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# ROBUST STABILITY OF VOLTERRA DISCRETE EQUATIONS UNDER PERTURBATIONS OF THEIR KERNELS

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**ABSTRACT.** Robust solutions for multidimensional Volterra difference equations under perturbations of their kernels are obtained. These properties of the solutions are formulated immediately in terms of the equations characteristics.

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## 1. INTRODUCTION

Control algorithms are designed with the assumption that the plant dynamics are exactly those of of one member of specified class of models.It is then natural to ask how the control system will behave when, as is inevitable in practice, the true plant is not perfectly described by any model in the given class. If the stability of the control system is guaranteed, provided only that the modeling error is sufficiently small, then we say that the control algorithm is robust, and we speak of robust stability. It is clear that robust stability is very important for the practical applicability of control algorithms.

Unfortunately, a stable control algorithm is not necessary stable. The reason is that the modeling error signal appears as a disturbance in the law and may cause the divergence of the adaptive process. The fact that the disturbance is correlated with the plant input and output signals and, in addition, is of the same order of magnitude, is part of the complexity of the robustness problem.

As a first step towards robustness results, the stability of control systems in the presence of *bounded external* disturbances has been investigated by several authors.This investigations were prompted by observations showing that a bounded external disturbance, even an asymptotically vanishing one, can can cause the divergence of the adaptive process, and thereby instability. To prevent the latter, four main approaches have been made.

In the first approach a dead zone is used in the law so that adaptation takes place only when the identification error exceeds a certain threshold. If the disturbance is

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bounded below this threshold, then it can be shown that the adaptation is always in the "right" direction and system stability is achieved. In order to choose the size of dead zone appropriately, a bound on disturbance must be known. In the second approach, a modification of the adaptive law is used, which comes into operation only when the norm of the estimated controller parameters exceeds a certain value and has the effect that the parameter estimates remain bounded for all time. Closed-loop system stability is thus obtained in the presence of bounded disturbances of arbitrary, unknown size. In this case a bound on the norm of the desired controller parameters must be known. In the third approach we suggested and analysed. again if the disturbance is known to be bounded, close-loop system stability is obtained. For plants of relative degree greater than two, this also requires the knowledge of a bound the norm of the(unknown)desired controller parameters. In the forth approach the idea is to produce persistency of excitation in order to make the control system exponentially stable, and to obtain stability in the presence of a bounded disturbance as a consequence of the exponential stability. The question of how the signal can be chosen to ensure the persistency of excitation in the presence of the disturbance has not been completely resolved as yet. We point out that in all four approaches the proof of stability depends crucially on the *a priory* boundedness of external disturbance.

In the robustness problem, the disturbance is internally generated and thus depends on the actual plant input and output signals. In particular, if the control system were unstable and the plant input and output signals were to grow without bound, then the caused by model-plant mismatch, would also grow without bound. In other words, the stability problem becomes the problem of an internal, and thus *potentially unbounded* disturbance.Therefore, when proving boundedness of the disturbance cannot be assumed *a priory, and,* consequently, the aforementioned approaches for bounded disturbances do not necessarily solve the problem..

In spite of this intrinsic difficulty, a number of robustness results have been obtained in the literature. The results which have been proved in this approach are also local in nature because,a sufficiently large disturbances could invalidate the persistency of excitation assumption. Another interesting approach, is to use a modified signal normalization, which suitably bounds the modeling error signal, which keeps the parameter estimates bounded, in the adaptive law. Stability is then shown subject to the assumptions that bounds on the unknown plants parameters are known and the estimated plant is uniformly controllable and observable.

Stability is then demonstrated subject to the assumption that the unknown plant parameters lie in a known set of the parameter space, throughout which no unstable pole-zero cancellation occurs.

A signal normalization is used to define the relative identification error. It is shown that if the model-plant mismatch is sufficiently small, then the relative modeling error signal is within the dead zone, and the law causes the parameter error to decrease monotonically as in the bounded disturbance case. However, the law can now only guarantee that the *relative*identification error becomes smaller than the dead zone eventually, and hence, if the close-loop system were unstable, the absolute identification error would still grow without bound. A crucial step in the proof of stability is to carry this potentially unbounded identification error in such a way that stability can be concluded and an expression for the admissible size of the dead zone is obtained. To achieve this, the above-mentioned *a priory* knowledge of bounds on the unknown plant and controller parameters and on the plant zeros appears to be necessary. In principle, these bounds can be extended arbitrarily to approach the assumptions made in the ideal non-robust case, but it is also true that their values affect the admissible size and hence the degree of robustness which can be guaranteed.

The relative zone approach gives rise to quite different properties of the overall adaptive system. The potential of parameter convergence to the true values is retained in the absence of modeling errors. However, in the presence of modeling errors, even when they are over the specified region.In contrast to that, in the relative zone approach the parameter estimates will not converge to their true values in general, even when no modeling errors are present. However, regardless of the presence of modeling errors, the parameter estimates converge to a certain neighborhood of the true parameter values, and no drift phenomena are encountered.

In summery, the main contributions of this paper are the introduction of the relative concept as a tool for dealing with robustness of solutions for discrete Volterra equations.

The paper is organized as follows: introduction, main theorems, stability under steady state acting perturbations, discussion and conclusion.

Let us consider a system of linear Volterra difference equations with initial condition $x_0$

$$x_{i+1} = \sum_{j=0}^{i} A(i,j)\, x_j,\ i \geq 0 \tag{1.1}$$

$$x_0 \in \mathbf{R}^n \tag{1.2}$$

Here $x_i \in \mathbf{R}^n,\ A(i,j) \in \mathbf{R}^{n\times n}$. The solution of equation (1.1) is denoted by $x(i,0,x_0)$. It is assumed that $x_0$ and the matrices $A(i,j)$ are given, $N$ is a set of integers.

**Definition.** *The solution $x(i,0,x_0)$of problem (1.1), (1.2) is called exponentially stable if there exist constant $C>0$ and $\gamma \in [0,1)$such that*

$$|x(i,0,x_0)| \leq C\,|x_0|\,\gamma^i,\ i \geq 0,\ \forall x_0 \in \mathbf{R}^n \tag{1.3}$$

*Here* $|\cdot|$ *is a norm in the space* $R^n$.

Conditions of exponential stability of the solutions $x(i,0,x_0)$ of equation (1.1) were investigated in some papers. In paper [1] were considered linear convolution type Volterra difference equations

$$x_{i+1} = \sum_{j=i_0}^{i} A(i-j)\, x_j,\ x_{i_0} = x_0 \in \mathbf{R}^n \tag{1.4}$$

In the paper [1] it was shown that if the system (1.4) is uniformly asymptotically stable with respect to the initial moment $i_0$, then system (1.4) exponentially stable if and only if the value $\|A(i)\|$ is decreasing exponentially:

$$\|A(i)\| \le c_1 \gamma_1^i,\ c_1 > 0,\ 0 \le \gamma_1 < 1,\ i \to \infty$$

The proof of the paper [1] was founded on Laplace transformation method.

In the paper [2] were formulated necessary and sufficient conditions of exponential stability for equation (1.1) in terms of mapping Banach space $L^\gamma$ into Banach space $C^\gamma$, $0 < \gamma < 1$. Here the Banach space $L^\gamma$ is defined as

$$\begin{gathered} L^\gamma = \left\{ \{f_i\}_{i\in N} : f_i \in \mathbf{R}^n, \sum_{i=0}^{\infty} |f_i|\, \gamma^{-i} < \infty \right\}, \\ \|f\|_{L^\gamma} = \sum_{i=0}^{\infty} |f_i|\, \gamma^{-i} \end{gathered} \tag{1.5}$$

The Banach space $C^\gamma, 0 < \gamma < 1$ is defined as

$$\begin{gathered} C^\gamma = \left\{ \{f_i\}_{i\in N} : f_i \in \mathbf{R}^n, \sup_{i\ge 0} |f_i|\, \gamma^{-1} < \infty \right\}, \\ \|f\|_{C^\gamma} = \sup_{i\ge 0} |f_i|\, \gamma^{-i} \end{gathered} \tag{1.6}$$

Further let us introduce the operator $F$ on $L^\gamma$

$$F : f \in L^\gamma \to y = \{y_i\},\ y_i \in \mathbf{R}^n \tag{1.7}$$

Here

$$y_0 = 0,\ y_i = \sum_{j=0}^{i-1} z(i, j+1)\, f_j,\ i \ge 1$$

The resolvent $z(i,j)$ of equation (1.1) is a matrix $(n \times n)$ satisfying the relations

$$z(i+1, j) = \sum_{l=j}^{i} A(i,l)\, z(l,j),\, i \ge j,\quad z(j,j) = I,$$

$$z(i,j) = z(i,l)\, z(l,j) + \sum_{k=l}^{i-1} z(i,k+1) \sum_{h=j}^{l-1} A(k,h)\, z(h,j),$$

$$i \ge l \ge j$$

where $I$ is identity $(n \times n)$ matrix.

Necessary and sufficient conditions of exponential stability of solutions $x_i$ of equation (1.1) is the following; the operator $F$ defined by the relation (1.7) must map Banach space $L^\gamma$ into space $C^\gamma, 0<\gamma<1$.

Let us consider along with equation (1.1) the perturbed equation

$$y_{i+1}=\sum_{j=0}^{i}\left(A\left(i,j\right)+B\left(i,j\right)\right)y_j,\ i\geq 0,\ y_0\in\mathbf{R}^n \tag{1.8}$$

Assume that the solutions of equation (1.1) are exponentially stable. It means that the resolvent $z\left(i,j\right)$ of equation (1.1) satisfies the inequality (see,e.g.,[2])

$$\left\|z\left(i,j\right)\right\|\leq\lambda\gamma^{i-j},\ z\left(i,i\right)=I,\ i\geq j, 0\leq\gamma<1,\lambda>0 \tag{1.9}$$

where $\gamma$ is a fixed number.

The statement of the problem that will be studied in this paper is the following: what conditions must be imposed on perturbations $B\left(i,j\right)\in\mathbf{R}^{n\times n}$ such that the resolvent $W\left(i,j\right)$ of equation (1.8) will satisfy the inequality

$$\left\|W\left(i,0\right)\right\|\leq\lambda_1\gamma_1^i,\ W\left(i,0\right)=I,\ \ i\geq 0, 0\leq\gamma_1<1,\lambda_1>0 \tag{1.10}$$

It is clear that if the inequality (1.10) be valid then the solutions of the perturbed equation (1.8) be exponentially stable as well.

## 2. MAIN THEOREMS

**Theorem 2.1.** *Assume that the resolvent $z\left(i,j\right)$ satisfies the inequality (1.9), matrix $B\left(j,l\right)$ equal zero for each $j$ and $l<j-m,\ l>j$, where $m\geq 0$. Let us introduce two positive numbers $\eta$ and $\mu$ such that*

$$\sup_{j\geq 0,l\geq 0}\left\|B\left(j,l\right)\right\|\leq\eta,\quad \eta>0,\ \mu=\lambda\eta\frac{\gamma^{-m}-1}{1-\gamma} \tag{2.1}$$

*Then under condition*

$$1>\gamma\exp\left(\mu\right) \tag{2.2}$$

*(where $\lambda>0,\gamma\in\left(0,1\right)$ from (1.9), $\eta>0$ are fixed numbers) the resolvent $W\left(i,0\right)$ satisfies the inequality (1.10), i. e. the solutions of perturbed equation (1.8) are exponentially stable.*

*Proof.* Using Cauchy formula for Volterra difference equations the solution of equation (1.8) can be represented either in the form

$$y\left(i,0,y_0\right)=W\left(i,0\right)y_0,\ i\geq 0 \tag{2.3}$$

or in the form

$$\begin{aligned} y\left(i+1,0,y_0\right)&=z\left(i+1,0\right)y_0\\ &+\sum_{j=0}^{i}z\left(i+1,j+1\right)\sum_{l=0}^{j}B\left(j,l\right)y\left(l,0,y_0\right)\end{aligned} \tag{2.4}$$

But

$$y(l,0,y_0)=W(l,0)\,y_0,\ \ l\geq 0$$

Therefore

$$y(i+1,0,y_0)=z(i+1,0)\,y_0 + \sum_{j=0}^{i} z(i+1,j+1)\sum_{l=0}^{j} B(j,l)\,W(l,0)\,y_0$$

Comparing relations (2.1) and (2.2) in view of the arbitrariness $y_0$ and $i\geq 0$ we can conclude that

$$W(i+1,0)=z(i+1,0) + \sum_{j=0}^{i} z(i+1,j+1)\sum_{l=0}^{j} B(j,l)\,W(l,0)\,,\ \ i\geq j\geq 0 \tag{2.5}$$

Further according to the assumption of theorem 1 the resolvent $z(i,k)$ satisfies the estimate (1.9).From this estimate and relations (2.1), (2.5) it follows that

$$\begin{aligned}\gamma^{-i-1}\left\|W(i+1,0)\right\| &\leq \lambda\left(1+\gamma^{-1}\sum_{j=0}^{i}\gamma^{-j}\left\|\sum_{l=0}^{j}B(j,l)\,W(l,0)\right\|\right)\\ &\leq \lambda\left(1+\eta\gamma^{-1}\sum_{j=0}^{i}\gamma^{-j}\sum_{l=0}^{j}\left\|W(l,0)\right\|\right)\\ &= \lambda\left(1+\eta\gamma^{-1}\sum_{l=0}^{i}\left\|W(l,0)\right\|\sum_{j=l}^{l+m}\gamma^{-j}\right)\\ &= \lambda+\mu\sum_{l=0}^{i}\left\|W(l,0)\right\|\gamma^{-l}\end{aligned}$$

Here the number $\mu$ is given by (2.1).

Denote by $\omega(i)$ the function

$$\omega(i)=\gamma^{-i}\left\|W(i,0)\right\|,i\geq 0$$

Now using discrete variant of Gronwall-Bellman lemma ([3], p.15) we obtain that

$$\left\|W(i+1,0)\right\|\leq \lambda\gamma\left(\gamma\exp(\mu)\right)^{i},i\geq 0$$

Hence under assumption of theorem 1 the solutions of the perturbed system (1.8) are exponentially stable. Theorem 1 is proven. □

**Theorem 2.2.** *Assume that the resolvent $z(i,j)$ of equation (1.1) is bounded for all $i\geq j$ by the number $q_1$:*

$$\left\|z(i,j)\right\|\leq q_1,\ \ i\geq j\geq 0 \tag{2.6}$$

*Then the resolvent $W(i,j)$ of the perturbed equation (1.8) is also bounded for all $i\geq j\geq 0$ under condition*

$$\sum_{j=0}^{\infty}\sum_{l=0}^{j}\left\|B(j,l)\right\|<\infty \tag{2.7}$$

*Proof.* Let us denote for any fixed $k$ the function

$$\lambda_k(j) = \sup_{k \le l \le j} \|W(l,j)\|$$

From equality (2.5) and inequality (2.6) it follows that

$$\|W(i+1,k)\| \le q_1 \left(1 + \sum_{j=k}^{i} \sup_{k \le l \le j} \|W(l,k)\| \sum_{l=k}^{j} \|B(j,l)\|\right) \tag{2.8}$$

Then using inequality (2.8) we have

$$\lambda_k(i+1) \le q_1 \left(1 + \sum_{j=k}^{i} \lambda_k(j) \sum_{l=k}^{j} \|B(j,l)\|\right)$$

From here and Gronwall-Bellman lemma it follows

$$\lambda_k(i) \le q_1 \exp\left\{q_1 \sum_{j=k}^{\infty} \sum_{l=k}^{j} \|B(j,l)\|\right\}, k \ge 0$$

Therefore by virtue of (2.7) the resolvent $W(i,j)$ of perturbed equation (1.8) is bounded. Theorem 2 is proven. □

## 3. STABILITY UNDER STEADY STATE ACTING PERTURBATIONS

Denote by $L_1$ and $C$ two Banach spaces of sequences defined by relations (1.5), (1.6) for $\gamma = 1$. Let us interpret elements $f \in L_1$ as a perturbations acting on system (1.1). Without loss of generality we can assume that initial condition $x_{i_0} = 0$ for equation (1.1).

**Theorem 3.1.** *For any element $f \in L_1$ corresponds the solution $x \in C$ of the problem*

$$x_{i+1} = \sum_{j=i_0}^{i} A(i,j)\, x_j + f_i, \quad x_{i_0} = 0 \tag{3.1}$$

*if and only if*

$$\|z(i,j)\| \le k, \; i \ge j \ge i_0 \tag{3.2}$$

*Here $k > 0$ is a positive constant, $x_i \in \mathbf{R}^n$, $z(i,j)$ is resolvent matrix $(n \times n)$ of homogeneous equation (3.1), perturbation $f_i \in \mathbf{R}^n$.*

*Proof.* Sufficiency. Assume that inequality (3.2) is valid and $f \in L_1$. The solution of the problem (3.1) is given by the formula

$$x_i = \sum_{j=i_0}^{i-1} z(i,j+1)\, f_j$$

Hence

$$|x_i| = \left|\sum_{j=i_0}^{i-1} z(i,j+1)\, f_j\right| \le k \sum_{j=i_0}^{\infty} |f_j| = k\, \|f\|_{L_1}$$

Therefore $x \in C$.

Necessity. Assume that for every element $f \in L_1$ will be valid equality $x = Ff \in C$, where $F$ is an operator acting from the space $L_1$ into $C$. Let us show that from here it follows relation (3.2). Take some fixed moment $l \geq i_0$ and put

$$f_1(l) = \omega \in \mathbf{R}^n,\ f_1(j) = 0,\ j \neq l$$

where $\omega$ is a constant vector from $\mathbf{R}^n$. Remind (see, e.g. [4], ch.3) that if the operator $F$ acts from the Banach space $L_1$ into Banach space $C$ then it will be bounded. Therefore for $f = f_1$ we have

$$\|(Ff_1)(i)\| = \|z_\omega(i,l)\| \leq k\,|\omega|$$

From here and arbitrariness of the vector $\omega$ it follows the estimate (3.2). Theorem 3 is proven. □

Consider Volterra difference equation under nonlinear perturbations

$$x_{i+1} = \sum_{j=i_0}^{i} A(i,j)\,x_j + F\left(i, \sum_{j=i_0}^{i} B(i,j)\,x_j\right),\ i \geq i_0 \tag{3.3}$$

Here $F : N_0 \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous function with respect to the both arguments. Denote by $\mu_1$ the value

$$\mu_1 = \sup_{i \geq i_0} \sum_{j=i_0}^{i-1} z(i,j+1)$$

**Theorem 3.2.** *Let the resolvent $z(i,j)$ of equation (1.1) satisfy the inequality (1.9), initial condition $x_{i_0}$ of equation (3.3) satisfy the estimate*

$$|x_{i_0}| < \varepsilon\,(2\lambda)^{-1},\ \varepsilon > 0$$

*and in the domain $\{|u| \leq H, i \geq i_0\}$ be met the inequalities*

$$\left|F\left(i, \sum_{j=i_0}^{i} B(i,j)\,u_j\right)\right| < \mu_2,\ \mu_1\mu_2 < \frac{\varepsilon}{2},\ i \geq i_0,\ |u| \leq H$$

*Then for the solution $x(i,i_0,x_{i_0})$ of equation (3.3) is valid estimate $x(i,i_0,x_{i_0}) < \varepsilon,\ i \geq i_0$.*

*Proof.* The solution of equation (3.3) has a form

$$x_i = z(i,i_0)\,x_0 + \sum_{j=i_0}^{i-1} z(i,j+1)\,F\left(j, \sum_{l=i_0}^{j} B(j,l)\,x_l\right)$$

Let us introduce the operator $D^\eta, 0 \leq \eta \leq 1$ acting in the space $L_1$ by the relation

$$(D^\eta x)(i) = z(i,i_0)\,x_0 + \eta \sum_{j=i_0}^{i-1} z(i,j+1)\,F\left(j, \sum_{l=i_0}^{j} B(j,l)\,x_l\right) \tag{3.4}$$

Taking into account assumption about function $F$ we can conclude that the operator $D^\eta$ completely continuous. Further using Schauder principle ([3],p.298) we can obtain apriori estimate of all solutions

$$x^\eta(i) = (D^\eta x)(i)\,,\ 0 \le \eta \le 1$$

of equation (3.4). In fact from (3.4) it follows

$$|x^\eta(i)| \le \lambda\gamma^{i-i_0}|x_0| + \mu_2 \sum_{j=i_0}^{i-1} z(i, j-1) \le \frac{\varepsilon}{2} + \mu_1\mu_2 \le \varepsilon$$

Also for $\eta = 0$ we have

$$\sup_{i \ge i_0} |z(i, i_0)\, x_0| < \frac{\varepsilon}{2} < \varepsilon_1,\ \varepsilon_1 > \varepsilon$$

Therefore all assumptions of Leray-Schauder theorem are satisfied. Consequently equation (3.3) has at least one solution and for all solutions of equation (3.3) with initial condition $|x_0| < \varepsilon \diagup (2\lambda)$ will be valid estimate $|x_i| \le \varepsilon,\ i \ge i_0$. Theorem 4 is proven. □

**Corollary**.*Consider equation (3.1) under perturbations $f_i$ that are bounded by some number $\delta > 0$*

$$|f_i| \le \delta,\ \delta > 0,\ i \ge i_0$$

*Further assume that the resolvent $z(i,j)$ of equation (1.1) satisfies exponential estimate*

$$\|z(i,j)\| \le \lambda\gamma^{i-j},\ 0 < \gamma < 1,\ i \ge j \ge i_0$$

*Then the solutions $x(i, i_0, x_0)$ of equation (3.1) are bounded for all $i \ge i_0$.*

*In fact from (3.1) it follows that*

$$x_i = z(i, i_0)\, x_0 + \sum_{j=i_0}^{i-1} z(i, j+1)\, f_j$$

*Hence*

$$|x_i| \le k\left(\gamma^{i-i_0}|x_0| + \delta \sum_{j=i_0}^{i-1} \gamma^{i-j-1}\right)$$
$$= \gamma^i k\left(\gamma^{-i_0}|x_0| + \delta\left(\gamma^{-i-1} - \gamma^{-i_0-1}\right) \diagup \left(\gamma^{-1} - 1\right)\right) < \infty$$

**Conclusion**. *In this paper, we have further extended the stability analysis results under exogenous disturbances and parametric/dynamic uncertainties. Sufficient conditions for robust stability of Volterra difference equations were obtained.*

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