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OPTIMAL CONTROL FOR PREDATOR-PREY SYSTEM WITH PREY-DEPENDENT FUNCTIONAL RESPONSE

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ABSTRACT. An optimal control problem is studied for a predator-prey system with logistic growth rate of the prey and a prey-dependent functional response of the predator. The control function has two components and signifies the rate of mixture between the individuals of the species. The form of the optimal control is determined according to Pontryagin's maximum principle. It is bang-bang and the number of switchings points depends on the choice of some specific parameters.

Keywords: Logistic growth rate, Predator's functional response, Bang-bang control, Bolza type cost functional

AMS Subject Classification: 49K15, 92D25, 93C10, 93C15, 34H05

1. INTRODUCTION

Consider the predator-prey system

$$\begin{cases} y_1' = ry_1\left(1-\dfrac{y_1}{k}\right) - my_2 h\left(qy_1\right) \\ y_2' = ny_2 h\left(qy_1\right) - ly_2, \quad t \geq 0, \end{cases}$$

where y_1, y_2 are the densities of prey and predators, respectively and $r, k, m, n, l, q > 0$ are given parameters. The growth rate of the prey, $f\left(y_1\right) = ry_1\left(1 - y_1/k\right)$ is logistic and the predator's functional response h depends only on the prey density. The functional response represents the prey consumption rate per predator as a fraction of the maximal consumption rate m. Parameters n and r denote the maximal per capita birth rates of predators and prey, respectively, l is the per capita predator death rate, and k is the prey carrying capacity. Constant q shows how fast the consumption rate saturates the predator when the prey density increases (see [4], [9]).

We work under the following hypothesis:

(H) $h \in C^1\left([0,\infty)\right)$, $h\left(0\right) = 0$, $h'\left(ay_1\right) > 0$, $(\forall)\, y_1 > 0$.

This model takes into account several well-known types of functional responses which depend only on the prey density y_1:

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- $h(y_1) = by_1$ (Holling type I). It is a linear function in y_1 and models the behavior of passive predators (for example, spiders). The number of caught prey is proportional to the prey density. This is the original functional response introduced by V. Volterra.
- $h(y_1) = \frac{by_1}{1+my_1}$ (Holling type II). This is the most frequently studied functional response and has been used in many works to fit ecological data. At low prey densities, predators spend most of their time on search, whereas at high prey densities, predators spend most of their time on prey handling. Predators of this type cause maximum mortality at low prey density. For example, small mammals destroy most of gypsy moth pupae in sparse populations of gypsy moth. However in high-density defoliating populations, small mammals kill a negligible proportion of pupae.
- $h(y_1) = \frac{by_1^2}{1+my_1^2}$ (Holling type III). It is employed if the predators are more efficient at higher prey densities and less efficient at lower prey densities. For example, many predators respond to chemicals emitted by prey and increase thus their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually.
- $h(y_1) = k\left(1 - e^{-by_1}\right)$ (Ivlev functional response). The equation describes a cyrtoid or type II functional response because the feeding rate declines with increasing resource abundance until it reaches a constant rate k. Ivlev's equation describes the effect of consumer satiation on the rate of resource consumption in a similar way to Holling's disk equation; i.e., it describes a type II functional response.

If we put $\widetilde{y}_1 = y_1/k$, $\widetilde{y}_2 = my_2/rk$, $\widetilde{t} = rt$ and denote $a = qk$, $b = n/r$, $c = l/r$ (see [5]), the above system can be written in the simpler form (after dropping the tildes)

$$\begin{cases} y_1' = y_1(1-y_1) - y_2 h(ay_1) \\ y_2' = by_2 h(ay_1) - cy_2, \quad t \geq 0. \end{cases}$$

The parameters a, b, c are positive. We study this system on a finite time interval $[0, T]$. Other problems connected with ordinary differential systems from population dynamics are treated in the papers [6, 7, 9] and in the monographs [4] and [10].

One introduces two control variables u and v, such that $0 \leq u(t) \leq 1$, $0 < v_0 \leq v(t) \leq 1$ a.e. on $[0, T]$, which represent the rate of mixture of the prey and predators (for u) and of prey individuals only (for v). More exactly, $1-u$ is the separation rate between prey and predators, while $1-v$ is the separation rate between the individuals of prey population. The constant v_0 is positive; this shows that the prey individuals can not be separated completely at any moment. The dynamics of the controlled ecosystem on $[0,T]$ is described by the differential system

$$\begin{cases} y_1' = y_1(1 - vy_1) - uy_2 h(ay_1) \\ y_2' = bu\ y_2 h(ay_1) - cy_2 \end{cases}, \quad t \in [0,T]. \tag{1.1}$$

We add to this system some initial conditions

$$y_1(0) = y_1^0 > 0, \quad y_2(0) = y_2^0 > 0. \tag{1.2}$$

The Cauchy problem admits a unique local solution $y = (y_1, y_2)$ defined on an interval $[0,\delta)$, $\delta \in (0,T]$. By a comparison result, we get $y_1 > 0$, $y_2 > 0$. Since the conditions imposed on h imply that $h(ay_1) > 0$ for $y_1 > 0$, then from (1.1)–(1.2) we infer that $y_1(t) \le y_1^0 e^t \le y_1^0 e^T$, $(\forall)\, t \in [0,\delta)$ and therefore, $y_2(t) \le y_2^0 e^{b\int_0^\delta h(ay_1)ds} \le y_2^0 e^{bTh\left(ay_1^0 e^T\right)}$, $(\forall)\, t \in [0,\delta)$. So $y = (y_1, y_2)$ is bounded on its maximal interval of definition. Consequently, $y = (y_1, y_2)$ is defined on the entire $[0,T]$, it is positive and bounded.

The aim of the paper is to study an optimal control problem associated with problem (1.1)–(1.2), namely:

Problem (P) : Minimize the cost functional

$$J(y_1, y_2, u, v) = -[\alpha y_1(T) + \beta y_2(T)] - \int_0^T [k_1 y_1(t) + k_2 y_2(t)]dt, \tag{1.3}$$

subject to the state system (1.1) and to constraints (1.2), $0 \le u(t) \le 1$, $0 < v_0 \le v(t) \le 1$ a.e. on $[0,T]$. The constants α, β, k_1 and k_2 are supposed to be non-negative, not all of them zero at the same time. Problem (1.1)–(1.3) is also referred to as a control problem of *Bolza*.

This problem contains in itself some special classes of optimal control problems associated with system (1.1)–(1.2). For example, if we take $\alpha = \beta = 1$ and $k_1 = k_2 = 0$, the goal is to minimize the functional

$$J_M(y_1, y_2, u, v) = -y_1(T) - y_2(T),$$

subject to the state system (1.1) and to constraints (1.2), $0 \le u(t) \le 1$, $0 < v_0 \le v(t) \le 1$ a.e. on $[0,T]$. In other words, we have to find necessary optimality conditions such that, in the end of the time interval $[0,T]$, the total density of the two populations is maximal. In the literature, a problem of this type is called the control problem of *Mayer*.

If $\alpha = \beta = 0$ and $k_1 = k_2 = 1$, then we have to minimize

$$J_L(y_1, y_2, u, v) = -\int_0^T [y_1(t) + y_2(t)]dt,$$

subject to (1.1)–(1.2) and $0 \le u(t) \le 1$, $0 < v_0 \le v(t) \le 1$ a.e. on $[0,T]$. This cost functional is of *Lagrange* type.

In the next section, one applies Pontryagin's maximum principle to find necessary optimality conditions for problem (1.1)–(1.3) and to establish the form of the optimal control. We show that u is bang-bang and $v = v_0$ on $[0,T]$. According to the sign of a specific constant, in Section 3 we find the number of the switchings of u.

For linear growth rate of the prey $f(y_1) = ry_1$ and linear predator's functional response h (i.e., Holling type I functional response), such problem was analyzed by S. Yosida ([15, 16]). A Mayer optimal control problem for an ecosystem composed by three species was studied in [1]. Other optimal control problems in population dynamics can be found in [3, 8, 11, 14]. For general theory in the optimal control field the reader may refer to [2, 12, 13].

2. NECESSARY OPTIMALITY CONDITIONS

In this section we find necessary optimality conditions for problem (P).

Under hypothesis (H), h is obviously positive for all $y_1 > 0$. Since the solution $y = (y_1, y_2)$ of the Cauchy problem (1.1)–(1.2) is bounded, we can take a compact target set at $t = T$ and then apply Theorem 1.2, page 43, [2], to deduce the existence of an optimal solution (y_1, y_2, u, v) for the optimal control problem (P).

To find the form of the optimal control (u, v), we apply Pontryagin's maximum principle. The Hamiltonian function is

$$H(y,p,u,v) = y_1p_1 - cy_2p_2 - vy_1^2p_1 + uy_2h(ay_1)(bp_2 - p_1) + (k_1y_1 + k_2y_2), \tag{2.1}$$

where $p = (p_1, p_2)$ is the solution of the adjoint system

$$\begin{cases} p_1' = -\dfrac{\partial H}{\partial y_1} = -p_1 - k_1 + 2vy_1p_1 + au\ y_2h'(ay_1)(p_1 - bp_2) \\ p_2' = -\dfrac{\partial H}{\partial y_2} = cp_2 - k_2 + uh(ay_1)(p_1 - bp_2),\ t \in [0,T], \end{cases} \tag{2.2}$$

subjected to the transversality conditions

$$p_1(T) = \alpha, \quad p_2(T) = \beta. \tag{2.3}$$

If y_1, y_2, p_1, p_2 are considered fixed, since $h(ay_1) > 0$, the maximum of H is reached when

$$u(t) = \begin{cases} 0,\ bp_2 - p_1 < 0 \\ 1,\ bp_2 - p_1 > 0 \end{cases}, \quad v(t) = \begin{cases} v_0,\ p_1 > 0 \\ 1,\ p_1 < 0 \end{cases}, \tag{2.4}$$

for almost all $t \in [0,T]$. This implies that

$$u(t)(p_1 - bp_2)(t) \le 0 \quad \text{a.e. } t \in [0,T]. \tag{2.5}$$

Observe that the solution of the linear boundary value problem $p' = f(t)\, p + g(t)$, $t \in [0,T]$, $p(T) = p_T$ is given by

$$p(t) = e^{-\int_t^T f(s)ds} \left\{ p_T - \int_t^T g(s)\, e^{\int_s^T f(\theta)d\theta} ds \right\}, \quad t \in [0,T]. \tag{2.6}$$

Regarding the first equation from (2.2)–(2.3) as a linear equation in p_1 with

$$f(t) = -1 + 2vy_1, \quad g(t) = [au\ y_2 h'(ay_1)(p_1 - bp_2) - k_1](t),$$

one gets

$$p_1(t) = e^{\int_t^T (1-2vy_1)ds} \left\{ \alpha - \int_t^T [au\ y_2 h'(ay_1)(p_1 - bp_2) - k_1](s)\, e^{-\int_s^T (1-2vy_1)d\theta} ds \right\}, \tag{2.7}$$

and similarly

$$p_2(t) = e^{-c(T-t)} \left\{ \beta - \int_t^T [u(s)\, h(ay_1(s))(p_1 - bp_2)(s) - k_2]\, e^{c(T-s)} ds \right\}, \ t \in [0,T]. \tag{2.8}$$

Then we can state the following auxiliary result.

Lemma 2.1. *Under hypothesis* (H)*, the second component of the control function* (u,v) *is* $v(t) = v_0$ *for all* $t \in [0,T]$.

Proof. By (2.7)–(2.8) one obtains that $p_1(t) \geq 0$ and $p_2(t) \geq 0$ on $[0,T]$. The limit case $p_1(t) = 0$ holds for some point $t \in [0,T)$ if and only if $\alpha = 0$, $k_1 = 0$, and $u = 0$ a.e. on $[t,T]$. Similarly $p_2(t) = 0$ if and only if $\beta = 0$, $k_2 = 0$, $u = 0$ a.e. on $[t,T]$. Since α, β, k_1, k_2 are not all zero, we deduce that $p_1(t)$, $p_2(t)$ cannot be zero at the same time.

We show that $p_1(t) > 0$. Indeed, supposing that $p_1(t) = 0$, one derives that $p_2(t) > 0$. Since $u = 0$ a.e. on $[t,T]$, this together with (2.4), gives a contradiction in the sign of $bp_2 - p_1$. Therefore $v(t) = v_0$ for all $t \in [0,T]$. The lemma is proved. □

3. THE NUMBER OF SWITCHINGS FOR u

This section is devoted to the number of switching points of the control u.

The form of u can be found in each of the following cases: $b\beta < \alpha$, $\beta b = \alpha$, $b\beta > \alpha$. From the adjoint system (2.2) with $v = v_0$ we get

$$(bp_2 - p_1)' = bcp_2 + p_1 + k_1 - bk_2 - 2v_0 y_1 p_1 - u(bp_2 - p_1)[bh(ay_1) - ay_2 h'(ay_1)], \tag{3.1}$$

for all $t \in [0,T]$. To establish the exact form of the component u of the optimal control, we impose the following additional hypotheses:

$$c + 1 - 2v_0 y_1^{\max} > 0, \quad k_1 - bk_2 \geq 0, \tag{3.2}$$

where $y_1^{\max} = \sup\{y_1(t),\ t \in [0,T]\}$.

The following simple lemma is the crucial point in the proof of our main result.

Lemma 3.1. *Assume that* α, β, k_1, k_2 *are non-negative constants, not all zero. Let* $\Phi(t) = bp_2(t) - p_1(t)$ *be the switching function for the optimal control* u. *If conditions* (3.2) *hold, then whenever* $\Phi(\tau) = 0$ *it follows that* $\Phi'(\tau) > 0$.

Proof. If $\Phi(\tau) = 0$, by (3.1) we derive that

$$\Phi'(\tau) = p_1(\tau)(c + 1 - 2v_0 y_1(\tau)) + k_1 - bk_2.$$

Inequality $p_1(\tau) > 0$ together with (3.2) leads to $\Phi'(\tau) > 0$, as claimed. □

Now we are able to find the number of switchings for u.

Theorem 3.2. *Suppose that condition* (3.2) *holds. If* (u, v) *is the optimal control of the problem* (P), *then* $v = v_0$ *on* $[0, T]$ *and for* u *we have two possibilities:*

I) *If* $b\beta \le \alpha$, *then* $u(t) = 0$, $(\forall)\, t \in [0, T]$. *The corresponding optimal state is the solution of* (1.1)–(1.2) *with* $u = 0$ *and* $v = v_0$.

II) *If* $b\beta > \alpha$, *then* u *has at most one switching time. If actually there exists a switching time* $\tau \in (0, T)$, *then* u *is bang-bang and has the form*

$$u(t) = \begin{cases} 0, & t \in [0, \tau] \\ 1, & t \in (\tau, T]. \end{cases}$$

Otherwise $u(t) = 1$, $(\forall)\, t \in [0, T]$.

Proof. Case 1. Let first $b\beta < \alpha$. In view of (2.3), we have $\Phi(T) = b\beta - \alpha < 0$. It follows that $\Phi = bp_2 - p_1 < 0$ at least in a left neighbourhood of T. Let $(\tau, T]$ be the maximal interval with this property. Then the optimal control is $u(t) = 0$ on $(\tau, T]$. We show that $\tau = 0$. Indeed, if this is not the case, then $\tau > 0$ and $\Phi(\tau) = (bp_2 - p_1)(\tau) = 0$. By Lemma 3.1, we obtain that $\Phi'(\tau) > 0$. This means that function Φ cannot change its sign at the left side of τ. Consequently, $\tau = 0$ and $u = 0$ on the whole interval $[0, T]$.

Case 2. Assume that $b\beta = \alpha$. Then $\Phi(T) = (bp_2 - p_1)(T) = 0$. In view of Lemma 3.1, it follows that $\Phi'(T) > 0$. This implies that Φ is increasing in a left neighbourhood of T and $\Phi(t) < 0$ for $t < T$, t close to T. As in the first case, we find that $u = 0$ on $[0, T]$.

Case 3. Let $b\beta > \alpha$. Then $\Phi = bp_2 - p_1 > 0$ on a left neighbourhood of T. We put

$$\tau = \inf\{s \in [0, T],\ (bp_2 - p_1)(t) > 0,\ (\forall)\, t \in (s, T]\}.$$

If $\tau = 0$, then $u = 1$ on $[0, T]$.

If $\tau \in (0, T)$, then $\Phi(\tau) = 0$, $\Phi > 0$ on $(\tau, T]$ and then $u = 1$ on $(\tau, T]$. Making use of Lemma 3.1, one derives that τ is a switching for u. Thus $\Phi < 0$ in a left neighbourhood (τ_1, τ) of τ, which can be chosen maximal. The control u is 0 on this interval.

If $\tau_1 > 0$, then $\Phi(\tau_1) = 0$. Applying again Lemma 3.1, we get $\Phi'(\tau_1) > 0$. This contradicts the fact that $\Phi(t) < 0$ on (τ_1, τ) and $\Phi(\tau_1) = 0$. The conclusion is that $\tau_1 = 0$, i.e. $u(t) = 0$ on $[0, \tau]$. The theorem is proved. □

Remark 3.3. Observe that, in order to minimize the cost functional (1.3), the prey individuals should have the smallest degree of mixture (their rate of separation is $1 - v_0$). Concerning u, we have two situations. If $b\beta \leq \alpha$, then the prey and predators are completely separated on the whole time interval. If $b\beta > \alpha$, then prey and predators should be either not separated at all ($u = 1$), or completely separated on a time interval $[0, \tau]$ (here $u = 0$) and next completely mixed on $(\tau, T]$ (where $u = 1$).

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