NEW OSCILLATION RESULTS FOR LINEAR MATRIX
HAMILTONIAN SYSTEMS

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ABSTRACT. Some new criteria have been established for the oscillation of the linear matrix Hamiltonian system
\[\begin{align*}
X' &= A(t)X + B(t)Y \\
Y' &= C(t)X - A^*(t)Y,
\end{align*}\]
under the hypothesis: \(A(t), B(t) = B^*(t) > 0\) and \(C(t) = C^*(t)\) are \(n \times n\) real continuous matrix functions on the interval \([t_0, \infty)\) \(t_0 > -\infty\). Our results are different from most known ones in the sense that they are given in the form \(\limsup_{t \to \infty} g[\cdot] > \text{const}\) rather than \(\limsup_{t \to \infty} \lambda_1[\cdot] = \infty\), where \(g\) is a positive linear functional on the linear space of \(n \times n\) matrices with real entries. Our results improve some previous ones. Two examples are worked out to illustrate the effectiveness of our results.

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1. INTRODUCTION

Consider the linear matrix Hamiltonian system
\[\begin{align*}
X' &= A(t)X + B(t)Y \\
Y' &= C(t)X - A^*(t)Y,
\end{align*}\]
where \(A(t), B(t) = B^*(t) > 0, C(t) = C^*(t)\) are \(n \times n\)-matrices of real valued continuous functions on \([t_0, \infty)\). By \(M^*\) we mean the conjugate transpose of the matrix \(M\).

For any solution \((X, Y)\) of (1.1), \(X^*(t)Y(t) - Y^*(t)X(t)\) is a constant matrix. The solution \((X, Y)\) of (1.1) is said to be nontrivial if \(\det X(t) \neq 0\) for at least one \(t \in [t_0, \infty)\). A nontrivial solution \((X, Y)\) of (1.1) is said to be prepared if \(X^*(t)Y(t) - Y^*(t)X(t) = 0\) for every \(t \in [t_0, \infty)\). A prepared solution \((X, Y)\) of (1.1) is said to be oscillatory on \([t_0, \infty)\) if \(\det X(t)\) has arbitrarily large zeros. System (1.1) is said to be oscillatory on \([t_0, \infty)\) if each nontrivial prepared solution of (1.1) is oscillatory.

In the case when \(A(t) \equiv 0, B(t) > 0\), system (1.1) reduces to the second order self-adjoint matrix differential system
\[\begin{align*}
(P(t)X')' + Q(t)X &= 0,
\end{align*}\]
with \(P(t) = B^{-1}(t), Q(t) = -C(t)\). Oscillation and nonoscillation of system (1.1) or (1.2) have been extensively studied by many authors [1-16, 19]. However, all the
results in [1-16, 19] are given in the form \( \lim_{t \to \infty} \sup \lambda_1[.] = \infty \), where \( \lambda_1[P] \) denotes the largest eigenvalue of an \( n \times n \) Hermitian matrix \( P \). In our recent papers [18, 20, 21], we gave several oscillation criteria in the form \( \lim_{t \to \infty} \sup \lambda_1[.] > \text{const} \). In this paper, we will further the investigation and establish some new oscillation criteria that are presented in the form \( \lim_{t \to \infty} \sup g[.] > \text{const} \) for the system (1.1) by using a class of particular functions \((t; s; r)\) defined by:

\[
(1.3) \quad \Phi(t, s, r) = (t - s)^\alpha (s - r)^\beta, \quad \text{for } t \geq s \geq r \geq t_0 \text{ and } \alpha, \beta > 1/2,
\]

where \( g \) is a positive linear functional on the linear space of \( n \times n \) matrices with real entries. Our results improve many known oscillation results even for the self-adjoint differential system (1.2), which can be illustrated by the examples given at the end of this paper.

In the sequel, Let \( \mathbb{R}^{n \times n} \) be the linear space of \( n \times n \) matrices with real entries, \( \mathcal{S} \subset \mathbb{R}^{n \times n} \) be the subspace of \( n \times n \) symmetric matrices, and \( g \) be a linear functional on \( \mathbb{R}^{n \times n} \). \( g \) is said to be positive if \( g(A) > 0 \) whenever \( A \in \mathcal{S} \) and \( A > 0 \).

2. MAIN RESULTS

The following lemma will be used to prove the main results of this paper.

Lemma 2.1. ([17]) If \( g \) is a positive linear functional on \( \mathbb{R}^{n \times n} \) then for all \( A, B \in \mathbb{R}^{n \times n} \), we have \( |g[A^*B]|^2 \leq g[A^*A]g[B^*B] \).

Now we give the main results of this paper.

Theorem 2.2. Let \( \Phi(t, s, r) \) be defined by (1.3). If there exist a positive linear functional \( g \) on \( \mathbb{R} \) and a function \( f(t) \in C^1[0, \infty) \) such that for each \( r \geq t_0 \)

\[
(2.1) \quad \lim_{t \to \infty} \sup \int_r^t g[M_1(t, s, r)] ds > 0,
\]

where

\[
M_1(t, s, r) = -\Phi^2(t, s, r)(C_1 + A^*B_1^{-1}A)(s) + 2\Phi(t, s, r)\Phi'_s(t, s, r)(B_1^{-1}A)(s) - \Phi'_s^2(t, s, r)B_1^{-1}(s),
\]

(2.2) \quad \begin{align*}
M_1(t, s, r) &= -\Phi^2(t, s, r)(C_1 + A^*B_1^{-1}A)(s) + 2\Phi(t, s, r)\Phi'_s(t, s, r)(B_1^{-1}A)(s) - \Phi'_s^2(t, s, r)B_1^{-1}(s),
\end{align*}

(2.3) \quad \begin{align*}
B_1(t) &= a^{-1}(t)B(t), \quad a(t) = \exp \left\{-2 \int_t^s f(s)ds\right\},
\end{align*}

and

(2.4) \quad C_1(t) = a(t) \left\{C(t) + f(t)[B^{-1}A + A^*B^{-1}](t) + [f(t)B^{-1}(t)]' - f^2(t)B^{-1}(t)\right\},

then the system (1.1) is oscillatory.
Proof. Assume to the contrary that (1.1) is nonoscillatory. Then there exists a non-trivial prepared solution \((X(t), Y(t))\) of (1.1) such that \(X(t)\) is nonsingular for all sufficiently large \(t\), say \(t \geq T \geq t_0\). This allows us to make a transformation

\[
(2.5) \quad W(t) = -a(t) \left[ Y(t)X^{-1}(t) + f(t)B^{-1}(t) \right], \quad t \geq T.
\]

From (1.1) and (2.3)-(2.5) we have

\[
(2.6) \quad W''(t) + A^*(t)W(t) + W^*(t)A(t) - W(t)B_1(t)W(t) + C_1(t) = 0,
\]

Multiplying (2.6), with \(t\) replaced by \(s\), by \(\Phi^2(t, s, T)\) and integrating from \(T\) to \(t\), we obtain

\[
(2.7) \quad \int_T^t \Phi^2(t, s, T)(-C_1(s))ds = -2 \int_T^t \Phi(t, s, T)\Phi'_s(t, s, T)W(s)ds
\]

\[
- \int_T^t \Phi^2(t, s, T)(W^*B_1W - A^*W - W^*A)(s)ds.
\]

Now the substitution

\[
P(t) = W(t) - B_1^{-1}(t)A(t)
\]

in the above equation (2.7) gives us

\[
(2.8) \quad \int_T^t M_0(t, s, T)ds = -2 \int_T^t \Phi(t, s, T)\Phi'_s(t, s, T)P(s)ds
\]

\[
- \int_T^t \Phi^2(t, s, T)(P^*B_1P)(s)ds,
\]

where

\[
M_0(t, s, T) = -\Phi^2(t, s, T)(C_1 + A^*B_1^{-1}A)(s) + 2\Phi(t, s, T)\Phi'_s(t, s, T)(B_1^{-1}A)(s).
\]

Applying the linear functional \(g\) on both sides of (2.8), we obtain

\[
(2.9) \quad \int_T^t g[M_0(t, s, T)]ds = \int_T^t 2\Phi(t, s, T)\Phi'_s(t, s, T)g[P(s)]ds
\]

\[
- \int_T^t \Phi^2(t, s, T)g[(P^*B_1P)(s)]ds.
\]

We now claim that for \(t \in [T, \infty)\),

\[
(2.10) \quad g[(P^*B_1P)(t)] \geq \left\{ g \left[ B_1^{-1}(t) \right] \right\}^{-1} \left\{ g \left[ P(t) \right] \right\}^2.
\]

In fact, by the lemma, for all \(t \in [T, \infty)\)

\[
g[B_1^{-1}(t)]g[(P^*B_1P)(t)] = g[(B_1^{-1/2}B_1^{-1/2})(t)]g[(B_1^{1/2}P)^*(B_1^{1/2}P)(t)]
\]

\[
\geq \left\{ g[(B_1^{-1/2}B_1^{1/2}P)(t)] \right\}^2
\]

\[
= \left\{ g[P(t)] \right\}^2.
\]
Hence the claim is true. By (2.9) and (2.10), we have that
\begin{align}
(2.11) \int_T^t g[M_0(t, s, T)] ds & \leq - \int_T^t 2\Phi(t, s, T)\Phi'_s(t, s, T)g[P(s)]ds \\
& \quad - \int_T^t \Phi^2(t, s, T) \left\{ g \left[ B_1^{-1}(s) \right] \right\}^{-1} \left\{ g([P(s)]) \right\}^2 ds \\
& = \int_T^t \Phi'_s^2(t, s, T)g[B_1^{-1}(s)]ds \\
& \quad - \int_T^t \left[ \frac{\Phi(t, s, T)}{\sqrt{g[B_1^{-1}(s)]}}g[P(s)] + \Phi'_s(t, s, T)\sqrt{g[B_1^{-1}(s)]} \right]^2 ds \\
& \leq \int_T^t \Phi'_s^2(t, s, T)g[B_1^{-1}(s)]ds.
\end{align}

From (2.2) and (2.11) we have
\[ \int_T^t g[M_1(t, s, T)] ds = \int_T^t g \left[ M_0(t, s, T) - \Phi'_s^2(t, s, T)B_1^{-1}(s) \right] ds \leq 0, \]
which implies a contradiction to the hypothesis (2.1). This completes the proof of Theorem 2.2.

If we choose an appropriate function \( f(t) \) and a positive linear functional \( g \) in Theorem 2.2 such that \( g[B_1^{-1}(t)] = g[a(t)B^{-1}(t)] \leq m \) for \( t \geq t_0 \), where \( m > 0 \) is a constant, and let \( \Phi(t, s, r) = (t-s)(s-r)\alpha \) for \( \alpha > 1/2 \), then we have the following theorem from Theorem 2.2:

**Theorem 2.3.** System (1.1) is oscillatory provided that for some \( \alpha > 1/2 \) and for each \( r \geq t_0 \),
\begin{align}
(2.12) \limsup_{t \to \infty} \frac{1}{t^{2\alpha+1}} \int_r^t g[M_2(t, s, r)] ds > \frac{ma}{(2\alpha - 1)(2\alpha + 1)},
\end{align}
where
\begin{align}
M_2(t, s, r) &= -(t-s)^2(s-r)^{2\alpha}g[(C_1 + A^*B_1^{-1}A)(s)] \\
& \quad + 2(t-s)(s-r)^{2\alpha-1}[\alpha t - (\alpha + 1)s + r]g[(B_1^{-1}A)(s)],
\end{align}
and \( B_1(t), C_1(t) \) are the same as in Theorem 2.2.

**Proof.** Assume to the contrary that (1.1) exists a nontrivial prepared solution \((X(t), Y(t))\) such that \( X(t) \) is nonsingular for \( t \geq T \geq t_0 \). Similar to the proof of Theorem 2.2, and noting that \( g[B_1^{-1}(t)] \leq m \) for \( t \geq t_0 \), we have for \( t \geq T \)
\begin{align}
(2.14) \int_T^t g[M_2(t, s, r)] ds & \leq \int_T^t [\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^{\alpha}]g[B_1^{-1}(s)]ds \\
& \leq m \int_T^t [\alpha(t-s)(s-T)^{\alpha-1} - (s-T)^{\alpha}]^2 ds.
\end{align}
Integrating by parts, we can easily obtain,

\[(2.15) \quad \int_{T}^{t} [\alpha(t - s)(s - T)^{\alpha - 1} - (s - T)^{\alpha} \alpha]^{2} ds = \frac{\alpha}{(2\alpha - 1)(2\alpha + 1)} (t - T)^{2\alpha + 1}.\]

Thus, from (2.13), (2.14) and (2.15) we have

\[
\lim_{t \to \infty} \sup_{t \geq t_0} \frac{1}{t^{2\beta + 1}} \int_{r}^{t} g[M_{2}(t, s, r)] ds \leq \frac{m\alpha}{(2\alpha - 1)(2\alpha + 1)},
\]

which contradicts the hypothesis (2.12). This completes the proof of Theorem 2.3. \(\square\)

If we choose an appropriate function \(f(t)\) and a positive linear functional \(g\) in Theorem 2.2 such that \(g[B_{1}^{-1}(t)] = g[a(t)B_{1}^{-1}(t)] \leq m\) for \(t \geq t_{0}\), where \(m > 0\) is a constant, and let \(\Phi(t, s, r) = (t - s)^{\alpha}(s - r)\) for \(\alpha > 1/2\), then, similar to the proof of Theorem 2.3, we have the following theorem:

**Theorem 2.4.** System (1.1) is oscillatory provided that for some \(\alpha > 1/2\) and for each \(r \geq t_{0}\),

\[(2.16) \quad \lim_{t \to \infty} \sup_{t \geq t_0} \frac{1}{t^{2\alpha + 1}} \int_{r}^{t} g[M_{3}(t, s, r)] ds > \frac{m\alpha}{(2\alpha - 1)(2\alpha + 1)},\]

where

\[
M_{3}(t, s, r) = -(t - s)^{2\alpha}(s - r)^{2}g[(C_{1} + A^\ast B_{1}^{-1}A)(s)] + 2(t - s)^{2\alpha - 1}(s - r)[t - (\alpha + 1)s + \alpha r]g[(B_{1}^{-1}A)(s)],
\]

and \(B_{1}(t), C_{1}(t)\) are the same as in Theorem 2.2.

In the sequel, we generalize Theorems 2.3 and 2.4 as the following theorem:

**Theorem 2.5.** If there exist a positive linear functional \(g\) on \(\mathcal{R}\), a function \(f(t) \in C^{1}[0, \infty)\) and a constant \(m > 0\) such that \(g[B_{1}^{-1}(t)] = g[a(t)B_{1}^{-1}(t)] \leq m\) for \(t \geq t_{0}\), and for each \(r \geq t_{0}\),

\[(2.17) \quad \lim_{t \to \infty} \sup_{t \geq t_0} \frac{1}{t^{2\alpha + 2\beta - 1}} \int_{r}^{t} g[M_{1}(t, s, r)] ds > 2m\alpha\beta(\alpha + \beta - 1) \frac{\Gamma(2\alpha - 1)\Gamma(2\beta - 1)}{\Gamma(2\alpha + 2\beta)},\]

where \(M_{1}(t, s, r)\) is defined as in Theorem 2.2, then system (1.1) is oscillatory.

**Proof.** Noting that

\[
\int_{T}^{t} \Phi_{s}^{2}(t, s, T) ds = \int_{T}^{t}[\beta(t - s)^{\alpha}(s - T)^{\beta - 1} - \alpha(t - s)^{\alpha - 1}(s - r)^{\beta}]^{2} ds
\]

\[= \int_{T}^{t} [\beta^{2}(t - s)^{2\alpha}(s - T)^{2(\beta - 1)} - 2\alpha\beta(t - s)^{2\alpha - 1}(s - r)^{2\beta - 1} + \alpha^{2}(t - s)^{2(\alpha - 1)}(s - r)^{2\beta}] ds.
\]
Integrating the above equality by parts and setting \( u = s - t \) and \( w = t - T \), we have eventually

\[
\int_T^t \Phi_s^2(t, s, T)ds = \int_T^t (s - T)^{2(\alpha - 1)}(s - T)^{2(\beta - 1)}[\beta(s - T) - \alpha(s - T)]
\]

(2.18)

\[
= \int_0^{t-T} (t - T - u)^{2(\alpha - 1)}u^{2(\beta - 1)}[\beta(t - T - u) - \alpha u]^2du
\]

\[
= \int_0^w (w - u)^{2(\alpha - 1)}u^{2(\beta - 1)}[\beta(w - u) - \alpha u]^2du.
\]

We now evaluate this integral using Euler’s Beta function

\[
\int_0^1 x^a(1 - x)^b dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}, \quad Re(a, b) > 0.
\]

Since

\[
\int_0^w (w - u)^{2(\alpha - 1)}u^{2(\beta - 1)}[\beta(w - u) - \alpha u]^2du = \int_0^w [\beta^2(w - u)^{2\alpha}u^{2(\beta - 1)}
\]

\[
- 2\alpha\beta(w - u)^{2\alpha - 1}u^{2\beta - 1} + \alpha^2(w - u)^{2(\alpha - 1)}u^{2\beta}]du,
\]

by evaluating the first integral by Euler’s Beta function and setting \( u = wx \), we obtain

\[
\int_0^w (w - u)^{2(\alpha - 1)}u^{2(\beta - 1)}du = w \int_0^1 (w - wx)^{2\alpha} (wx)^{2(\beta - 1)}dx
\]

\[
= w^{2(\alpha + \beta) - 1} \int_0^1 (1 - x)^{2\alpha} x^{2(\beta - 1)}dx
\]

\[
= w^{2\alpha + 2\beta - 1} \frac{\Gamma(2\alpha + 1)\Gamma(2\beta - 1)}{\Gamma(2\alpha + 2\beta)}.
\]

Similarly, we can evaluate the second and the third integral as

\[
\int_0^w (w - u)^{2\alpha - 1}u^{2\beta - 1}du = w^{2\alpha + 2\beta - 1} \frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(2\alpha + 2\beta)}
\]

and

\[
\int_0^w (w - u)^{2(\alpha - 1)}u^{2\beta}du = w^{2\alpha + 2\beta - 1} \frac{\Gamma(2\alpha - 1)\Gamma(2\beta + 1)}{\Gamma(2\alpha + 2\beta)}.
\]

Thus, we have

\[
\int_0^w (w - u)^{2(\alpha - 1)}u^{2(\beta - 1)}[\beta(w - u) - \alpha u]^2du
\]

\[
= \frac{w^{2\alpha + 2\beta - 1}}{\Gamma(2\alpha + 2\beta)} [\beta^2\Gamma(2\alpha + 1)\Gamma(2\beta - 1) - 2\alpha\beta\Gamma(2\alpha)\Gamma(2\beta)
\]

\[
+ \alpha^2\Gamma(2\alpha - 1)\Gamma(2\beta + 1)]
\]

(2.20)

\[
= 2\alpha\beta(\alpha + \beta - 1) \frac{\Gamma(2\alpha - 1)\Gamma(2\beta - 1)}{\Gamma(2\alpha + 2\beta)} w^{2\alpha + 2\beta - 1}.
\]

From (2.18)-(2.20), we get

\[
\int_T^t \Phi_s^2(t, s, T)ds = 2\alpha\beta(\alpha + \beta - 1) \frac{\Gamma(2\alpha - 1)\Gamma(2\beta - 1)}{\Gamma(2\alpha + 2\beta)} (t - T)^{2\alpha + 2\beta - 1}.
\]
The following proof is similar to that of Theorem 2.2, and hence is omitted. This completes the proof of Theorem 2.5.

The following two examples illustrate our results.

Example 2.6. Consider the Euler differential system

\[(2.21) \quad Y'' + \text{diag} \left( \frac{\gamma}{t^2}, \frac{\beta}{t^2} \right) Y = 0, \quad t \geq 1,\]

where \(\gamma \geq \beta > 0\) are constants. Our Theorem 2.3 can be applied to (2.21) and easily reveal the well-known fact that (2.21) is oscillatory for \(\gamma > 1/4\). In fact, if we choose \(f(t) = 0, \Phi(t, s, r) = (t - s)(s - r)^\alpha\) for \(\alpha > 1/2\), then we have

\[M_2(t, s, r) = (t - s)^2(s - r)^{2\alpha} g \left[ \text{diag} \left( \frac{\gamma}{t^2}, \frac{\beta}{t^2} \right) \right].\]

Let the positive linear functional \(g[A] = a_{11}\), where \(A = (a_{ij})\) is a 2 \(\times\) 2 matrix. Note that \(g[B_1^{-1}(t)] = 1\) and for each \(r \geq 1\)

\[\lim_{t \to \infty} \frac{1}{t^{2\alpha+1}} \int_r^t (t - s)^2(s - r)^{2\alpha} \frac{\gamma}{s^2} ds = \frac{\gamma}{\alpha(2\alpha - 1)(2\alpha + 1)}.\]

For any \(\gamma > 1/4\), there exists an constant \(\alpha > 1/2\) such that

\[
\frac{\gamma}{\alpha(2\alpha - 1)(2\alpha + 1)} > \frac{\alpha}{(2\alpha - 1)(2\alpha + 1)},
\]

i.e., \(\gamma > \alpha^2\). This means that (2.12) holds. By Theorem 2.3, we find that (2.21) is oscillatory for \(\gamma > 1/4\). However, we can easily see that criteria in [3, 6, 8] fail to reveal this fact.

Example 2.7. Consider the 4–dimensional matrix Hamiltonian system (2.22) with system parameters

\[(2.22) \quad A(t) = \begin{bmatrix} 0 & -1/t \\ 2/t & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix}, \quad C(t) = -\begin{bmatrix} \theta/t^3 & 0 \\ 0 & \eta/t^3 \end{bmatrix},\]

where \(t \geq 1, \eta \geq \theta > 0\) are constants. If we let \(f(t) = \frac{1}{2t^3}, \Phi(t, s, r) = (t - s)(s - r)^\alpha\) for \(\alpha > 1/2\), and the positive linear functional\(g[A] = a_{22}\), where \(A = (a_{ij})\) is a 2 \(\times\) 2 matrix, then we have

\[a(t) = t, \quad g \left[ M_2(t, s) \right] = (t - s)^2(s - r)^{2\alpha}(\eta - 11/8)/s^2,\]

and \(g[B_1^{-1}(t)] = g[a(t)B^{-1}(t)] = 1/2\) for \(t \geq 1\). Similar to the proof of Example 2.6, we can obtain that system (2.22) is oscillatory for \(\eta > 3/2\) by Theorem 2.3.
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