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ON POSITIVE SOLUTIONS OF A NONLINEAR FOURTH ORDER BOUNDARY VALUE PROBLEM VIA A FIXED POINT THEOREM IN ORDERED SETS

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ABSTRACT. This paper presents sufficient conditions for the existence and uniqueness of a positive solution to a nonlinear fourth-order differential equation under Lidstone boundary conditions. Our analysis relies on a fixed point theorem in partially ordered sets.

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1. INTRODUCTION AND PRELIMINARIES

The fourth-order differential equation

$$u^{(4)}(t) = \lambda \, f\left(t, u, u', u'', u'''\right), \quad t \in (0,1), \tag{1.1}$$

with Lidstone boundary conditions

$$u(0) = u(1) = 0 = u''(0) = u''(1), \tag{1.2}$$

has received a lot of attention in the last decades since it models the stationary states of the deflection of an elastic beam with both ends hinged (see [20]).

Some of the main tools of nonlinear analysis have been applied in the literature devoted to the study of problem (1.1)–(1.2): lower and upper solutions [3, 7, 8, 17], monotone iterative technique [1, 9, 11], Krasnoselskii fixed point theorem [5], fixed point index [2, 15, 21], Leray-Schauder degree [10, 13] and bifurcation theory [16, 19].

In this paper we are interested in the positive solutions of problem

$$\begin{gathered} u^{(4)}(t) = f(t,u), \quad t \in (0,1) \\ u(0) = u(1) = 0 = u''(0) = u''(1). \end{gathered} \tag{1.3}$$

A particular case of this problem is

$$\begin{gathered} u^{(4)}(t) = \lambda \, h(t) f(u), \quad t \in (0,1), \; \lambda > 0 \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered}$$

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which has been recently studied in [6] by using a slight variant of a fixed point theorem proved in [4].

In [6] the uniqueness of the solution is not treated.

The main tool in our work is a fixed point theorem in partially ordered sets which appears in [12].

In what follows, we present some results about the fixed point theorems which we will use later. These results appear in [12].

Definition 1.1. An altering distance function is a function $\psi\colon [0,\infty) \to [0,\infty)$ which satisfies:

(a) ψ is continuous and nondecreasing.
(b) $\psi(t) = 0$ is and only if $t = 0$.

Theorem 1.2 (Theorem 2.2 of [12])**.** *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies the following condition*

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x \text{ then } x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{1.4}$$

Let $T : X \to X$ be a nondecreasing mapping such that

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \text{for } x \geq y,$$

where ψ and ϕ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ then T has a fixed point.

If we consider that $(X, \leq)$ satisfies the following condition:

$$\text{for } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{1.5}$$

then we have the following result.

Theorem 1.3 (Theorem 2.3 of [12])**.** *Adding condition* (1.5) *to the hypotheses of Theorem 1.2 we obtain uniqueness of the fixed point of T.*

In our considerations, we will work in the Banach space

$$\mathcal{C}[0,1] = \{x : [0,1] \to \mathbb{R},\ \text{continuous}\},$$

with the standard norm $\|x\| = \max_{0\leq t\leq 1} |x(t)|$.

Note that this space can be equipped with a partial order given by

$$x, y \in \mathcal{C}[0,1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0,1].$$

In [18] it is proved that $\big(\mathcal{C}[0,1],\leq\big)$ with the classic metric given by

$$d(x,y)=\max_{0\leq t\leq 1}\big|x(t)-y(t)\big|$$

satisfies condition (1.4) of Theorem 1.2. Moreover, for $x,y\in\mathcal{C}[0,1]$, as the function $\max\{x,y\}$ is continuous, $\big(\mathcal{C}[0,1],\leq\big)$ satisfies condition (1.5).

On the other hand, the boundary value problem (1.3) can be rewritten as the integral equation (see, for example, [14])

$$u(t)=\int_0^1 G(t,s)f\big(s,u(s)\big)\mathrm{d}s,$$

where $G(t,s)$ is the Green function given by

$$G(t,s)=\begin{cases}\frac{1}{6}s(1-t)(2t-s^2-t^2), & s\leq t\\ \frac{1}{6}t(1-s)(2s-t^2-s^2), & s>t.\end{cases} \tag{1.6}$$

Note that $G(t,s)$ is a continuous function on $[0,1]\times[0,1]$ and $G(t,s)=G(s,t)$. Moreover, $G(t,s)\geq 0$ for $t,s\in[0,1]$. In fact, for $s\leq t$ we have

$$\begin{aligned}G(t,s)=\frac{1}{6}s(1-t)(2t-s^2-t^2) &\geq \frac{1}{6}s(1-t)(2t-t^2-t^2)\\ &= \frac{1}{6}s(1-t)(2t-2t^2)\\ &= \frac{1}{6}s(1-t)2t(1-t)\geq 0.\end{aligned}$$

The case $t<s$ is a direct consequence from $G(t,s)=G(s,t)$.

On the other hand, $G(t,s)=G(1-t,1-s)$. In fact, for $s\leq t$ we have $1-t\leq 1-s$ and, consequently,

$$\begin{aligned}G(1-t,1-s)&=\frac{1}{6}(1-t)s\big[2(1-s)-(1-t)^2-(1-s)^2\big]\\ &=\frac{1}{6}s(1-t)\big[2t-t^2-s^2\big]\\ &=G(t,s).\end{aligned}$$

From $G(t,s)=G(s,t)$ we can obtain the case $t<s$.

Finally, a straightforward calculation gives us

$$\begin{aligned}\int_0^1 G(t,s)^2\mathrm{d}s&=\int_0^t G(t,s)^2\mathrm{d}s+\int_t^1 G(t,s)^2\mathrm{d}s\\ &=\frac{1}{945}t^2(t-1)^2\left(3t^4-6t^3-t^2+4t+2\right).\end{aligned}$$

Moreover, if in the integral $\int_0^1 G(1-t,s)^2 \mathrm{d}s$ we make the change of variables $s=1-u$ and taking into account $G(t,s)=G(1-t,1-s)$ we get

$$\begin{aligned}\int_0^1 G(1-t,s)^2 \mathrm{d}s &= -\int_1^0 G(1-t,1-u)^2 \mathrm{d}u \\ &= \int_0^1 G(1-t,1-u)^2 \mathrm{d}u \\ &= \int_0^1 G(t,u)^2 \mathrm{d}u \\ &= \int_0^1 G(t,s)^2 \mathrm{d}s.\end{aligned}$$

This means that the polynomial

$$\int_0^1 G(t,s)^2 \mathrm{d}s = \frac{1}{945} t^2 (t-1)^2 \left(3t^4 - 6t^3 - t^2 + 4t + 2\right)$$

is a symmetric function with respect to $t_0 = 1/2$ and, consequently, as $t(t-1)$ is also symmetric function with respect to $t_0 = 1/2$ then

$$q(t) = 3t^4 - 6t^3 - t^2 + 4t + 2$$

must be symmetric with respect to $t_0 = 1/2$.

Therefore

$$\max_{0\le t\le 1} q(t) = \max_{0\le t\le \frac{1}{2}} q(t).$$

If the maximum is reached at $\left(0,\frac{1}{2}\right)$, the symmetric character of $q(t)$ says us that the derivative of $q(t)$ has at least two zeroes in $(0,1)$ and consequently, the second derivative of $q(t)$ has at least one zero in $(0,1)$. But

$$\begin{aligned}q'(t) &= 12t^3 - 18t^2 - 2t + 4 \\ q''(t) &= 36t^2 - 36t - 2\end{aligned}$$

and $q''(t)$ does not vanish in $(0,1)$ and, consequently, the maximum of $q(t)$ is reached in $t_0 = 0$ or $t_0 = 1/2$. As $q(0) = 2$ and $q(1/2) = 51/16$, thus,

$$\max_{0\le t\le 1} q(t) = \max_{0\le t\le 1/2} q(t) = q\left(\frac{1}{2}\right) = \frac{51}{16}.$$

Moreover, as the maximum of $h(t) = t^2(1-t)^2$ is reached at $t_0 = 1/2$ we get

$$\begin{aligned}\max_{0\le t\le 1} \int_0^1 G(t,s)^2 \mathrm{d}s &= \max_{0\le t\le 1/2} \int_0^1 G(t,s)^2 \mathrm{d}s \\ &= \frac{1}{945} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot q\left(\frac{1}{2}\right) \\ &= \frac{17}{80640}\ .\end{aligned}$$

2. MAIN RESULT

In this section, we present the main results of the paper.

Theorem 2.1. *Consider problem* (1.3) *with* $f\colon [0,1]\times[0,\infty)\to[0,\infty)$ *continuous and non-decreasing with respect to the second variable and suppose that there exists* $0<\alpha\le\sqrt{\frac{80640}{17}}$ *such that, for* $x,y\in[0,\infty)$ *with* $y\ge x$,

$$f(t,y)-f(t,x)\le\alpha\,\sqrt{\ln\big[(y-x)^2+1\big]}. \tag{2.1}$$

Then problem (1.3) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P=\big\{x\in\mathcal{C}[0,1]:x(t)\ge 0\big\}.$$

Obviously, $\big(P,d\big)$ with $d(x,y)=\sup\big\{|x(t)-y(t)|:t\in[0,1]\big\}$ is a complete metric space. Moreover, $\big(P,\le\big)$ with the partial order defined by

$$x\le y\Longleftrightarrow x(t)\le y(t),\quad \text{for } t\in[0,1]$$

satisfies condition (1.4) and (1.5) $\big($see, Section 1$\big)$. (Notice that for $x,y\in P$ then $\max\{x,y\}\in P$.)

Consider the operator defined by

$$\big(Tu\big)(t)=\int_0^1 G(t,s)f\big(s,u(s)\big)\mathrm{d}s,\quad \text{for } u\in P,$$

where $G(t,s)$ is the Green function defined in Section 1.

Since $f(t,x)$ and $G(t,s)$ are nonnegative continuous functions T applies P into itself.

In what follows, we check that hypotheses in Theorems 1.2 and 1.3 are satisfied.

The operator T is nondecreasing, since for $u,v\in P,\ u\ge v$ and $t\in[0,1]$, we have

$$\big(Tu\big)(t)=\int_0^1 G(t,s)f\big(s,u(s)\big)\mathrm{d}s\ge\int_0^1 G(t,s)f\big(s,v(s)\big)\mathrm{d}s=\big(Tv\big)(t).$$

(Notice that f is nondecreasing with respect to the second variable.)

Besides, for $u,v\in P$ and $u\ge v$, and, taking into account our assumptions and the above mentioned properties of the Green function (see, Section 1), we can get

$$\begin{aligned}
d\big(Tu,Tv\big)&=\sup_{0\le t\le 1}\Big|\big(Tu\big)(t)-\big(Tv\big)(t)\Big|\\
&=\sup_{0\le t\le 1}\Big(\big(Tu\big)(t)-\big(Tv\big)(t)\Big)\\
&=\sup_{0\le t\le 1}\int_0^1 G(t,s)\Big(f\big(s,u(s)\big)-f\big(s,v(s)\big)\Big)\mathrm{d}s\\
&\le\sup_{0\le t\le 1}\int_0^1 G(t,s)\ \alpha\sqrt{\ln\Big[\big(u(s)-v(s)\big)^2+1\Big]}\mathrm{d}s.
\end{aligned}$$

Using the Cauchy-Schwarz inequality in the last integral we get

$$\int_0^1 G(t,s)\alpha\sqrt{\ln\left[\left(u(s)-v(s)\right)^2+1\right]}\mathrm{d}s \le \left(\int_0^1 \left(G(t,s)\right)^2\mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \alpha^2\ln\left[\left(u(s)-v(s)\right)^2+1\right]\mathrm{d}s\right)^{\frac{1}{2}}$$

and, as

$$\int_0^1 \alpha^2\ln\left[\left(u(s)-v(s)\right)^2+1\right]\mathrm{d}s \le \alpha^2\ln\left[\left\|u-v\right\|^2+1\right] = \alpha^2\ln\left[d\left(u,v\right)^2+1\right],$$

$\sup_{t\in I}\int_0^1 \left(G(t,s)\right)^2\mathrm{d}s = \frac{17}{80640}$, and $\alpha \le \sqrt{\frac{80640}{17}}$, we can get

$$\begin{aligned} d\left(Tu,Tv\right) &\le \sup_{t\in I}\left(\int_0^1 \left(G(t,s)\right)^2\mathrm{d}s\right)^{\frac{1}{2}} \cdot \left(\alpha^2\ln\left[d\left(u,v\right)^2+1\right]\right)^{\frac{1}{2}} \\ &= \alpha\left(\ln\left[d\left(u,v\right)^2+1\right]\right)^{\frac{1}{2}}\sqrt{\frac{17}{80640}} \\ &\le \left(\ln\left[d\left(u,v\right)^2+1\right]\right)^{\frac{1}{2}}. \end{aligned}$$

The last inequality gives us

$$\begin{aligned} d\left(Tu,Tv\right)^2 &\le \ln\left[d\left(u,v\right)^2+1\right] \\ &= d(u,v)^2 - \left(d(u,v)^2 - \ln\left[d\left(u,v\right)^2+1\right]\right). \end{aligned}$$

Put $\psi(x) = x^2$ and $\phi(x) = x^2 - \ln(x^2+1)$. Obviously, ψ and ϕ are altering distance functions, and we have

$$\psi\left(d\left(Tu,Tv\right)\right) \le \psi\left(d\left(u,v\right)\right) - \phi\left(d\left(u,v\right)\right), \quad \text{for } u,v\in P \text{ with } u\ge v.$$

Finally, as f and G are nonnegative functions

$$\left(T0\right)(t) = \int_0^1 G(t,s)f\left(s,0\right)\mathrm{d}s \ge 0,$$

and Theorems 1.2 and 1.3 give us that T has a unique fixed point or, equivalently, our problem (1.3) has a unique nonnegative solution. □

Remark 2.2. Note that $G(1,s) = G(0,s) = 0$ and for $t \ne 0,1$, $G(t,\cdot) \ne 0$ a.e. because $G(t,\cdot)$ is given by a polynomial.

In the sequel, we present a sufficient condition for the existence and uniqueness of positive solutions for our problem (1.3) (positive solution means $x(t) > 0$ for $t\in(0,1)$).

Theorem 2.3. *Under assumptions of Theorem* 2.1 *and suppose that* $f(t,0)\ne 0$ *for* $t\in A\subset[0,1]$ *with* $\mu(A) > 0$, *then our problem* (1.3) *has a unique positive solution.* (*Here* μ *denotes the classical Lebesgue measure in* $[0,1]$).

Proof. Notice that the nonnegative solution given by Theorem 2.1 for our problem (1.3) satisfies

$$x(0) = \int_0^1 G(0,s) f\big(s, x(s)\big) \mathrm{d}s = \int_0^1 0 \cdot f\big(s, x(s)\big) \mathrm{d}s = 0$$

$$x(1) = \int_0^1 G(1,s) f\big(s, x(s)\big) \mathrm{d}s = \int_0^1 0 \cdot f\big(s, x(s)\big) \mathrm{d}s = 0.$$

Moreover, taking into account our assumptions, the zero function is not solution of problem (1.3). In fact, in contrary case, we have

$$0 = \int_0^1 G(t,s) f\big(s, 0\big) \mathrm{d}s, \quad \text{for } t \in [0,1].$$

This and the nonnegativity of the functions $G(t,s)$ and $f(s,0)$ gives us

$$G(t,s) \cdot f\big(s,0\big) = 0 \quad \text{a.e.,} \quad \text{for } t \in [0,1].$$

As $G(t,-) > 0$ a.e. $\big($see Remark 2.2$\big)$ then

$$f(s,0) = 0 \quad \text{a.e.}$$

which contradicts our hypothesis.

Now, suppose that there exists $0 < t_0 < 1$ such that $x(t_0) = 0$. As $x(t)$ is a fixed point of the operator T which appears in Theorem 2.1, we have

$$x(t_0) = \int_0^1 G(t_0,s) f\big(s, x(s)\big) \mathrm{d}s = 0.$$

As $x(s) \geq 0$ and the fact that f is nondecreasing with respect to the second variable, we can obtain

$$0 = x(t_0) = \int_0^1 G(t_0,s) f\big(s, x(s)\big) \mathrm{d}s \geq \int_0^1 G(t_0,s) f\big(s, 0\big) \mathrm{d}s \geq 0$$

and the last inequality gives us

$$\int_0^1 G(t_0,s) f\big(s,0\big) \mathrm{d}s = 0.$$

By using an analogous argument to the used one previously, we can conclude that

$$f\big(s,0\big) = 0 \quad \text{a.e.}$$

and this contradicts our hypothesis. Therefore, for $t \in (0,1)$, $x(t) > 0$. This finishes the proof. □

Remark 2.4. Notice that as $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, the fact $f(\cdot,0) \neq 0$ give us automatically the assumption that appears in Theorem 2.3. Indeed, as $f(\cdot,0) \neq 0$ there exists $t_0 \in [0,1]$ such that $f(t_0,0) > 0$ and using the continuity of f, we can find a set $A \subset [0,1]$ with $\mu\big(A\big) > 0$ such that $f(t,0) > 0$ for $t \in A$.

In [6] the authors obtained sufficient conditions for the existence of positive solutions for the following fourth-order equation with Lidstone boundary conditions

$$\begin{aligned} &u^{(4)}(t) = \lambda\, h(t)g(u), \quad t \in (0,1),\ \lambda > 0 \\ &u(0) = u(1) = 0 = u''(0) = u''(1). \end{aligned} \tag{2.2}$$

More precisely, the main result in [6] is the following theorem.

Theorem 2.5 (Theorem 3.1 [6]). *Suppose that* $h\colon [0,1] \to [0,\infty)$ *is continuous and not identically zero in* $\left[\frac{1}{4},\frac{3}{4}\right]$, $g\colon \mathbb{R} \to [0,\infty)$ *continuous,* $\lim_{s\to\infty} \frac{g(s)}{s} = +\infty$, *and there exists* $B \in [0,+\infty)$ *such that* g *is nondecreasing on* $[0,B)$. *If*

$$0 < \lambda < \sup_{s\in(0,B)} \frac{s}{\gamma^* g(s)}\ ,$$

where

$$\gamma^* = \max_{t\in(0,B)} \int_0^1 G(t,s)h(s)\mathrm{d}s,$$

then problem (2.2) *has at least a positive solution.*

Note that in [6] the uniqueness of the solution is not treated.

Some particular cases of problem (2.2) can be treated by our Theorem 2.3 and we additionally obtain uniqueness of the solution. For example, if $g(0) > 0$ and $g(y) - g(x) \leq \sqrt{\ln\left[(y-x)^2+1\right]}$, for $0 \leq x \leq y$, with $g\colon [0,\infty) \to [0,\infty)$ continuous and nondecreasing with respect to the second variable, and $h\colon [0,1] \to [0,\infty)$ continuous and not identically zero $\big($ in particular if h is not identically zero in $\left[\dfrac{1}{4},\dfrac{3}{4}\right]$ $\big)$. In this case, $f(t,u) = \lambda\, h(t)g(u)$ and, obviously, $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ is continuous and nondecreasing with respect to the second variable. Moreover, for $0 \leq x \leq y$ we have

$$\begin{aligned} f(t,y) - f(t,x) &= \lambda\cdot h(t)\ [g(y)-g(x)] \\ &\leq \lambda\cdot h(t)\sqrt{\ln\left[(y-x)^2+1\right]} \\ &\leq \lambda\cdot \|h\|\sqrt{\ln\left[(y-x)^2+1\right]}, \end{aligned}$$

where $\|h\| = \sup\big\{h(t) : t \in [0,1]\big\}$. As $f(t,0) = \lambda\, h(t)g(0)$, $g(0) > 0$ and h is not identically zero, Remark 2.4 says that $f(t,0)$ satisfies the assumption appearing in Theorem 2.3. Consequently, for $\lambda \leq \frac{1}{\|h\|}\sqrt{\frac{80640}{17}}$, Theorem 2.3 says us that problem (2.2), for this particular case, has a unique positive solution.

Remark 2.6. Notice that in [6] the authors allow to the nonlinear part to be zero at $u = 0$, but their results don't ensure the uniqueness of the solution.

3. ACKNOWLEDGEMENT

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