Theta}) + 3t\int_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(f(s, X(s)), \hat{\theta}\big)ds \times dP \\
&\quad + 3\int_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(g(s, X(s)), \hat{\theta}\big)ds \times dP \\
&\leq 3D^2_{L^2}(X_0, \hat{\Theta}) + 6C^2t(t+1) + 6C^2(t+1)\int_0^t D^2_{L^2}\big(X(s), \hat{\Theta}\big)ds
\end{aligned}$$

Now it is enough to apply the Gronwall lemma to get the result. □

Next we will show a continuous dependence on initial conditions of the solution to (3.3). Let X, Y denote the solutions of the equations

$$X(t) = X_0 + (\mathcal{F})\int_0^t f(s, X(s))ds + (\mathcal{F})\int_0^t g(s, X(s))dB(s) \text{ for } t \in I, \tag{3.4}$$

$$Y(t) = Y_0 + (\mathcal{F}) \int_0^t f(s, Y(s))ds + (\mathcal{F}) \int_0^t g(s, Y(s))dB(s) \text{ for } t \in I, \tag{3.5}$$

respectively. Assume that X_0, Y_0 and f, g satisfy the conditions as in Theorem 3.7.

Theorem 3.9. *For the solutions X, Y of the equations* (3.4), (3.5) *it holds*

$$\sup_{t\in I} D^2_{L^2}(X(t), Y(t)) \leq 3D^2_{L^2}(X_0, Y_0)e^{3K^2T(T+1)}.$$

Proof. It is enough to notice that for every $t \in I$ one has

$$\begin{aligned} &D^2_{L^2}\big(X(t), Y(t)\big) \\ &\quad\leq 3D^2_{L^2}(X_0, Y_0) + 3t \int_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(f(s, X(s)), f(s, Y(s))\big)ds \times dP \\ &\qquad + 3 \int_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(g(s, X(s)), g(s, Y(s))\big)ds \times dP \\ &\quad\leq 3D^2_{L^2}(X_0, Y_0) + 3K^2(t+1) \int_0^t D^2_{L^2}\big(X(s), Y(s)\big)ds. \end{aligned}$$

Now application of the Gronwall lemma gives the result. □

Finally we present the stability property of solutions to the system of fuzzy stochastic integral equations.

Let us consider the following equations:

$$X(t) = X_0 + (\mathcal{F}) \int_0^t f(s, X(s))ds + (\mathcal{F}) \int_0^t g(s, X(s))dB(s) \text{ for } t \in I,$$

and for $n = 1, 2, \ldots$

$$X_n(t) = X_{0,n} + (\mathcal{F}) \int_0^t f_n(s, X_n(s))ds + (\mathcal{F}) \int_0^t g_n(s, X_n(s))dB(s)$$

for $t \in I$.

Theorem 3.10. *Let $f, g, f_n, g_n \colon I \times \Omega \times \mathcal{F}^b_c(L^2) \to \mathcal{F}^b_c(\mathbb{R}^d)$ satisfy the conditions* (f1)-(f3) *with the same constants K, C. Let also $X_0, X_{0,n} \in \mathcal{F}^b_c(L^2_0)$ for every $n \in \mathbb{N}$. Assume that*

(i) $D_{L^2}\big(X_{0,n}, X_0\big) \to 0$,
(ii) $D_{\mathbb{R}^d}\big(f_n(t, u), f(t, u)\big) \to 0$, *for every* $(t, \omega, u) \in I \times \Omega \times \mathcal{F}^b_c(L^2)$,
(iii) $D_{\mathbb{R}^d}\big(g_n(t, u), g(t, u)\big) \to 0$, *for every* $(t, \omega, u) \in I \times \Omega \times \mathcal{F}^b_c(L^2)$.

Then

$$\sup_{t\in I} D_{L^2}\big(X_n(t), X(t)\big) \to 0, \quad \text{as } n \to \infty.$$

Proof. By virtue of Corollary 3.4, assumptions (f1)-(f3), let us note that for every $t \in I$ one can show the following estimations:

$$\begin{aligned}
&D^2_{L^2}\big(X_n(t), X(t)\big)\\
&\quad\leq 3D^2_{L^2}\big(X_{0,n}, X_0\big) + 3t \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(f_n(s, X_n(s)), f(s, X(s))\big) ds \times dP\\
&\qquad + 3 \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(g_n(s, X_n(s)), g(s, X(s))\big) ds \times dP\\
&\quad\leq 3D^2_{L^2}\big(X_{0,n}, X_0\big) + 6t \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(f_n(s, X(s)), f(s, X(s))\big) ds \times dP\\
&\qquad + 6 \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(g_n(s, X(s)), g(s, X(s))\big) ds \times dP\\
&\qquad + 6K^2(t+1) \int_0^t D^2_{L^2}\big(X_n(s), X(s)\big) ds.
\end{aligned}$$

Application of Gronwall's lemma yields

$$\begin{aligned}
&D^2_{L^2}\big(X_n(t), X(t)\big)\\
&\quad\leq \Big(3D^2_{L^2}\big(X_{0,n}, X_0\big) + 6t \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(f_n(s, X(s)), f(s, X(s))\big) ds \times dP\\
&\qquad + 6 \int\limits_{[0,t]\times\Omega} D^2_{\mathbb{R}^d}\big(g_n(s, X(s)), g(s, X(s))\big) ds \times dP\Big) e^{6K^2 t(t+1)}.
\end{aligned}$$

Using the assumptions and the Lebesgue Dominated Convergence Theorem we end the proof. □

4. **SET-VALUED STOCHASTIC INTEGRAL EQUATIONS**

In this section we consider set-valued stochastic integral equations. The motivation of the study of such equations comes from the deterministic case, where the study of them has been used as an alternative approach to the issue of fuzzy differential equations (see e.g. [26, 28]). The idea used there was to repleace an original fuzzy differential equation by the system of set-valued differential equations generated from the original fuzzy setup. Such procedure avoided the main disadvantage of studying directly the fuzzy differential equation in the original formulation, i.e, the lack of a reflection of the rich behaviour of corresponding differential equations without fuzziness. As it was shown, it had been caused by the fact that the diameter of any solution of a fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved. Taking these remarks into consideration, let $F, G\colon I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d)$ and let $X_0 \in \mathcal{K}^b_c(L^2_0)$. By a set-valued stochastic integral equation we mean the following relation in the space $\mathcal{K}_c^b(L^2)$:

$$X(t) = X_0 + L_0^t(F(X)) + J_0^t(G(X)) \text{ for } t \in I, \tag{4.1}$$

where $F(X), G(X)\colon I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ are defined by $F(X)(t,\omega) = F(t,\omega,X(t))$, $G(X)(t,\omega) = G(t,\omega,X(t))$ for $(t,\omega) \in I \times \Omega$.

Definition 4.1. By a solution to (4.1) we mean a continuous mapping $X\colon I \to \mathcal{K}_c^b(L^2)$ that satisfies (4.1). A solution $X\colon I \to \mathcal{K}_c^b(L^2)$ to (4.1) is unique if

$$X(t) = Y(t) \text{ for every } \ t \in I,$$

where $Y\colon I \to \mathcal{K}_c^b(L^2)$ is any solution of (4.1).

Now we formulate the conditions required from the equation coefficients.

Assume that $F, G\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ satisfy:

(s1) for every $u \in \mathcal{K}_c^b(L^2)$ the mappings

$$F(\cdot,\cdot,u), G(\cdot,\cdot,u)\colon I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$$

are the nonanticipating set-valued stochastic processes,

(s2) there exists a constant $K > 0$ such that

$$H_{\mathbb{R}^d}\big(F(t,u), F(t,v)\big) + H_{\mathbb{R}^d}\big(G(t,u), G(t,v)\big) \le K H_{L^2}(u,v)$$

for every $(t,\omega) \in I$, and every $u, v \in \mathcal{K}_c^b(L^2)$,

(s3) there exists a constant $C > 0$ such that

$$H_{\mathbb{R}^d}\big(F(t,u), \theta\big) + H_{\mathbb{R}^d}\big(G(t,u), \theta\big) \le C\big(1 + H_{L^2}(u,\Theta)\big),$$

for every $(t,\omega) \in I \times \Omega$, and every $u \in \mathcal{K}_c^b(L^2)$.

Using arguments which are similar to those from Section 3, we obtain the results for set-valued stochastic integral equations.

Theorem 4.2. *$X_0 \in \mathcal{K}_c^b(L_0^2)$, and $F, G\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ satisfy the conditions* (s1)-(s3). *Then equation* (4.1) *has a unique solution.*

Theorem 4.3. *Let $X_0 \in \mathcal{K}_c^b(L_0^2)$ and $F, G\colon I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ satisfy the assumptions of Theorem* 4.2. *Then the solution X of* (4.1) *satisfies*

$$\sup_{t\in I} H_{L^2}^2\big(X(t),\Theta\big) \le \big(3H_{L^2}^2\big(X_0,\Theta\big) + 6C^2T(T+1)\big)e^{6C^2T(T+1)}.$$

Also a continuous dependence on initial conditions of solution to (4.1) holds: let X, Y denote the solutions of the equations

$$X(t) = X_0 + L_0^t(F(X)) + J_0^t(G(X)) \text{ for } t \in I, \tag{4.2}$$

$$Y(t) = Y_0 + L_0^t(F(Y)) + J_0^t(G(Y)) \text{ for } t \in I, \tag{4.3}$$

respectively. Assume that X_0, Y_0 and f, g satisfy the conditions as in Theorem 4.2.

Theorem 4.4. *For the solutions X, Y of the equations* (4.2), (4.3) *it holds*

$$\sup_{t\in I} H^2_{L^2}(X(t), Y(t)) \leq 3H^2_{L^2}(X_0, Y_0)e^{3K^2T(T+1)}.$$

Notice that a stability property of solutions to the system of set-valued stochastic integral equations holds. Indeed, let us consider the following equations:

$$X(t) = X_0 + L^t_0(F(X)) + J^t_0(G(X)) \ \text{ for } \ t \in I,$$

and for $n = 1, 2, \ldots$

$$X_n(t) = X_{0,n} + L^t_0(F_n(X_n))ds + J^t_0(G_n(X_n)) \ \text{ for } \ t \in I,$$

Theorem 4.5. *Let $F, G, F_n, G_n \colon I \times \Omega \times \mathcal{K}^b_c(L^2) \to \mathcal{K}^b_c(\mathbb{R}^d)$ satisfy the the conditions* (s1)-(s3) *with the same constants K, C. Let also $X_0, X_{0,n} \in \mathcal{K}^b_c(L^2_0)$ for every $n \in \mathbb{N}$. Assume that*

(i) $H_{L^2}\big(X_{0,n}, X_0\big) \to 0$,
(ii) $H_{\mathbb{R}^d}\big(F_n(t,u), F(t,u)\big) \to 0$, *for every* $(t,\omega,u) \in I \times \Omega \times \mathcal{K}^b_c(L^2)$,
(iii) $H_{\mathbb{R}^d}\big(F_n(t,u), F(t,u)\big) \to 0$, *for every* $(t,\omega,u) \in I \times \Omega \times \mathcal{K}^b_c(L^2)$.

Then

$$\sup_{t\in I} H_{L^2}\big(X_n(t), X(t)\big) \to 0, \ \ \textit{as } n \to \infty.$$

REFERENCES

[1] R.P. Agarwal, D. O'Regan and V. Lakshmikantham, A stacking theorem approach for fuzzy differential equations, *Nonlinear Anal.*, 55:299–312, 2003.

[2] R.P. Agarwal, D. O'Regan and V. Lakshmikantham, Viability theory and fuzzy differential equations, *Fuzzy Sets and Systems*, 151:536–580, 2005.

[3] N.U. Ahmed, Nonlinear stochastic differential inclusions on Banach spaces, *Stoch. Anal. Appl.*, 12:1–10, 1994.

[4] N.U. Ahmed, Impulsive perturbation of C_0 semigroups and stochastic evolution inclusions, *Discuss. Math. DICO*, 22:125–149, 2002.

[5] N.U. Ahmed, Optimal relaxed controls for nonlinear infinite dimensional stochastic differential inclusions, *Optimal Control of Differential Equations*, M. Dekker Lect. Notes, 160:1–19, 1994.

[6] N.U. Ahmed, Differential inclusions operator valued measures and optimal control, *Dynamic Syst. Appl.*, 16:13–36, 2007.

[7] B. Bede and S. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151:581–599, 2007.

[8] T.G. Bhaskar, V. Lakshmikantham and V. Devi, Revisiting fuzzy differential equations, *Nonlinear Anal.*, 58:351–358, 2004.

[9] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994.

[10] N. Dunford and J.T. Schwartz, *Linear Operators I*, Int. Publ., INC, New York, 1967.

[11] W. Fei, Existence and uniqueness of solution for fuzzy random differential equations with non-Lipschitz coefficients, *Inform. Sci.*, 177:4329–4337, 2007.

[12] Y. Feng, Fuzzy stochastic differential systems, *Fuzzy Sets and Systems*, 115:351–363, 2000.

[13] F. Hiai and H. Umegaki, Integrals, conditional expectations and martingales for multivalued functions, *J. Multivar. Anal.*, 7:147–182, 1977.

[14] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Vol.1, Theory*, Kluwer Acad. Publ., Boston, 1997.

[15] E. Hüllermeier, An approach to modelling and simulation of uncertain dynamical systems, *Int. J. Uncertainty Fuzziness Knowledge-Based Syst.*, 5:117–137, 1997.

[16] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland Publ., Amsterdam, 1981.

[17] E.J. Jung and J.H. Kim, On set-valued stochastic integrals, *Stoch. Anal. Appl.*, 21:401–418, 2003.

[18] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24:301–317, 1987.

[19] B.K. Kim and J.H. Kim, Stochastic integrals of set-valued processes and fuzzy processes, *J. Math. Anal. Appl.*, 236:480–502, 1999.

[20] J.H. Kim, On fuzzy stochastic differential equations, *J. Korean Math. Soc.*, 42:153–169, 2005.

[21] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Acad. Publ., Dordrecht, 1991.

[22] M. Kisielewicz, Backward stochastic differential inclusions, *Dynamic Syst. Appl.*, 16:121–140, 2007.

[23] M. Kisielewicz, Weak compactness of weak solutions to backward stochastic differential inclusions, *Dynamic Syst. Appl.*, 17:351–370, 2008.

[24] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part I. Existence and regularity properties, *Dynamic Syst. Appl.*, 12:405–431, 2003.

[25] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part II. Viability and semimartingale issue, *Dynamic Syst. Appl.*, 12:433–466, 2003.

[26] V. Lakshmikantham, S. Lela and A. Vatsala, Interconnection between set and fuzzy differential equations, *Nonlinear Anal.*, 54:351–360, 2003.

[27] V. Lakshmikantham and R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis Publishers, London, 2003.

[28] V. Lakshmikantham and A.A. Tolstonogov, Existence and interrelation between set and fuzzy differential equations, *Nonlinear Anal.*, 55:255–268, 2003.

[29] J. Li and S. Li, Set-valued stochastic Lebesgue integral and representation theorems, *Int. J. Comp. Intelligence Syst.*, 1:177–187, 2008.

[30] S. Li and A. Ren, Representation theorems, set-valued and fuzzy set-valued Ito integral, *Fuzzy Sets and Systems*, 158:949–962, 2007.

[31] M.T. Malinowski, On random fuzzy differential equations, *Fuzzy Sets and Systems*, 160:3152–3165, 2009.

[32] M.T. Malinowski and M. Michta, Stochastic fuzzy differential equations with an application, submitted.

[33] M. Michta and J. Motyl, Weak solutions to a stochastic inclusion with multivalued integrator, *Dynamic Syst. Appl.*, 14:323–334, 2005.

[34] M. Michta and J. Motyl, Second order stochastic inclusions and equations, *Dynamic Syst. Appl.*, 15:301–315, 2006.

[35] M. Michta and J. Motyl, Set valued Stratonovich integral and Stratonovich type stochastic inclusion, *Dynamic Syst. Appl.*, 16:141–154, 2007.

[36] C.V. Negoita and D.A. Ralescu, *Applications of Fuzzy Sets to System Analysis*, Wiley, New York, 1975.

[37] J.J. Nieto and R. Rodríguez-López, Bounded solutions for fuzzy differential and integral equations, *Chaos Solitons & Fractals*, 27:1376–1386, 2006.

[38] J.J. Nieto and R. Rodríguez-López, Analysis of a logistic differential model with uncertainty, *Int. J. Dynamical Systems and Differential Equations*, 1:164–176, 2008.

[39] J.J. Nieto, R. Rodríguez-López and D. Franco, Linear first-order fuzzy differential equations, *Int. J. Uncertainty Fuzziness Knowledge-Based Syst.*, 14:687–709, 2006.

[40] J.J. Nieto, R. Rodríguez-López and D.N. Georgiou, Fuzzy differential systems under generalized metric spaces, *Dynamic Syst. Appl.*, 17:1–24, 2008.

[41] Y. Ogura, On stochastic differential equations with fuzzy set coefficients. In: *Soft Methods for Handling Variability and Imprecision*, pp. 263–270, Springer, Berlin, 2008.

[42] M.L. Puri and D.A. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.*, 91:552–558, 1983.

[43] D. Qiu and L. Shu, Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings, *Inform. Sci.*, 178:3595–3604, 2008.

[44] J. Zhang, Set-valued stochastic integrals with respect to a real valued martingale. In: *Soft Methods for Handling Variability and Imprecision*, pp. 253–259, Springer, Berlin, 2008.