Dynamic Systems and Applications 19 (2010) 545-556

BOUNDED SOLUTIONS FOR A DERIVATIVE DEPENDENT BOUNDARY VALUE PROBLEM ON THE HALF-LINE

SMAÏL DJEBALI AND SAMIRA ZAHAR

Lab. EDP&HM, E.N.S., P.B. 92, 16050 Kouba, Algiers. Algeria
Department of Mathematics, University A.E. Mira, 06000. Bejaia, Algeria

ABSTRACT. This paper is devoted to the existence of bounded solutions to a nonlinear second-order boundary value problem on the positive half-line where the nonlinearity depends on the first derivative. We employ topological degree theory combined with the method of upper and lower solutions on compact domains to prove existence of solution on truncated domains. Solutions are then extended to unbounded domains using sequential arguments. A uniqueness result is also obtained and two illustrative examples end the paper.

AMS (MOS) Subject Classification. 34B15, 34B40

1. INTRODUCTION

In this note, we are concerned with the existence of solutions to the following boundary value problem

$$(1.1)\qquad \begin{cases} -x''(t)+a(t)x(t) &= f(t,x(t),x'(t)), \quad t>0,\\ x(0) &= x_0, \quad x \text{ bounded on } [0,\infty), \end{cases}$$

where x_0 is a given real number, the nonlinearity $f : I\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous, and the function $a : I \to (0,\infty)$ is continuous and satisfies

$$(H_0)\quad \exists\, a_0\in I,\ a(t)\geq a_0, \quad \forall\, t\geq 0.$$

Here I refers to the positive half-line. Boundary value problems (BVPs for short) on unbounded intervals arise in many applications in physics, combustion theory, biology,… (see e.g. the monographs by Agarwal and O'Regan [1, 2] and the references therein). During the last couple of years, BVPs on unbounded intervals have been intensively studied. The particular case of ordinary differential equations with constant coefficients has recently attracted the attention of many researchers. Indeed, in this case, the Green's function associated with the corresponding BVP can be computed explicitly and its main properties derived. As a consequence, the BVP may be transformed into an integral equation of Hammerstein type and thus arguments

Received May 17, 2009

from fixed point theory in Banach spaces can be applied. For instance, various BVPs associated with the generalized Fisher-like equations

$$-x''(t) + cx'(t) + \lambda x(t) = h(t, x(t)) \text{ and } -x''(t) + k^2 x(t) = h(t, x(t))$$

on the half and the whole real line are discussed in [7, 8, 9]. Notice that the main difficulty in dealing with BVPs on unbounded intervals is the lack of compactness for such problems. To overcome this difficulty, very recent papers make use of some compactness criteria on unbounded domains such that Corduneanu [6], Fréchet-Kolmogorov [17] and Zima [20] criteria, extending the classical Ascoli-Arzéla theorem. In addition, the method of upper and lower solutions turns out to be a powerful tool in dealing with such BVPs. In [3], this method was used to study a class of second-order BVP on infinite interval; see also [4, 16]. In [10, 16, 19], the authors appeal to this method to discuss some singular BVPs on the half-line. In these works, upper and lower solutions are first sought on bounded intervals; then sequential arguments are developed to extend the obtained solutions on unbounded domains. Recall that on bounded intervals of the real line, the method of upper and lower solutions is well developed in the recent book by De Coster and Habets [11].

However, when the ordinary differential equation has time varying coefficients, deep investigation is needed to find a Green's function and then study its fundamental properties. In case of Problem (1.1), the Green's function together with its main properties are investigated in [12, 13, 14]. This equation has the particularity that it is of limit-point case (see [5] for more details on the limit-point and the limit-circle cases in ordinary differential equations). This means that the corresponding homogeneous problem has only the trivial solution, which in turn implies existence and uniqueness of the Green's function. When $f = f(t, x)$ does not depend on the first derivative, Problem (1.1) is studied in [15] and existence of multiple solutions is obtained.

Our aim, in this work, is to consider the more general case of a derivative depending nonlinearity $f = f(t, x, x')$. For this purpose, further to the assumptions in [15], the additional assumption (H_3) both with a Nagumo-Bernstein type assumption (H_2) will be considered; this extends by the way some known results for BVPs on bounded domains for which a rich bibliography is available (see e.g. [11] and the references therein). In Section 2, we will first discuss Problem (1.1) on a bounded interval $(0, b)$ providing estimates for possible solutions and their first derivatives. Such a priori estimates are exploited in Section 3 to prove existence of solution on $(0, b)$. To this end, we shall appeal to the topological degree of Leray and Schauder (se e.g. [18] for the main properties). Sequential arguments are then used to extend the obtained solution to the half-line and a uniqueness result is provided under an additional hypothesis (H_4). We end this paper with two illustrative examples in Section 4. Throughout this paper, $C^k(I\times\mathbb{R}^2,\mathbb{R})$ will refer to the space of functions whose $k-th$ derivatives are continuous on $I\times\mathbb{R}^2$ and $CB^k(I,\mathbb{R})$ denotes the space of functions whose derivatives, up to the order k, are bounded and continuous on I. For each $x\in CB(I,\mathbb{R})$, denote by $\|x\|=\sup\limits_{t\in I}|x(t)|$.

2. THE PROBLEM ON A TRUNCATED DOMAIN

2.1. **Preliminaries.** Let $b>b_0$ for some fixed $b_0>0$ and consider the problem on a bounded domain

$$\left\{\begin{array}{rcl} -x''(t)+a(t)x(t) &=& f(t,x(t),x'(t)), \quad 0<t<b,\\ x(0) &=& x_0, \quad x'(b)=0. \end{array}\right. \tag{2.1}$$

Definition 2.1. (a) We say that α_b is a $\mathcal{C}^0$-lower solution of Problem (2.1) if $\alpha_b\in\mathcal{C}^0([0,b])$, $\alpha_b'(b)$ exists, and for each $t\in(0,b)$, there exists an open interval $I_t\subset(0,b)$ with $t\in I_t$ and a function $\alpha_t\in\mathcal{C}^2(I_t)$ such that

$$\left\{\begin{array}{rcl} \alpha_t(t) &=& \alpha_b(t),\\ \alpha_t(s) &\le& \alpha_b(s), \quad s\in I_t,\\ -\alpha_t''(s)+a(s)\alpha_t(s) &\le& f(s,\alpha_t(s),\alpha_t'(s)), \quad s\in I_t,\\ \alpha_b(0) &\le& x_0, \quad \alpha_b'(b)<0. \end{array}\right.$$

(b) A function β_b is a $\mathcal{C}^0$-upper solution of Problem (2.1) if $\beta_b\in\mathcal{C}^0([0,b])$, $\beta_b'(b)$ exists, and for each $t\in(0,b)$, there exists an open interval $I_t\subset(0,b)$ with $t\in I_t$ and a function $\beta_t\in\mathcal{C}^2(I_t)$ such that

$$\left\{\begin{array}{rcl} \beta_t(t) &=& \beta_b(t),\\ \beta_t(s) &\ge& \beta_b(s), \quad s\in I_t,\\ -\beta_t''(s)+a(s)\beta_t(s) &\ge& f(s,\beta_t(s),\beta_t'(s)), \quad s\in I_t,\\ \beta_b(0) &\ge& x_0, \quad \beta_b'(b)>0. \end{array}\right.$$

The following auxiliary lemmas will be crucial in the sequel.

Lemma 2.2 ([14], Theorem 2.3)**.** *Let $a\in\mathcal{C}(I)$ satisfy (H_0). Then there exists a unique Green's function $G=G(t,s)$ such that $u_0(t)=\int_0^\infty G(t,s)ds$ is the unique solution of the problem*

$$\left\{\begin{array}{rcl} x''(t)-a(t)x(t) &=& 1, \quad t>0,\\ x(0) &=& 0, \quad x(t) \textit{ bounded on } (0,\infty). \end{array}\right.$$

Moreover G satisfies the integrability property:

$$\int_0^\infty |G(t,s)|ds<\frac{1}{a_0}, \quad \forall\, t\ge 0.$$

Lemma 2.3 ([14], Lemma 3.2)**.** *Let $a \in \mathcal{C}(I)$ satisfy (H_0). Then, for any real number x_0, the problem*

$$\begin{cases} x''(t) - a(t)x(t) &= 0, \quad t > 0, \\ x(0) &= x_0, \quad x(t) \text{ bounded on } (0,\infty) \end{cases}$$

has a unique solution p_0 which satisfies

$$|p_0(t)| \leq |x_0|, \quad \forall\, t \geq 0.$$

It follows that for any bounded and continuous function h on I, the problem

$$\begin{cases} x''(t) - a(t)x(t) &= h, \quad t > 0, \\ x(0) &= x_0, \quad x(t) \text{ bounded on } (0,\infty) \end{cases}$$

has the unique solution u with the representation

$$u(t) = p_0(t) + \int_0^\infty G(t,s)h(s)ds, \quad t > 0.$$

Finally consider $G(b;t,s)$ the Green's function associated with the right focal problem

$$\begin{cases} x''(t) - a(t)x(t) &= 0, \quad 0 < t < b, \\ x(0) &= 0, \quad x'(b) = 0, \end{cases}$$

and $p_b(t)$ is the unique solution of the problem

$$\begin{cases} x''(t) - a(t)x(t) &= 0, \quad 0 < t < b, \\ x(0) &= x_0, \quad x'(b) = 0. \end{cases}$$

2.2. **General assumptions and modified problem.** We first enunciate some assumptions:

(H_1) *There exist $\alpha, \beta \in CB^1(0,\infty)$, $(\alpha \leq \beta)$ and $b_0 > 0$ such that for each $b > b_0$ the functions $\alpha_b := \alpha_{|[0,b]}$ and $\beta_b := \beta_{|[0,b]}$ are $\mathcal{C}^0$-lower and upper solutions of Problem* (2.1) *respectively and*

$$f(t,\alpha(t),\alpha'(t)) \leq 0 \leq f(t,\beta(t),\beta'(t)), \quad t \in (0,b). \tag{2.2}$$

(H_2) *There exist $c \geq 0$, $q : (0,\infty) \to I$ integrable and $\psi : I \to [1,\infty)$ continuous, with $\frac{1}{\psi}$ integrable over bounded intervals and $\int_0^\infty \frac{ds}{\psi(s)} = +\infty$ such that*

$$|f(t,x,y)| \leq \psi(|y|)(q(t) + c|y|),\ \forall\,(t,x,y) \in D_\alpha^\beta \times \mathbb{R}, \tag{2.3}$$

where D_α^β is defined by

$$D_\alpha^\beta := \{(t,x) \in (0,\infty) \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}.$$

(H_3)

$$A := \int_0^\infty a(t)\max(|\alpha(t)|, |\beta(t)|)\,dt < \infty.$$

Let $Q := \int_0^\infty q(t)dt$ and $K_0 := \max\{\|\alpha\|, \|\beta\|\}$. (H_2) implies the existence of a real number K_1 such that $K_1 > \max\{\|\alpha'\|, \|\beta'\|\}$ and

$$\int_0^{K_1} \frac{ds}{\psi(s)} > Q + 2cK_0 + A. \tag{2.4}$$

For $t \in [0,b]$, define the truncation function $\widetilde{f}$ by

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,\beta(t),T_{K_1}(y)), & \beta(t) < x, \\ f(t,x,T_{K_1}(y)), & \alpha(t) \le x \le \beta(t), \\ f(t,\alpha(t),T_{K_1}(y)), & x < \alpha(t), \end{cases}$$

where

$$T_K(y) = \begin{cases} -K, & y < -K, \\ y, & -K \le y \le K, \\ K, & K < y, \end{cases}$$

is the truncation function at level K. Then consider the family of problems

$$\begin{cases} -x''(t) + a(t)x(t) = \lambda\widetilde{f}(t,x(t),x'(t)), & t \in (0,b), \\ x(0) = x_0, \quad x'(b) = 0, \end{cases} \tag{2.5}$$

where the parameter λ lies in the interval $[0,1]$.

2.3. **A priori bounds on solutions.** We prove two results giving estimates on solutions of Problem (2.5) and on their first derivatives respectively.

Proposition 2.4. *Under Assumption* (H_1)*, all possible solutions of Problem* (2.5) *satisfy the estimates*

$$\alpha_b(t) \le x(t) \le \beta_b(t), \quad \forall\, t \in [0,b].$$

Proof. Suppose, on the contrary that there is some $t_0 \in [0,b]$ such that $x_b(t_0) - \alpha_b(t_0) = \min_t(x - \alpha_b)(t) < 0$. We have:

(a) $t_0 \neq 0$ since $(x-\alpha_b)(0) = x_0 - \alpha_b(0) \ge 0$.
(b) $t_0 \neq b$ since $(x-\alpha_b)'(b) = -\alpha_b'(b) > 0$ and if $(x-\alpha_b)$ achieves its minimum at $t_0 = b$, then $(x-\alpha_b)'(b) \le 0$, which is a contradiction.
(c) So $t_0 \in (0,b)$. By definition of a $\mathcal{C}^0-$ lower-solution, there exists an open interval I_{t_0} with $t_0 \in I_{t_0} \subset (0,b)$ and a function $\alpha_{t_0} \in \mathcal{C}^2(I_{t_0})$ such that $\alpha_{t_0}(t_0) = \alpha_b(t_0)$, $\alpha_{t_0}(s) \le \alpha_b(s)$ and $\alpha_{t_0}''(s) - a(s)\alpha_{t_0}(s) + f(s,\alpha_{t_0}(s),\alpha_{t_0}'(s)) \ge 0$, on I_{t_0}. As a consequence, we have the estimates:

$$\begin{aligned}(x'' - \alpha_{t_0}'')(t_0) &= a(t_0)x(t_0) - \lambda\widetilde{f}(t_0,x(t_0),x'(t_0)) - \alpha_{t_0}''(t_0) \\ &\le a(t_0)x(t_0) - \lambda\widetilde{f}(t_0,x(t_0),x'(t_0)) - a(t_0)\alpha_{t_0}(t_0) + f(t_0,\alpha_{t_0}(t_0),\alpha_{t_0}'(t_0)) \\ &= a(t_0)(x-\alpha_b)(t_0) - \lambda\widetilde{f}(t_0,x(t_0),x'(t_0)) + f(t_0,\alpha_b(t_0),\alpha_b'(t_0)) \\ &= a(t_0)(x-\alpha_b)(t_0) + (1-\lambda)f(t_0,\alpha_b(t_0),\alpha_b'(t_0))] < 0,\end{aligned}$$

where the last inequality follows from the first inequality in (2.2). Since the function $x-\alpha_{t_0}$ achieves its minimum at t_0, we deduce that

$$(x''-\alpha''_{t_0})(t_0)\geq 0,$$

leading to a contradiction. Similarly, we can prove that $x(t)\leq\beta_b(t)$, for all $t\in[0,b]$.

□

Proposition 2.5. *Under Assumptions* (H_1)–(H_3)*, all possible solutions of Problem* (2.5) *satisfy the estimate*

$$\|x'\|\leq K_1,$$

where K_1 *is as defined by* (2.4)*.*

Proof. Let x be a solution of Problem (2.5). Suppose, on the contrary, that there exists $\tau\in(0,b)$ such that $|x'(\tau)|\geq K_1$. Then, there exist t_0, t_1 $(t_0<t_1)$ such that either one of the following situations holds:

$$\begin{cases} x'(t_0)=0,\ x'(t_1)=K_1, & \text{and} \quad 0<x'(t)<K_1, & \text{for } t\in(t_0,t_1);\\ x'(t_0)=K_1,\ x'(t_1)=0, & \text{and} \quad 0<x'(t)<K_1, & \text{for } t\in(t_0,t_1);\\ x'(t_0)=0,\ x'(t_1)=-K_1, & \text{and} \quad -K_1<x'(t)<0, & \text{for } t\in(t_0,t_1);\\ x'(t_0)=-K_1,\ x'(t_1)=0, & \text{and} \quad -K_1<x'(t)<0, & \text{for } t\in(t_0,t_1).\end{cases}$$

For simplicity, we only study the first case. By Proposition 2.4, we have since $x'(t)>0$ on (t_0,t_1),

$$\begin{aligned}|x''(t)|-a(t)|x(t)| &\leq |x''(t)-a(t)x(t)|\\ &=\lambda|\widetilde{f}(t,x(t),x'(t))|\\ &=\lambda|f(t,x(t),x'(t))|\\ &\leq\psi(x'(t))(q(t)+cx'(t)).\end{aligned}$$

Since $\psi(s)\geq 1$, for all $s\in I$, we infer that

$$\begin{aligned}|x''(t)| &\leq \psi(x'(t))(q(t)+cx'(t))+a(t)|x(t)|\\ &\leq\psi(x'(t))(q(t)+cx'(t)+a(t)|x(t)|)\end{aligned}$$

and so

$$\frac{x''(t)}{\psi(x'(t))}\leq q(t)+cx'(t)+a(t)|x(t)|.$$

Integrating from t_0 to t_1 yields

$$\begin{aligned}\int_0^{K_1}\frac{ds}{\psi(s)} &= \int_{t_0}^{t_1}\frac{x''(t)}{\psi(x'(t))}dt\\ &\leq\int_{t_0}^{t_1}[q(t)+cx'(t)+a(t)\max\{|\alpha(t)|,|\beta(t)|\}]dt\leq Q+2cK_0+A,\end{aligned}$$

which is a contradiction to the definition of K_1 in (2.4). □

3. THE PROBLEM ON THE HALF LINE

3.1. **Existence result.** Our main existence result in this paper is

Theorem 3.1. *Let Assumptions* (H_0)–(H_3) *hold. Then Problem* (1.1) *has at least one solution* x *such that, for each* $b > 0$,

$$\alpha_b(t) \leq x(t) \leq \beta_b(t), \quad \forall\, t \in [0, b].$$

Moreover, if f *is bounded, then* x *has the representation:*

$$x(t) = p_0(t) + \int_0^\infty G(t,s) f(s, x(s), x'(s)) ds.$$

Proof. Let $b > 0$. We first use the Leray-Schauder topological degree to prove existence of solution on the bounded interval $[0, b)$. Then a diagonalization process is employed to ensure the solution can be extended to $[0, \infty)$.

Step 1. *Problem* (2.1) *has at least one solution in* $\mathcal{C}^1[0, b]$.

Define the linear operator $L : D(L) \longrightarrow \mathcal{C}^0[0, b]$ by $Lx(t) = x''(t) - a(t)x(t)$ with $D(L) = \{x \in \mathcal{C}^2[0, b] : x(0) = x_0, x'(b) = 0\}$ and the Nemytskii operator $N : \mathcal{C}^1[0, b] \longrightarrow \mathcal{C}^0[0, b]$ by $Nx(t) = \widetilde{f}(t, x(t), x'(t))$. Notice that solving Problem (2.5) is equivalent to proving existence of a fixed point for the abstract nonlinear operator $H_\lambda := \lambda L^{-1}N$, for $\lambda = 1$, where the map

$$H_\lambda : \mathcal{C}^1[0, b] \longrightarrow \mathcal{C}^1[0, b]$$

is defined by

$$(H_\lambda x)(t) = \lambda p_b(t) + \lambda \int_0^b G(b; t, s) \widetilde{f}(s, x(s), x'(s)) ds. \tag{3.1}$$

Set

$$\Omega = \{x \in \mathcal{C}^1[0, b] : \|x\|_1 < K\}$$

with $K := K_0 + K_1 + 1$ and

$$\|x\|_1 = \max\left(\sup_{0 \leq t \leq b} |x(t)|, \sup_{0 \leq t \leq b} |x'(t)|\right).$$

It is clear that H_λ is compact (see e.g., [6, 17]). By Propositions 2.4 and 2.5 and the definition of Ω, H_λ has no fixed point on $\partial\Omega$ for all $\lambda \in [0, 1]$. Since $H_0 = 0$, it follows that $1 = \deg(I - H_0, \Omega, 0) = \deg(I - H_1, \Omega, 0)$. As a consequence, $H_1 := L^{-1}N$ has a fixed point in Ω, i.e. Problem (2.5) has at least one solution $x_b \in \mathcal{C}^1[0, b]$ for $\lambda = 1$. By Propositions 2.4 and 2.5 and the definition of $\widetilde{f}$, we obtain that $f(t, x_b(t), x_b'(t)) = \widetilde{f}(t, x_b(t), x_b'(t))$ and then x_b is a solution of Problem (2.1).

Step 2. *Problem* (1.1) *has at least one solution in* $\mathcal{C}^1[0, \infty)$.

We will use a diagonalization argument. Let x_b be a solution of Problem (2.1). Define

$$u_b(t) = \begin{cases} x_b(t), & 0 \le t \le b, \\ x_b(b), & t > b. \end{cases}$$

Notice first that K is independent of b, so the family $\{u_b, \text{ for } b \in (0,\infty)\}$ is uniformly bounded in $\mathcal{C}^1[0,\infty)$. In addition, for any $t_0, t_1 \in (0,\infty)$ $(t_0 < t_1)$, we have

$$\begin{aligned} u_b'(t_0) - u_b'(t_1) &= \int_{t_0}^{t_1} u_b''(s)ds \\ &\le \int_{t_0}^{t_1} [a(s)u_b(s) + \psi(|u_b'(s)|)(q(t) + c|u_b'(s)|)]ds \\ &\le \int_{t_0}^{t_1} a(s)\max\{|\alpha(s)|, |\beta(s)|\}ds + K_2 \int_{t_0}^{t_1} q(s)ds \\ &\quad + cK_1K_2(t_1 - t_0), \end{aligned} \tag{3.2}$$

where $K_2 := \sup\limits_{0 \le s \le K_1} \psi(s)$. Let $\{b_i\}$ be an increasing sequence of real numbers such that $\lim\limits_{i\to\infty} b_i = +\infty$. The family $\{u_{b_i}\}_{i\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^1[0,b_1]$, so it is relatively compact in $\mathcal{C}^0[0,b_1]$. Moreover, it is equicontinuous from (3.2), (H_3) and the fact that q is integrable. This implies that $\{u_{b_i}'\}$ is relatively compact in $\mathcal{C}^0[0,b_1]$. The Arzéla-Ascoli lemma guarantees the existence of a subsequence $\Delta_1 \subset \mathbb{N}^*$ and a function $\omega_1 \in \mathcal{C}^1[0,b_1]$ such that the sequence $\{u_{b_i}^j\}$, for $i \in \Delta_1$, converges uniformly to ω_1^j on $[0,b_1]$, for $j = 0,1$.

Consider now the family $\{u_{b_i}\}$, $i \in \Delta_1 \backslash \{1\}$ defined on the interval $[0,b_2]$. By the same argument, there exists a subsequence $\Delta_2 \subset \Delta_1\backslash\{1\}$ and a function $\omega_2 \in \mathcal{C}^1[0,b_2]$ such that the sequence $\{u_{b_i}^j\}$, $i \in \Delta_2$ converges uniformly to ω_2^j on $[0,b_2]$, for $j = 0,1$.

By induction, we obtain, for any integer k, the existence of $\Delta_k \subset \Delta_{k-1}\backslash\{k-1\}$ and a function $\omega_k \in \mathcal{C}^1[0,b_k]$ such that $\{u_{b_i}^j\}$, $i \in \Delta_k$ converges uniformly to ω_k^j on $[0,b_k]$, for $j = 0,1$. Note that $\omega_k = \omega_{k-1}$ on $[0,b_{k-1}]$. Finally, define the function

$$x(t) = \bigcup_{k\in\mathbb{N}} \omega_k(t).$$

Then x is a solution of Problem (1.1) since $x(0) = \omega_1(0) = x_0$, $\|x\| < K$. Moreover, for any fixed $t \in (0,\infty)$, we can choose $b_i > t$. Then

$$x''(t) = \omega_{b_i}''(t) = f(t, \omega_{b_i}(t), \omega_{b_i}'(t)) = f(t, x(t), x'(t)),$$

which completes the proof of the theorem. □

Remark 3.2. (a) By Lemmas 2.2 and 2.3 together with (3.1), any solution of Problem (2.5) satisfies the estimate

$$|x_b(t)| \le |p_b(t)| + N/a_0,$$

where $N = \max \widetilde{f}(t,x,y)$ for $t \in [0,b]$ and $x, y \in \mathbb{R}^2$.
(b) Assume that for each $M > 0$, there exists a function $f_M : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, bounded such that $|f(t,x,y)| \le f_M(t)$, for $t \ge 0$, $|x| + |y| \le M$. Then any solution of Problem (1.1) satisfies

$$|x(t)| \le |x_0| + F_M/a_0, \ \forall\, t \ge 0$$

where $F_M = \sup\limits_{t \ge 0} f_M(t)$.

3.2. **Uniqueness result.** The following result complements Theorem 3.1.

Theorem 3.3. *Assume that $f = f(t,x,y)$ is continuously differentiable in x, y for each $t \ge 0$. Assume that Assumptions (H_0)–(H_3) hold together with*

(H_4) *$f(t,x,y)$ is nonincreasing in x for each t and y fixed,*
and $f(t,x,y)$ nonincreasing in y for each t and x fixed.

Then Problem (1.1) has a unique solution x such that, for each $b \in (0,\infty)$,

$$\alpha_b(t) \le x(t) \le \beta_b(t), \quad \forall\, t \in [0,b].$$

Proof. Suppose that there exist two distinct solutions x_1, x_2 to Problem (1.1) and let $z = x_1 - x_2$. By the Mean Value Theorem, there exist θ, φ such that

$$f(t,x_2,x_2') = f(t,x_1,x_1') - z\frac{\partial f}{\partial y}(t,\theta,\varphi) - z'\frac{\partial f}{\partial z}(t,\theta,\varphi).$$

Assume that $z(t_1) > 0$ for some t_1 and that z has a positive maximum at some $t_0 < \infty$. Then

$$\begin{aligned} 0 \ge z''(t_0) &= a(t_0)z(t_0) + f(t_0,x_2(t_0),x_2'(t_0)) - f(t_0,x_1(t_0),x_1'(t_0)) \\ &= (a(t_0) - \frac{\partial f}{\partial y}(t_0,\theta,\varphi))z(t_0) - \frac{\partial f}{\partial z}(t_0,\theta,\varphi)z'(t_0) \\ &= (a(t_0) - \frac{\partial f}{\partial y}(t_0,\theta,\varphi))z(t_0) > 0, \end{aligned}$$

leading to a contradiction. Hence $\sup z(t) = \lim_{t\to\infty} z(t)$. Let

$$T = \sup\{t \ge 0, \ z(t) \le 0\} < \infty.$$

Then z satisfies

$$\begin{cases} -z''(t) + a(t)z(t) &= f(t,x_1(t),x_1'(t)) + f(t,x_2(t),x_2'(t)) \quad t > T, \\ z(T) &= 0, \quad z \text{ bounded on } (0,\infty). \end{cases}$$

We claim that $z'(t)$ is positive, increasing on $[T,+\infty)$. On the contrary, assume that there exists some $T_1 > T$ such that $z''(T_1) = 0$. Let $T_1 = \inf\{t \ge 0, \ z''(t) = 0\}$; then

$z'(t) > 0$ on $[T, T_1]$. Since $z(T) = 0$, it follows that $z(t) > 0$ on $[T, T_1]$. Thus, for $t = T_1$, we have that

$$0 = z''(T_1) = \left(a(T_1) - \frac{\partial f}{\partial y}(T_1, \theta, \varphi)\right) z(T_1) - \frac{\partial f}{\partial z}(T_1, \theta, \varphi) z'(T_1) > 0,$$

leading to a contradiction. Hence for $t \in (T, +\infty)$

$$z(t) \geq z(T) + z'(T)(t - T).$$

This is a contradiction to the boundedness of z, which completes the proof of the theorem. $\square$

4. **EXAMPLES**

4.1. **Example 1.** Consider the problem

$$\left\{ \begin{array}{rcl} -x''(t) + a(t)x(t) & = & x(t) - x'(t), \quad t > 0, \\ x(0) & = & 1, \quad x \text{ bounded on } [0, \infty), \end{array} \right. \tag{4.1}$$

where

$$a(t) = \left\{ \begin{array}{rl} t + 1, & t \geq 2 \\ 3, & 0 \leq t \leq 2. \end{array} \right.$$

Then $a(t) \geq a_0 = 3$ and the functions $\alpha(t) = -e^{-t}$, $\beta(t) = e^{-t}$ are lower and upper solutions respectively. Then Assumptions (H_0)–(H_3) are satisfied. As a consequence, Problem (4.1) has at least one solution x such that

$$-e^{-t} \leqslant x(t) \leqslant e^{-t}, \quad \forall\, t \geq 0.$$

In particular, we know the limit $\lim\limits_{t \to +\infty} x(t) = 0$.

4.2. **Example 2.** Consider the problem

$$\left\{ \begin{array}{rcl} -x''(t) + a(t)x(t) & = & q(t)(-x')^{\sigma}(t), \quad t > 0, \\ x(0) & = & x_0, \quad x \text{ bounded on } [0, \infty), \end{array} \right. \tag{4.2}$$

where $0 < \sigma = \frac{1}{2p+1} < 1$, $a(t) := k_1 \frac{(t+\frac{1}{2})^n}{(1+t)^n}$, and

$$q(t) := -\frac{m(m+1)k_2}{(mk_2)^{\sigma}}(1+t)^{-m-2+m\sigma+\sigma} + \frac{k_1 k_2}{(mk_2)^{\sigma}} \left(t + \frac{1}{2}\right)^n (1+t)^{-m-n+m\sigma+\sigma}$$

for some positive constants k_1, k_2 and some positive integers m, n, p satisfying

$$\left\{ \begin{array}{c} m > \frac{1+\sigma}{1-\sigma} \\ k_1 > m(m+1)2^n \\ k_2 > x_0 > 0. \end{array} \right.$$

Then it is clear that $f(t, x, y) = q(t)(-y)^{\sigma}$ satisfies a Nagumo condition in the argument y since

$$f(t, x, y) \leq q(t)(1 + |y|)^{\sigma}.$$

We take $c = 0$ and $\psi(s) = (1+s)^\sigma$ noting that q is integrable on $(0, +\infty)$ since $m > \frac{1+\sigma}{1-\sigma}$. Moreover $\beta(t) = \frac{k_2}{(1+t)^m}$ is a positive upper-solution while $\alpha(t) = 0$ is a lower-solution, hence $\alpha \leq \beta$. Finally, using the fact that $\left(t+\frac{1}{2}\right)^n(1+t)^{-n} \geq 2^{-n}$, we obtain that

$$f(t, \alpha(t), \alpha'(t)) \leq 0 \leq f(t, \beta(t), \beta'(t)), \quad \forall\, t > 0$$

as well as

$$A := k_1 k_2 \int \frac{(t+\frac{1}{2})^n}{(1+t)^{n+m}}\, dt < \infty.$$

Therefore all Assumptions $(H_0) - (H_3)$ are met and Problem (4.2) has at least one solution x such that

$$0 \leqslant x(t) \leqslant \frac{k_2}{(1+t)^m}, \quad \forall\, t \geq 0.$$

Acknowledgment. The authors are indebted to the referee for his careful reading of the first version of the manuscript.

REFERENCES

[1] R.P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations,* Kluwer Academic Publisher, Dordrecht, 2001.

[2] R.P. Agarwal and D. O'Regan, *Singular Differential and Integral Equations with Applications,* Kluwer Academic Publisher, Dordrecht, 2003.

[3] J.W. Bebernes and L.K. Jacksonn, *Infinite interval boundary value problems for $y'' = f(x, y)$*, Duke Math. J., **34** (1967), 39–47.

[4] A. Boucherif and J. Henderson, *Positive solutions of second order boundary value problem with changing signs Carathéodory nonlinearities*, Elec. J. Qual. Th.. Dif. Eq., (**6**) (2006), 1–14.

[5] E. Coddington and N. Levinson, *The Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[6] C. Corduneanu, *Integral Equations and Stability of Feedback Systems,* Academic Press, New York, 1973.

[7] S. Djebali, K. Mebarki, *Multiple positive solutions for singular BVPs on the positive half-line,* Comp. Math. Appl., **55(12)** (2008) 2940–2952

[8] S. Djebali and T. Moussaoui, *A class of second order BVPs on infinite intervals,* Elec. J. Qual. Th. Dif. Eq., **4** (2006) 1–19.

[9] S. Djebali, O. Saifi, *Positive solutions for singular BVPs on the positive half-line with sign changing and derivative depending nonlinearity,* Acta Applicandae Mathematicae, **110(2)** (2010) 639–665.

[10] S. Djebali, O. Saifi and B. Yan, *Positive solutions for singular BVPs on the positive half-line arising from epidemiology and combustion theory,* submitted

[11] C. De Coster and P. Habets, *Two Point Boundary Value Problems: Upper and Lower solutions*, Mathematics in Sc. and Eng., **205**, Elsevier, 2006.

[12] P.W. Eloe, *The quasilinearization method on an unbounded domain,* Proc. Amer. Math. Soc., **131(5)** (2002), 1481–1488.

[13] P.W. Eloe and E.R. Kaufmann, *A singular boundary value problem for a right disfocal operator*, Dynamical Systems Appl. **5** (1996), 174–182.

[14] P.W. Eloe, L.J. Grimm and J.D. Mashburn, *A boundary value problem on an unbounded domain*, Differential Equations and Dynamical Systems, **8(2)**, (2000), 125–140.

[15] P.W. Eloe, E.R. Kaufmann and C.C. Tisdell, *Multiple solutions of a boundary value problem on an unbounded domain*, Dynamical Systems Appl. **15(1)** (2006), 53–63.

[16] A. Granas, R.B. Guenther, J.W. Lee and D. O'Regan, *Boundary value problems on infinite intervals and semiconductor devices*, J. Math. Anal. Appl. **116** (1986), 335–348.

[17] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, The Netherlands, 1964.

[18] N.G. Lloyd, *Degree Theory*, Cambridge Tracts in Mathematics **73**, 1978.

[19] B. Yan, D. O'Regan and R.P. Agarwal, *Positive solutions to singular boundary value problems with sign changing nonlinearities on the half-Line via upper and lower solutions, Acta Math. Sinica. English Series,* **23(8)** (2007) 1447–1456.

[20] M. Zima, *On Positive Solutions of Boundary Value Problems on the Half-Line,* J. Math. Anal. Appli. 259 (2001), 127-136.