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MCKEAN-VLASOV STOCHASTIC DIFFERENTIAL EQUATIONS IN HILBERT SPACES UNDER CARATHEÓDORY CONDITIONS

N. I. MAHMUDOV[1] AND M. A. MCKIBBEN[2]

[1]Department of Mathematics, Eastern Mediterranean University
Gazimagusa, Mersin 10, TURKEY
E-mail: nazim.mahmudov@emu.edu.tr
[2]Goucher College, Mathematics and Computer Science Department
Baltimore, MD 21204 USA
E-mail: mmckibben@goucher.edu

ABSTRACT. We establish results concerning the global existence, uniqueness, and controllability of mild solutions for a class of first-order abstract McKean-Vlasov stochastic evolution equations with variable delay in a real separable Hilbert space. We allow the nonlinearities at a given time t to depend on the probability distribution at time t corresponding to the solution at time t. The results are obtained by imposing a so-called Carathéodory condition on the nonlinearities, which is weaker than the classical Lipschitz condition. Examples illustrating the applicability of the general theory are also provided.

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1. INTRODUCTION

In this paper we are concerned with the existence, uniqueness, and controllability of mild solutions to McKean-Vlasov stochastic semilinear differential equations with variable time delay of the form

$$\begin{aligned} dX(t) &= \left[AX(t) + Bu(t) + f(t, X(\theta(t)), \mu(t))\right] dt \\ &\quad + \sigma(t, X(\theta(t)), \mu(t))\, dW(t),\ t \in [0,T] \\ X(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t) : t \geq 0\}$ in a separable Hilbert space H, $W(t)$ is a Wiener process on a separable Hilbert space K, $B : U \to H$ is a linear bounded operator from a separable Hilbert space U to H, $u(t)$ is a control and $X(t)$ is a state process, $f : [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H) \to H$ and $\sigma : [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H) \to L_2^0$ are given functions to be specified later, $\theta : [0,\infty) \to [-r,\infty)$ is a suitable delay function,

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$\phi : [-r, 0] \times \Omega \to H$ is the initial datum, and $\mu(t)$ is the probability distribution of $X(t)$ at time t.

Stochastic partial functional differential equations with finite delay arise naturally in the mathematical modelling of phenomena in the natural sciences (see [28]). A recent survey article [15] recounts the work on such problems in the finite dimensional setting during the past 3 decades. Researchers have recently begun to extend this work to infinite dimensional stochastic evolution equations with delay (see [13], [17]).

It is known that if the nonlinearities f and σ do not depend on the probability distribution $\mu(t)$ of the state process, then the process described by (1.1) is a standard Markov process [1]. The introduction of the dependence of the nonlinearities on $\mu(t)$ is not superficial and, in fact, such problems arising in the study of diffusion processes have been studied extensively in the finite dimensional setting [11], [12], [21]. Ahmed and Ding [1] established an abstract formulation of such problems in a Hilbert space. Subsequently, Keck and McKibben [16] considered a Sobolev-type variant of the equation considered in [3], [13], [18], [24] and more recently, have extended this theory to a class of integro-differential stochastic evolution equations with finite delay related to (1.1) under Lipschitz growth conditions (see [17]). This was the first attempt at developing a general theory of abstract McKean-Vlasov equations with finite delay. A discussion of numerical schemes related to convergence issues for such finite-dimensional McKean-Vlasov equations have also been considered, first by Bossy and Talay (cf. [5], [6], and the references therein) and subsequently by Antonelli and Kohatsu-Higa [2]. These results were established assuming standard Lipschitz growth restrictions on the data; the question concerning the convergence the numerical scheme in the case in which the Lipschitz conditions are replaced by the weaker Caratheodory growth conditions (as in the current manuscript) is still open even in the finite dimensional case, as far as the authors are aware.

The results presented in the current manuscript constitute a continuation and generalization of existence, uniqueness, and controllability results from [1], [3], [13], [14], [17], [19], [20] in two ways. For one, we incorporate a so-called variable delay function (as in [13], [17]) into (1.1). And two, more importantly, we replace the Lipschitz growth conditions by more general Carathéodory-type conditions of the type introduced by [23] and subsequently adapted in [3], [7], [13]. The point of interest here is that the convergence scheme used in the proof still enables us to conclude uniqueness without any additional restriction on the operator A or the data. As such, the results in the references mentioned above are recovered as corollaries of the main results in this manuscript.

The following is the outline of the paper. First, we make precise the necessary notation, function spaces, and definitions, and gather certain preliminary results in

Section 2. We then formulate the main results in Section 3, while we devote Section 4 to a discussion of some concrete examples.

2. PRELIMINARIES

Let us first introduce some notation. For details, we refer the reader to [8], [10], [11], [16], [22] and the references therein. Throughout this paper, H and K shall denote real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_K$, while L_2^0 denotes the space of all Hilbert-Schmidt operators from K into H (the norm will be denoted as $\|\cdot\|_{L_2^0}$). Let $(\Omega, \mathfrak{F}_T, P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t : t \geq 0\}$ generated by the Wiener process W. For brevity, we suppress the dependence of all mappings on ω throughout the manuscript.

The function spaces needed in this manuscript coincide with those used in [1], [16]; we recall them here for convenience. First, $B(H)$ stands for the Borel class on H and $\mathfrak{M}(H)$ represents the space of all probability measures defined on $B(H)$ equipped with the weak convergence topology. Let $\lambda(x) = 1 + \|x\|, x \in H$ and define the space

$$\mathfrak{C}_\rho(H) = \{\varphi : H \to H : \varphi \text{ is continuous and}$$
$$\|\varphi\|_{C_\rho} = \sup_{x\in H} \frac{\|\varphi(x)\|}{\lambda^2(x)} + \sup_{x\neq y \text{ in } H} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|} < \infty \Big\}$$

For $p \geq 1$, we let

$$\mathfrak{M}^s_{\lambda^p}(H) = \{m : H \to \mathbb{R} \,|m \text{ is a signed measure on } H \text{ such that}$$
$$\|m\|_{\lambda^p} = \int_H \lambda^p(x) \, |m| \, (dx) < \infty \Big\}$$

where $|m| = m^+ + m^-$, $m = m^+ - m^-$ is the Jordan decomposition of m. Then, we can define the space

$$\mathfrak{M}_{\lambda^2}(H) = \mathfrak{M}^s_{\lambda^2}(H) \cap \mathfrak{M}(H)$$

equipped with the metric ρ given by

$$\rho(\nu_1, \nu_2) = \sup \left\{ \int_H \varphi(x) (\nu_1 - \nu_2)(dx) : \|\varphi\|_{C_\rho} \leq 1 \right\}.$$

It is has been shown that $(\mathfrak{M}_{\lambda^2}(H), \rho)$ is a complete metric space. The space of all continuous $\mathfrak{M}_{\lambda^2}(H)$-valued measures defined on $[0,T]$, denoted by $C([0,T]; (\mathfrak{M}_{\lambda^2}(H), \rho))$, is complete when equipped with the metric

$$D_T(\nu_1, \nu_2) = \sup_{t\in[0,T]} \rho(\nu_1(t), \nu_2(t)), \quad \nu_1, \nu_2 \in C([0,T]; (\mathfrak{M}_{\lambda^2}(H), \rho)). \tag{2.1}$$

Throughout the paper, $L^2_{\mathfrak{F}}(0,T;U)$ denotes the space square integrable and $\mathfrak{F}_t$-adapted processes from $[0,T] \times \Omega$ into U. $C([0,T]; L_2(\Omega, \mathfrak{F}_T, H))$ denotes the Banach space of continuous maps from $[0,T]$ into $L_2(\Omega, \mathfrak{F}_T, H)$ satisfying $\sup_{t\in[0,T]} \mathbf{E}\|X(t)\|^2 < \infty$, and $\mathfrak{H}_2$ stands for the closed subspace of $C([0,T]; L_2(\Omega, \mathfrak{F}_T, H))$ consisting of measurable, $\mathfrak{F}_t$-adapted processes. It is known that $\mathfrak{H}_2$ is a Banach space with the norm topology given by $\|X\|_{\mathfrak{H}_2} = \sup_{t\in[0,T]} \mathbf{E}\|X(t)\|^2$.

3. MAIN RESULTS

In this section we study the existence, uniqueness, and approximate controllability of mild solutions of (1.1) in the sense of the following definition.

Definition 3.1. A continuous stochastic process $X : [-r,T] \times \Omega \to H$ is a mild solution of (1.1) if the following conditions are satisfied:

(i) $X(t)$ is measurable and $\mathfrak{F}_t$-adapted, for all $-r \le t \le T$,
(ii) $\int_0^T \|X(s)\|^2 ds < \infty, \quad a.s.[P]$,
(iii) X satisfies the equation

$$X(t) = S(t)\phi(0) + \int_0^t S(t-s)Bu(s)\,ds + \int_0^t S(t-s) f(s, X(\theta(s)), \mu(s))\,ds$$
$$+ \int_0^t S(t-s)\sigma(s, X(\theta(s)), \mu(s))\,dW(s), 0 \le t \le T,$$
$$X(t) = \phi(t), \quad t \in [-r, 0].$$

The following are the main assumptions assumed in the manuscript.

(A1): A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \ge 0\}$ on H.

(A2): $(f,\sigma) : [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H) \to H \times L_2^0$ are $\mathfrak{F}_t$-measurable mappings satisfying:

- **(i):** There exists $K : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that
  - **(a):** $K(\cdot,\cdot)$ is continuous, monotone nondecreasing, and concave,
  - **(b):** $\|f(t,x,\mu)\|^2 + \|\sigma(t,x,\mu)\|_{L_2^0}^2 \le K\left(\|x\|^2, \|\mu\|_{\lambda^2}^2\right)$, for all $(t,x,\mu) \in [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H)$.
- **(ii):** There exists $N : [0,\infty) \to [0,\infty)$ such that
  - **(a):** N is continuous, monotone nondecreasing and concave, and $N(0) = 0$,
  - **(b):** $\|f(t,x,\mu) - f(t,y,\nu)\|^2 + \|\sigma(t,x,\mu) - \sigma(t,y,\nu)\|_{L_2^0}^2 \le N\left(\|x-y\|^2\right) + \rho^2(\mu,\nu)$, for all $(t,x,\mu), (t,y,\nu) \in [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H)$.

(A3): The function N of **(A2)**(ii) is such that if a nonnegative, continuous function $z(t)$ satisfies $z(0) = 0$ and

$$z(t) \le D \int_0^t N(z(s))\,ds,$$

for all $t \in [0,T]$, where $D > 0$, then $z(t) = 0$, for all $t \in [0,T]$.

(A4): For any fixed $T > 0$, $\beta > 0$, and $z \ge 0$ the initial-value problem

$$u'(t) = \beta K(u,z), \; u(0) = u_0 \ge 0, \tag{3.1}$$

has a global solution on $[0,T]$.

(A5): For each $0 \leq t < T$, the operator $\left(\alpha I + \Gamma_t^T\right)^{-1} \to 0$ as $\alpha \to 0^+$ in the strong operator topology, where $\Gamma_t^T = \int_t^T S\left(T-s\right) DD^* S^*\left(T-s\right) ds$ is the controllability Grammian.

(A6): $\theta : [0,\infty) \to [-r,\infty)$ is a continuously differentiable function of delay satisfying the conditions that

$$\theta'\left(t\right) \geq 1, \;\; -r \leq \theta\left(t\right) \leq t, \;\; \text{for } r > 0 \text{ and} t \geq 0.$$

(Observe that there exists a constant $k > 0$ such that $\theta^{-1}\left(t\right) \leq t + k, \;\; \text{for all } t \geq -r.$)

(A7): The function $\phi\left(t\right) : [-r,0] \times \Omega \to H$ is an $\mathfrak{F}_0$-measurable random variable independent of W with almost surely continuous paths.

Observe that the linear deterministic system

$$x'(t) = Ax(t) + Bu\left(t\right), \;\; 0 \leq t \leq T,$$

$$x(0) = x_0,$$

corresponding to (1.1) is approximately controllable on $[t,T]$ if and only the operator $\left(\alpha I + \Gamma_t^T\right)^{-1} \to 0$ strongly as $\alpha \to 0^+$ (see [19], [20]).

Definition 3.2. The system (1.1) is approximately controllable on $[0,T]$ if $\overline{R(T)} = L_2\left(\Omega, \mathfrak{F}_T, H\right)$, where

$$R(T) = \left\{X(T;u) : X\left(t,u\right) \text{ is a solution of (1.1) corresponding to } u \in L^2_{\mathfrak{F}}(0,T;U)\right\}.$$

It is known that for any $h \in L_2\left(\Omega, \mathfrak{F}_T, H\right)$ there exists $\varphi \in L^2_{\mathfrak{F}}(0,T;L_2^0)$ such that

$$h = \mathbf{E}h + \int_0^t \varphi\left(s\right) dW\left(s\right).$$

Now, using this presentation for any $(\alpha,h,z,\mu) \in (0,\infty) \times L_2\left(\Omega, \mathfrak{F}_T, H\right) \times \mathfrak{H}_2 \times \mathfrak{M}_{\lambda^2}\left(H\right)$, we define the control function by

$$\begin{aligned} u^\alpha\left(t,z\right) &= B^*S^*\left(T-t\right)\left(\alpha I + \Gamma_0^T\right)^{-1}\left(\mathbf{E}h - S(T)\phi\left(0\right)\right) \\ &+ B^*S^*\left(T-t\right)\int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1}\varphi\left(s\right)W(s) \\ &- B^*S^*\left(T-t\right)\int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) f\left(s, z\left(\theta(s)\right), \mu(s)\right) ds \\ &- B^*S^*\left(T-t\right)\int_0^t \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) g\left(s, z\left(\theta(s)\right), \mu(s)\right) dW(s). \end{aligned} \tag{3.2}$$

To present the result concerning the approximate controllability of mild solutions of (1.1), we fix $\alpha > 0$ and define the operator $\Phi_\alpha : \mathfrak{H}_2 \to \mathfrak{H}_2$ by

$$\left(\Phi_\alpha X\right)(t) = S(t)\phi(0) + \Gamma_0^t S^*(T-t)\left(\alpha I + \Gamma_0^T\right)^{-1}\left(\mathbf{E}h - S(T)\phi(0)\right)$$
$$+\int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\right] f\left(s, X\left(\theta(s)\right), \mu(s)\right) ds$$
$$+\int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\right] g\left(s, X\left(\theta(s)\right), \mu(s)\right) dW(s)$$
$$+\int_0^t \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} \varphi(s) dW(s). \tag{3.3}$$

The operator $\Phi_\alpha : \mathfrak{H}_2 \to \mathfrak{H}_2$ is naturally obtained by inserting the control (3.2) into the variation of parameters formula in Definition 3.1.

Lemma 3.3. *Under the conditions (A1) and (A2), the operator $\Phi_\alpha : \mathfrak{H}_2 \to \mathfrak{H}_2$ is well-defined and there exist positive constants $M_1(\alpha), M_2(\alpha)$, and $\overline{M}_T(\alpha)$ such that*

$$\mathbf{E}\left\|\left(\Phi_\alpha X\right)(t)\right\|^2 \le M_1(\alpha) + M_2(\alpha)\int_0^t K\left(\mathbf{E}\left\|X(u)\right\|^2, \gamma\right) du, \tag{3.4}$$

$$\mathbf{E}\left\|\left(\Phi_\alpha X\right)(t) - \left(\Phi_\alpha Y\right)(t)\right\|^2 \le \overline{M}_T(\alpha)\int_0^t N\left(\mathbf{E}\left\|X(u) - Y(u)\right\|^2\right) du, \tag{3.5}$$

for each $t \in [0, T]$ and $X, Y \in \mathfrak{H}_2$, where

$$M_1(\alpha) = 5M_S^2\mathbf{E}\left\|\phi(0)\right\|^2 + \frac{5}{\alpha^2}M_\Gamma^2 M_S^2\left(2\left\|\mathbf{E}h\right\|^2 + 2M_S^2\mathbf{E}\left\|\phi(0)\right\|^2 + \int_0^t \mathbf{E}\left\|\varphi(s)\right\|_{L_2^0}^2 ds\right),$$
$$M_2(\alpha) = 10\left(1 + \frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right) M_S^2(T+1),$$
$$\overline{M}_T(\alpha) = 2\left(1 + \frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right) M_S^2(T+1), \gamma = \sup_{t\in[0,T]}\left\|\mu(t)\right\|_{\lambda^2}^2,$$
$$M_S = \sup_{t\in[0,T]}\left\|S(t)\right\|, M_\Gamma = \sup_{0\le s\le t\le T}\left\|\Gamma_s^t\right\|.$$

Proof. Observe that standard computations yield

$$\mathbf{E}\left\|\left(\Phi_\alpha X\right)(t)\right\|^2 \le 5\mathbf{E}\left\|S(t)\phi(0)\right\|^2 + 5\left\|\Gamma_0^t S^*(T-t)\left(\alpha I + \Gamma_0^T\right)^{-1}\left(\mathbf{E}h - S(T)\phi(0)\right)\right\|^2$$
$$+5\mathbf{E}\left\|\int_0^t \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1}\varphi(s) dW(s)\right\|^2$$
$$+5\mathbf{E}\left\|\int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\right] f\left(s, X\left(\theta(s)\right), \mu(s)\right) ds\right\|^2$$
$$+5\mathbf{E}\left\|\int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\right] \sigma\left(s, X\left(\theta(s)\right), \mu(s)\right) dW(s)\right\|^2$$
$$\le 5M_S^2\mathbf{E}\left\|\phi(0)\right\|^2 + \frac{5}{\alpha^2}M_\Gamma^2 M_S^2\left(2\left\|\mathbf{E}h\right\|^2 + 2M_S^2\mathbf{E}\left\|\phi(0)\right\|^2 + \int_0^t \mathbf{E}\left\|\varphi(s)\right\|_{L_2^0}^2 ds\right)$$

$$+5\left(2+\frac{2}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2 T\int_0^t \mathbf{E}\left\|f\left(s,X\left(\theta\left(s\right)\right),\mu\left(s\right)\right)\right\|^2 ds$$
$$+5\left(2+\frac{2}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2 \int_0^t \mathbf{E}\left\|\sigma\left(s,X\left(\theta\left(s\right)\right),\mu\left(s\right)\right)\right\|_{L_2^0}^2 ds$$

$$\leq 5M_S^2\mathbf{E}\left\|\phi\left(0\right)\right\|^2+\frac{5}{\alpha^2}M_\Gamma^2 M_S^2\left(2\left\|\mathbf{E}h\right\|^2+2M_S^2\mathbf{E}\left\|\phi\left(0\right)\right\|^2+\int_0^t \mathbf{E}\left\|\varphi(s)\right\|_{L_2^0}^2 ds\right)$$
$$+10\left(1+\frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2\left(T+1\right)\int_0^t K\left(\mathbf{E}\left\|X\left(\theta\left(s\right)\right)\right\|^2,\gamma\right)ds$$
$$\leq 5M_S^2\mathbf{E}\left\|\phi\left(0\right)\right\|^2+\frac{5}{\alpha^2}M_\Gamma^2 M_S^2\left(2\left\|\mathbf{E}h\right\|^2+2M_S^2\mathbf{E}\left\|\phi\left(0\right)\right\|^2\right)$$
$$+10\left(1+\frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2\left(T+1\right)\int_{\theta(0)}^{\theta(t)} K\left(\mathbf{E}\left\|X\left(u\right)\right\|^2,\gamma\right)\frac{1}{\theta'\left(\theta^{-1}\left(u\right)\right)}du$$
$$\leq M_1\left(\alpha\right)+M_2\left(\alpha\right)\int_0^t K\left(\mathbf{E}\left\|X\left(u\right)\right\|^2,\gamma\right)du<\infty.$$

This proves (3.4). Next, (3.5) follows since

$$\mathbf{E}\left\|\left(\Phi_\alpha X\right)\left(t\right)-\left(\Phi_\alpha Y\right)\left(t\right)\right\|^2$$
$$\leq 2\left(1+\frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2 T\int_0^t \mathbf{E}\left\|f\left(s,X\left(\theta\left(s\right)\right),\mu\left(s\right)\right)-f\left(s,Y\left(\theta\left(s\right)\right),\mu\left(s\right)\right)\right\|^2 ds$$
$$+2\left(1+\frac{1}{\alpha^2}M_\Gamma^2 M_S^2\right)M_S^2\int_0^t \mathbf{E}\left\|\sigma\left(s,X\left(\theta\left(s\right)\right),\mu\left(s\right)\right)-\sigma\left(s,Y\left(\theta\left(s\right)\right),\mu\left(s\right)\right)\right\|_{L_2^0}^2 ds$$
$$\leq \overline{M}_T\left(\alpha\right)\int_0^t N\left(\mathbf{E}\left\|X\left(\theta\left(s\right)\right)-Y\left(\theta\left(s\right)\right)\right\|^2\right)ds$$
$$=\overline{M}_T\left(\alpha\right)\int_{\theta(0)}^{\theta(t)} N\left(\mathbf{E}\left\|X\left(u\right)-Y\left(u\right)\right\|^2\right)\frac{1}{\theta'\left(\theta^{-1}\left(u\right)\right)}du$$
$$\leq \overline{M}_T\left(\alpha\right)\int_0^t N\left(\mathbf{E}\left\|X\left(u\right)-Y\left(u\right)\right\|^2\right)du.$$

This completes the proof. □

We now construct successive approximations using a Picard-type iteration. For any fixed $T>0$, let

$$X_0\left(t\right)=S\left(t\right)\phi\left(0\right)+\Gamma_0^t S^*\left(T-t\right)\left(\alpha I+\Gamma_0^T\right)^{-1}\left(\mathbf{E}h-S(T)\phi\left(0\right)\right)$$
$$+\int_0^t \Gamma_s^t S^*\left(T-t\right)\left(\alpha I+\Gamma_s^T\right)^{-1}\varphi(s)dW\left(s\right),$$

and let $X_n(t)$ be the sequence defined recursively by

$$\begin{aligned} X_n(t) &= X_0(t) \\ &\quad + \int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) \right] \\ &\qquad \times f(s, X_{n-1}(\theta(s)), \mu(s))\, ds \\ &\quad + \int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t) \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) \right] \\ &\qquad \times \sigma(s, X_{n-1}(\theta(s)), \mu(s))\, dW(s). \end{aligned} \tag{3.6}$$

Lemma 3.4. *Under the conditions (A1) and (A2), the sequence* $\{X_n : n \geq 0\}$ *satisfies the following inequality for all* $0 \leq t \leq T$*:*

$$\mathbf{E} \|X_n(t)\|^2 \leq u(t). \tag{3.7}$$

Proof. It follows from Lemma 3.3 that

$$\mathbf{E} \|X_n(t)\|^2 \leq M_1(\alpha) + M_2(\alpha) \int_0^t K\left(\mathbf{E} \|X_{n-1}(s)\|^2, \gamma\right) ds, \tag{3.8}$$

where $M_1(\alpha)$ and $M_2(\alpha)$are positive constants independent of n. Let $u(t)$ be the global solution of the equation (3.1) with $u_0 \geq \max\left(M_1(\alpha), \sup_{t\in[0,T]} \mathbf{E} \|X_0(t)\|^2\right)$. We will establish inequality (3.7) using mathematical induction. To begin, note that for $n = 0$ the inequality (3.7) holds by the definition of u. Indeed, we have

$$\begin{aligned} u(t) &= u_0 + M_2(\alpha) \int_0^t K(u(s), \gamma)\, ds \\ &\geq \max\left(M_1(\alpha), \sup_{t\in[0,T]} \mathbf{E} \|X_0(t)\|^2 \right) + M_2(\alpha) \int_0^t K(u(s), \gamma)\, ds \geq \mathbf{E} \|X_0(t)\|^2. \end{aligned}$$

Next, suppose that

$$\mathbf{E} \|X_{n-1}(t)\|^2 \leq u(t), \quad \text{for all} \quad 0 \leq t \leq T.$$

Then, from (3.1) and (3.8), we conclude that

$$u(t) - \mathbf{E} \|X_n(t)\|^2 \geq M_2(\alpha) \int_0^t \left[K(u(s), \gamma) - K\left(\mathbf{E} \|X_{n-1}\|^2, \gamma\right) \right] ds \geq 0.$$

Hence, (3.7) holds for all n (thanks to **(A2)**). ☐

Lemma 3.5. *Under the conditions **(A1)** and **(A2)**,* $\{X_n : n \geq 1\}$ *is a Cauchy sequence in* $\mathfrak{H}_2$*.*

Proof. Define the sequence of functions $r_n : [0, T] \to \mathbb{R}$ by

$$r_n(t) = \sup_{m\geq n} \mathbf{E} \|X_{m+n}(t) - X_n(t)\|^2, \quad t \in [0, T], n \geq 1.$$

Note that for each $n \geq 1$, r_n is well-defined, uniformly bounded, and monotone nondecreasing (in t). Since $\{r_n : n \geq 1\}$is a monotone nonincreasing sequence, for each $t \in [0,T]$, there exists a monotone nondecreasing function $r : [0,T] \to \mathbb{R}$ such that

$$\lim_{n\to\infty} r_n(t) = r(t). \tag{3.9}$$

It follows from Lemma 3.3 that for any $n, m \geq 1$,

$$\mathbf{E}\left\|X_m(t) - X_n(t)\right\|^2 \leq \overline{M}_T(\alpha)\int_0^t N\left(\mathbf{E}\left\|X_{m-1}(s) - X_{n-1}(s)\right\|^2\right) ds,$$

from which we subsequently obtain

$$r(t) \leq r_n(t) \leq \overline{M}_T(\alpha)\int_0^t N(r_{n-1}(s))\, ds,$$

for any $n \geq 1$. Using (3.9), together with the Lebesgue dominated convergence theorem, then yields

$$r(t) \leq \overline{M}_T(\alpha)\int_0^t N(r(s))\, ds.$$

But, $\sup_{t\in[0,T]} \mathbf{E}\left\|X_{m+n}(t) - X_n(t)\right\|^2 \leq r_n(T)$ and $\lim_{n\to\infty} r_n(T) = r(T) = 0$. Therefore, $\sup_{t\in[0,T]} \mathbf{E}\left\|X_{m+n}(t) - X_n(t)\right\|^2 = 0$, so that $\{X_n, n \geq 1\}$ is indeed a Cauchy sequence in $\mathfrak{H}_2$. This completes the proof. □

Theorem 3.6. *If the conditions (A1)-(A7) hold, then (1.1) has a unique mild solution in $\mathfrak{H}_2$ with probability distribution $\mu \in C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$.*

Proof. Let $\mu \in C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$ be fixed. The completeness of $\mathfrak{H}_2$ guarantees the existence of a process X such that

$$\lim_{n\to\infty}\sup_{t\in[0,T]} \mathbf{E}\left\|X_n(t) - X(t)\right\|^2 = 0.$$

Further, we may infer from **(A2)** that

$$N\left(\sup_{t\in[0,T]} \mathbf{E}\left\|X_n(t) - X(t)\right\|^2\right) \to N(0) = 0,$$

and hence,

$$\lim_{n\to\infty}\sup_{t\in[0,T]} \mathbf{E}\left\|(\Phi_\alpha X_n)(t) - (\Phi_\alpha X)(t)\right\|^2 = 0.$$

Thus, X is a fixed point of Φ_α which is, in fact, a mild solution to (1.1) on $[0,T]$. Further, if $X, Y \in \mathfrak{H}_2$ are two fixed points of Φ_α, then

$$\sup_{s\in[0,t]} \mathbf{E}\left\|(\Phi_\alpha X)(s) - (\Phi_\alpha Y)(s)\right\|^2 \leq \overline{M}_T(\alpha)\int_0^t N\left(\sup_{r\in[0,s]} \mathbf{E}\left\|X(r) - Y(r)\right\|^2\right) ds,$$

so that **(A3)** would imply that $\sup_{t\in[0,T]} \mathbf{E}\left\|(\Phi_\alpha X)(t) - (\Phi_\alpha Y)(t)\right\|^2 = 0$. Consequently, $X = Y$ in $\mathfrak{H}_2$. Hence, Φ_α has a unique fixed point.

We now show that μ is, in fact, the probability law of X_μ employing the approach used in [1]. Toward this end, let $\mathfrak{L}(X_\mu) = \{\mathfrak{L}(X_\mu(t)) : t \in [0,T]\}$ denote the probability law of X_μ and define an operator Ψ on $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$ by $\Psi(\mu) =$

$\mathfrak{L}(X_\mu)$. We first prove that $\mathfrak{L}(X_\mu) \in C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$; that is, Ψ maps $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$ into itself. Indeed, note that since $X_\mu \in \mathfrak{H}_2$, $\mathfrak{L}(X_\mu(t)) \in \mathfrak{M}_{\lambda^2}(H)$, for any $t \in [0,T]$. As such, we need only show that $t \to \mathfrak{L}(X_\mu(t))$ is continuous. To do so, first let $-r \leq c \leq 0$ and $|h| > 0$ be small enough so that $-r \leq c+h \leq 0$. For all such c and h,

$$\mathbf{E}\left\|X_\mu(c+h) - X_\mu(c)\right\|^2 = \mathbf{E}\left\|\phi(c+h) - \phi(c)\right\|^2,$$

which approaches 0 as $h \to 0$ due to the sample path continuity of ϕ. Next, let $0 \leq c \leq T$ and observe that for sufficiently small $|h| > 0$, the continuity of X_μ, K and N ensures that

$$\lim_{h\to 0} \mathbf{E}\left\|X_\mu(c+h) - X_\mu(c)\right\|^2 = 0, \qquad \text{for all } -r \leq c \leq T.$$

Next, for all $c \in [-r,T]$ and $\varphi \in C_{\lambda^2}(H)$, the definition of the metric ρ yields

$$\begin{aligned} \left|\left\langle \varphi, \mathfrak{L}(X_\mu(c+h)) - \mathfrak{L}(X_\mu(c))\right\rangle\right| &= \left|\mathbf{E}\left[\varphi(X_\mu(c+h)) - \varphi(X_\mu(c))\right]\right| \\ &\leq \left\|\varphi\right\|_{C_{\lambda^2}} \mathbf{E}\left\|X_\mu(c+h) - X_\mu(c)\right\|. \end{aligned}$$

So, we may conclude that

$$\begin{aligned} &\lim_{t\to s} \rho\left(\mathfrak{L}(X_\mu(c+h)), \mathfrak{L}(X_\mu(c))\right) \\ &= \lim_{t\to s} \sup_{\|\varphi\|\leq 1} \int_H \varphi(x)\left(\mathfrak{L}(X_\mu(c+h)) - \mathfrak{L}(X_\mu(c))\right) dx = 0, \end{aligned}$$

thereby showing that $\mathfrak{L}(X_\mu) \in C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$. Secondly, we show that Ψ has a unique fixed point in $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$. If X is a fixed point of Φ_α, then clearly its probability law $\mathfrak{L}(X) = \mu$ is a fixed point of Ψ. Conversely, if μ is a fixed point of Ψ, then the variation of parameters formula (cf. Definition 3.1) parametrized by μ defines a solution X_μ which, in turn, has a probability law μ belonging to the space $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$. Thus, in order to complete the proof it suffices to show that the operator Ψ has a unique fixed point in $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$. To this end, let μ,ν be any two elements of $C([0,T];(\mathfrak{M}_{\lambda^2}(H),\rho))$ and let X_μ and X_ν be the corresponding fixed points of Φ_α. Using **(A2)**, in conjunction with the technique used to establish Lemma 3.3, we arrive at

$$\mathbf{E}\left\|X_\mu(t) - X_\nu(t)\right\|^2 \leq \overline{M}_T(\alpha) \int_0^t \left[N\left(\mathbf{E}\left\|X_\mu(s) - X_\nu(s)\right\|^2\right) + \rho^2(\mu(s),\nu(s))\right] ds.$$

Since $N(u)$ is concave on $[0,\infty)$, there exist positive constants a and b such that $N(u) \leq au + b$. So,

$$\begin{aligned}
&\mathbf{E}\left\|X_{\mu}(t)-X_{\nu}(t)\right\|^{2} \\
&\leq \overline{M}_{T}(\alpha) \int_{0}^{t}\left[a \mathbf{E}\left\|X_{\mu}(s)-X_{\nu}(s)\right\|^{2}+b+\rho^{2}(\mu(s), \nu(s))\right] d s \\
&\leq \overline{M}_{T}(\alpha) \int_{0}^{t}\left[b+\rho^{2}(\mu(s), \nu(s))\right] d s+a \overline{M}_{T}(\alpha) \int_{0}^{t} \mathbf{E}\left\|X_{\mu}(s)-X_{\nu}(s)\right\|^{2} d u,
\end{aligned}$$

from which it follows from an application of Gronwall's inequality that

$$\begin{aligned}
&\mathbf{E}\left\|X_{\mu}(t)-X_{\nu}(t)\right\|^{2} \\
&\leq \overline{M}_{T}(\alpha) \int_{0}^{t}\left[b+\rho^{2}(\mu(s), \nu(s))\right] d s \exp \left(a \overline{M}_{T}(\alpha) t\right) \\
&\leq \overline{M}_{T}(\alpha)\, b t \exp \left(a \overline{M}_{T}(\alpha) t\right)+\overline{M}_{T}(\alpha) \exp \left(a \overline{M}_{T}(\alpha) t\right) t D_{T}^{2}(\mu, \nu) .
\end{aligned}$$

For sufficiently small $t>0$, we have

$$\overline{M}_{T}(\alpha)\, b t \exp \left(a \overline{M}_{T}(\alpha) t\right)+\overline{M}_{T}(\alpha) \exp \left(a \overline{M}_{T}(\alpha) t\right) t D_{T}^{2}(\mu, \nu)<C D_{T}^{2}(\mu, \nu),$$

for some $0<C<1$. Thus, we are guaranteed the existence of $0<T_{1} \leq T$ such that

$$\mathbf{E}\left\|X_{\mu}(t)-X_{\nu}(t)\right\|^{2} \leq C D_{T}^{2}(\mu, \nu), \text { for all } t \in\left[0, T_{1}\right] .$$

Hence,

$$\|\Psi(\mu)-\Psi(\nu)\|_{C_{\lambda^{2}}}^{2}=D_{T}^{2}(\Psi(\mu), \Psi(\nu)) \leq\left\|X_{\mu}-X_{\nu}\right\|^{2}<C D_{T}^{2}(\mu, \nu),$$

for all $t \in\left[0, T_{1}\right]$. Thus, Ψ is a contraction on

$$C\left(\left[0, T_{1}\right] ;\left(\mathfrak{M}_{\lambda^{2}}(H), \rho\right)\right)$$

and therefore has a unique fixed point. As such, (1.1) has a unique mild solution on $\left[0, T_{1}\right]$ with probability distribution $\mu \in C\left(\left[0, T_{1}\right] ;\left(\mathfrak{M}_{\lambda^{2}}(H), \rho\right)\right)$. This procedure can be repeated in order to extend the solution, by continuity, to the entire interval $[0, T]$ in finitely many steps, thereby completing the proof. □

The following corollary follows immediately from Theorem 3.6 and provides a generalization and extension of the main existence result in [13].

Corollary 3.7. Assume that $B=0$ in the equation (1.1). Then, the equation (1.1) (without the control operator) has a unique mild solution.

We conclude from Theorem 3.6 that for any $\alpha>0$, the operator Φ_{α} has a unique fixed point which is clearly a mild solution of the following equation:

$$\begin{gathered}
x^{\alpha}(t)=S(t) \phi(0) \\
+\Gamma_{0}^{t} S^{*}(T-t)\left(\alpha I+\Gamma_{0}^{T}\right)^{-1}(\mathbf{E} h-S(T) \phi(0)) \\
+\int_{0}^{t}\left[S(t-s)-\Gamma_{s}^{t} S^{*}(T-t)\left(\alpha I+\Gamma_{s}^{T}\right)^{-1} S(T-s)\right] f\left(s, x^{\alpha}(\theta(s)), \mu(s)\right) d s
\end{gathered}$$

$$+\int_0^t \left[S(t-s) - \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\right] g\left(s, x^\alpha\left(\theta(s)\right), \mu(s)\right) dW(s)$$

$$+\int_0^t \Gamma_s^t S^*(T-t)\left(\alpha I + \Gamma_s^T\right)^{-1} \varphi(s) dW(s). \tag{3.10}$$

The next main result in this section concerning the approximate controllability of mild solutions of (1.1) can now be stated as follows:

Theorem 3.8. *Assume that conditions (A1)-(A7) hold. If the functions f and σ are uniformly bounded on their respective domains, and the semigroup $\{S(t) : t > 0\}$ is compact, then the system (1.1) is approximately controllable on $[0, T]$.*

Proof. It is easily follows from (3.10) that

$$X^\alpha(T) = h - \alpha\left(\alpha I + \Gamma_0^T\right)^{-1}\left(Eh - S(T)\phi(0)\right)$$

$$+\alpha\int_0^T \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s) f\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) ds$$

$$+\alpha\int_0^T \left(\alpha I + \Gamma_s^T\right)^{-1} S(T-s)\left[\sigma\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) - \varphi(s)\right] dW(s). \tag{3.11}$$

It follows from properties of f and σ that

$$\left\|f\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right)\right\|^2 + \left\|\sigma\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right)\right\|^2 \le N,$$

a.e. on $[0, T] \times \Omega$. Then, there exists a subsequence, still denoted by

$$\left\{f\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right), \sigma\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right)\right\},$$

which converges weakly to, say, $\{f(s), \sigma(s)\}$ in $H \times L_2^0$. The compactness of $\{S(t) : t > 0\}$ then implies that

$$\begin{cases} S(T-s) f\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) \to S(T-s) f(s), \\ S(T-s)\sigma\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) \to S(T-s)\sigma(s), \end{cases}$$

a.e on $[0, T] \times \Omega$. On the other hand, by assumption **(A5)**, for all $0 \le s < T$, $\alpha\left(\alpha I + \Gamma_s^T\right)^{-1} \to 0$ strongly as $\alpha \to 0^+$. Moreover, since $\left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\| \le 1$, it follows from (3.11) (using the Lebesgue dominated convergence theorem) that

$$E\left\|X^\alpha(T) - h\right\| \le 4\left\|\alpha\left(\alpha I + \Gamma_0^T\right)^{-1}\left(Eh - S(T)\phi(0)\right)\right\|^2$$

$$+4T\int_0^T \left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\|^2 \left\|S(T-s)\left[f\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) - f(s)\right]\right\|^2 ds$$

$$+4\int_0^T \left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\right\|^2 \left\|S(T-s)\left[\sigma\left(s, X^\alpha\left(\theta(s)\right), \mu(s)\right) - \sigma(s)\right]\right\|^2 ds$$

$$+4\int_0^T \left\|\alpha\left(\alpha I + \Gamma_s^T\right)^{-1}\varphi(s)\right\|^2 ds,$$

as $\alpha \to 0^+$, thereby establishing the approximate controllability of (1.1). □

Corollary 3.9. Assume that (A1), (A5), and (A6) hold. If the functions f and σ are uniformly bounded and globally Lipschitz on their respective domains, and the semigroup $\{S(t) : t > 0\}$ is compact, then the system (1.1) is approximately controllable on $[0, T]$.

Remark 3.10. Theorem 3.8 is new even when $\theta(t) = t$, and it provides a generalization of the results presented in [19], [20].

4. APPLICATIONS

Example 4.1 Let D be a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂D. Consider the following initial boundary value problem.

$$\frac{\partial}{\partial t}x(t,z) = \Delta_z x(t,z) + F_1(t,z,x(t-r,z)) + \int_{L^2(D)} F_2(t,z,y)\mu(t,z)(dy)$$
$$+g(s,z,x(s-r,z))d\beta(s), \quad \text{a.e. on } (0,T)\times D$$
$$x(t,z) = 0, \text{ a.e. on } (0,T)\times \partial D,$$
$$x(t,z) = \psi(t,z), \ -r \le t \le 0, \text{ a.e. on } \partial D, \tag{4.1}$$

where $x : [0,T]\times D \to \mathbb{R}$, $F_1 : [0,T]\times D\times\mathbb{R} \to \mathbb{R}$, $F_2 : [0,T]\times D\times L^2(D) \to L^2(D)$, $\mu(t,\cdot) \in \mathfrak{M}_{\lambda^2}(L^2(D))$ is the probability law of $\mu(t,\cdot)$, $a : \Delta \to \mathbb{R}$, $g : [0,T]\times D\times\mathbb{R} \to L_2^0(\mathbb{R}^N, L^2(D))$, β is a standard $N-$dimensional Brownian motion, and $\psi : [0,T]\times D \to \mathbb{R}$. We impose the following conditions:

(A8): F_1 satisfies the Carathéodory conditions (i.e., measurable in (t,z) and continuous in the third variable) such that

- **(i):** $|F_1(t,z,y)| \le \xi_1(z)\Gamma_1(y)$, for all $0 \le t \le T$, $z \in D, y \in \mathbb{R}$, where
  - **(a):** $\xi_1 \in L^2(D)$,
  - **(b):** $\Gamma_1(\cdot)$ is a continuous, concave, nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\Gamma_1(0) = 0$, $\Gamma_1(y) > 0$ for $y > 0$, and $\int_{0^+} \frac{1}{\Gamma_1(y)} = +\infty$,
- **(ii):** $|F_1(t,z,y_1) - F_1(t,z,y_2)| \le \overline{\xi_1}(z)\overline{\Gamma_1}(|y_1 - y_2|)$, for all $0 \le t \le T$, $z \in D, y_1, y_2 \in \mathbb{R}$, where $\overline{\xi_1}$ and $\overline{\Gamma_1}$satisfy (i) (a) and (b), respectively.

(A9): F_2 satisfies the Carathéodory conditions and

- **(i):** $\|F_2(t,y,z)\|_{L^2(D)} \le \bar{M}_{F_2}\left[1 + \|z\|_{L^2(D)}\right]$, for all $0 \le t \le T, y \in D, z \in L^2(D)$, and some $\bar{M}_{F_2} > 0$,
- **(ii):** $F_2(t,y,\cdot) : L^2(D) \to L^2(D)$ is in C, for each $0 \le t \le T, y \in D$.

(A10): g satisfies the Caratheodory conditions and

- **(i):** $\|g(t,z,y)\|_{L_2^0(\mathbb{R}^N,L^2(D))} \le \xi_2(z)\Gamma_2(y)$, for all $0 \le t \le T$, $z \in D, y \in \mathbb{R}$, where ξ_2 and Γ_2 satisfy (A8)(i) (a) and (b), respectively.
- **(ii):** $\|g(t,z,y_1) - g(t,z,y_2)\|_{L_2^0(\mathbb{R}^N,L^2(D))} \le \overline{\xi_2}(z)\overline{\Gamma_2}(|y_1 - y_2|)$, for all $0 \le t \le T$, $z \in D$, $y_1, y_2 \in \mathbb{R}$, where $\overline{\xi_2}$ and $\overline{\Gamma_2}$satisfy (A8)(i) (a) and (b), respectively.

(A11): ψ is an $\mathfrak{F}_0-$measurable random variable independent of β with almost surely continuous paths.

We have the following theorem:

Theorem 4.1. *If (A8)-(A11) are satisfied, then (4.1) has a unique mild solution* $x \in C([-r,T]; L^2(\Omega, L^2(D)))$ *with probability law* $\{\mu(t,\cdot) : 0 \le t \le T\}$.

Proof. Let $H = L^2(D)$ and $K = \mathbb{R}^N$ and denote $\frac{\partial x}{\partial t}$ by $x'(t)$. Define the operator

$$Ax(t,\cdot) = \Delta_z x(t,\cdot), \quad x \in H^2(D) \cap H_0^1(D). \tag{4.2}$$

It is known that A generates a strongly continuous semigroup $\{S(t)\}$ on $L^2(D)$ (see [22]). Define the maps $f : [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H) \to H$, $\sigma : [0,T] \times H \times \mathfrak{M}_{\lambda^2}(H) \to L_2^0(K,H)$, and $\phi : [0,T] \times D \to \mathbb{R}$ respectively by

$$f(t, x(\theta(t)), \mu(t))(z) = F_1(t, z, x(t-r,z)) + \int_{L^2(D)} F_2(t,z,y)\mu(t,z)(dy), \tag{4.3}$$

$$\sigma(t, x(\theta(t)), \mu(t))(z) = g(t, z, x(t-r,z)), \tag{4.4}$$

$$\phi(t,z) = \psi(t)(z), \tag{4.5}$$

for all $0 \le t \le T$, $z \in D$, and $x(\theta(t)) \in H$. Taking $B \equiv 0$, observe that these identifications enable us to view (4.1) in the abstract form (1.1). It is easy to see that (A1), (A6), and (A7) are satisfied. We now show that f and σas defined in (4.3) and (4.4) satisfy (A2) and (A3). To this end, observe that from (A8)(i), it follows (with the help of Jensen's inequality) that

$$\begin{aligned} \|F_1(t,\cdot,x(\theta(t),\cdot)\|_{L^2(D)}^2 &\le \|\xi_1(\cdot)\Gamma_1(x(\theta(t),\cdot))\|_{L^2(D)}^2 \\ &\le m(D)\,\|\xi_1(\cdot)\|_{L^2(D)}^2\,\|\Gamma_1(x(\theta(t),\cdot))\|_{L^2(D)}^2 \\ &\le \widehat{\alpha_1}\Gamma_1\left(\|x(\theta(t),\cdot)\|_{L^2(D)}^2\right), \end{aligned} \tag{4.6}$$

for all $0 \le t \le T$, $x(\theta(t)) \in H$, where $\widehat{\alpha_1} = m(D)\,\|\xi_1(\cdot)\|_{L^2(D)}^2$. (Here, m denotes Lebesgue measure in $\mathbb{R}^N$.) A similar computation (with the help of (A8)(ii)) enables us to conclude that

$$\|F_1(t,\cdot,x(\theta(t),\cdot) - F_1(t,\cdot,y(\theta(t),\cdot)\|_{L^2(D)} \le \widehat{\beta}_1\overline{\Gamma_1}\left(\|x(\theta(t),\cdot) - y(\theta(t),\cdot)\|_{L^2(D)}^2\right), \tag{4.7}$$

for all $0 \le t \le T$, $x(\theta(t))$, $y(\theta(t)) \in H$, where $\widehat{\beta}_1 = m(D)\left\|\overline{\xi_1}(\cdot)\right\|_{L^2(D)}^2$. Next, using (A9)(i) together with Hölder, we observe that

$$\left\|\int_{L^2(D)} F_2(t,\cdot,y)\mu(t,\cdot)(dy)\right\|_{L^2(D)} = \left[\int_D \left[\int_{L^2(D)} F_2(t,z,y)\mu(t,z)(dy)\right]^2 dz\right]^{\frac{1}{2}}$$

$$\le \left[\int_D \int_{L^2(D)} \|F_2(t,z,y)\|^2_{L^2(D)} \mu(t,z)(dy)dz\right]^{\frac{1}{2}}$$

$$\le \bar{M}_{F_2}\left[\int_D \left(\int_{L^2(D)} (1+\|y\|_{L^2(D)})^2 \mu(t,z)(dy)\right) dz\right]^{\frac{1}{2}}$$

$$\le \bar{M}_{F_2}\sqrt{m(D)}\sqrt{\|\mu(t)\|_{\lambda^2}}$$

$$\le \bar{M}_{F_2}\sqrt{m(D)}(1+\|\mu(t)\|_{\lambda^2}), \tag{4.8}$$

for all $0 \le t \le T$, $\mu \in \mathfrak{M}_{\lambda^2}(H)$. Also, invoking (A9)(ii) enables us to see that for all $\mu, \nu \in \mathfrak{M}_{\lambda^2}(H)$,

$$\left\|\int_{L^2(D)} F_2(t,\cdot,y)\mu(t,\cdot)(dy) - \int_{L^2(D)} F_2(t,\cdot,y)\nu(t,\cdot)(dy)\right\|_{L^2(D)}$$

$$= \left\|\int_{L^2(D)} F_2(t,\cdot,y)(\mu(t,\cdot)-\nu(t,\cdot))(dy)\right\|_{L^2(D)}$$

$$\le \|\rho(\mu(t),\nu(t))\|_{L^2(D)}$$

$$\le \sqrt{m(D)}\rho(\mu(t),\nu(t)), \tag{4.9}$$

for all $0 \le t \le T$. Along similar lines, we use (A9) (i) and (ii) to obtain the following estimates on g:

$$\|g(t,\cdot,x(\theta(t),\cdot)\|^2_{L^0_2(K,L^2(D))} \le \widehat{\alpha_2}\Gamma_2\left(\|x(\theta(t),\cdot)\|^2_{L^2(D)}\right), \tag{4.10}$$

$$\|g(t,\cdot,x(\theta(t),\cdot) - g(t,\cdot,y(\theta(t),\cdot)\|_{L^0_2(K,L^2(D))}$$

$$\le \widehat{\beta_2}\overline{\Gamma_2}\left(\|x(\theta(t),\cdot) - y(\theta(t),\cdot)\|^2_{L^2(D)}\right), \tag{4.11}$$

where $\widehat{\alpha_2} = m(D)\|\xi_2(\cdot)\|^2_{L^2(D)}$and $\widehat{\beta_2} = m(D)\left\|\overline{\xi_2}(\cdot)\right\|^2_{L^2(D)}$. Now, define the mappings $K : [0,\infty)\times[0,\infty) \to [0,\infty)$ and $N : [0,\infty) \to [0,\infty)$ by

$$K(w_1,w_2) = \widehat{\alpha_1}\Gamma_1\left(w_1^2\right) + \widehat{\alpha_2}\Gamma_2\left(w_1^2\right) + \bar{M}^2_{F_2}m(D)(1+w_2^2), \tag{4.12}$$

$$N(w_1) = \widehat{\beta_1}\overline{\Gamma_1}\left(w_1^2\right) + \widehat{\beta_2}\overline{\Gamma_2}\left(w_1^2\right). \tag{4.13}$$

One can show that these two mappings satisfy the criterion in (A2) – (A4) (see [8]). (The reader can find particular examples of the mappings Γ_i in [8] as well.) Further, combining (4.6) - (4.9), we see that $f$and σsatisfy (A2). Thus, we can invoke Theorem 3.6 to conclude that (4.1) has a unique mild solution $x \in C([-r,T];L^2(\Omega,L^2(D)))$ with probability law $\{\mu(t,\cdot) : 0 \le t \le T\}$. □

Example 4.2 Consider the following initial-boundary value problem of Sobolev type:

$$\frac{\partial}{\partial t}\left(x(t,z)-x_{zz}(t,z)\right)-x_{zz}(t,z)=F_1(t,z,x(t-r,z))+\int_{L^2(0,\pi)}F_2(t,z,y)\mu(t,z)(dy)$$
$$+g\left(s,z,x(s-r,z)\right)dW(s),\quad 0\le z\le\pi,\ 0\le t\le T,$$
$$x(t,0)=x(t,\pi)=0,\ \ 0\le t\le T,$$
$$x(t,z)=\psi(t,z),\ \ 0\le z\le\pi,\ \ -r\le t\le 0, \tag{4.14}$$

where $x:[0,T]\times[0,\pi]\to\mathbb{R}$, $F_1:[0,T]\times[0,\pi]\times\mathbb{R}\to\mathbb{R}$, $F_2:[0,T]\times[0,\pi]\times L^2(0,\pi)\to L^2(0,\pi)$, $\mu(t,\cdot)\in\mathfrak{M}_{\lambda^2}(L^2(0,\pi))$is the probability law of $\mu(t,\cdot)$, $g:[0,T]\times[0,\pi]\times\mathbb{R}\to L_2^0(\mathbb{R}^N,L^2(0,\pi))$, W is a standard $L^2(0,\pi)-$valued Wiener process, and $\psi:[0,T]\times[0,\pi]\to\mathbb{R}$ are mappings satisfying (A8) – (A11) (in the appropriate spaces). We have the following theorem.

Theorem 4.2. *Under the above assumptions, (4.14) has a unique mild solution* $x\in C\left([-r,T];L^2(\Omega,L^2(0,\pi))\right)$ *with probability law* $\{\mu(t,\cdot):0\le t\le T\}$.

Proof. Let $H=L^2(0,\pi),K=\mathbb{R}$, and define the operators $A:D(A)\subset H\to H$ and

$$B:D(B)\subset H\to H,$$

respectively, by

$$Ax(t,\cdot)=-x_{zz}(t,\cdot),\quad Bx(t,\cdot)=x(t,\cdot)-x_{zz}(t,\cdot),$$

with domains

$$D(A)=D(B)=\big\{x\in L^2(0,\pi):x,x_z\text{are absolutely continuous,}$$
$$x_{zz}\in L^2(0,\pi),\,x(0)=x(\pi)=0\big\}.$$

Define $f,\sigma,$and ϕ as in Example 4.1 (with $L^2(0,\pi)$ in place of $L^2(D)$). Then, (4.14) can be written in the abstract form

$$(Bx(t))'+Ax(t)=f\left(t,X\left(\theta\left(t\right)\right),\mu\left(t\right)\right)dt$$
$$+\sigma\left(t,X\left(\theta\left(t\right)\right),\mu\left(t\right)\right)dW\left(t\right),\ 0\le t\le T, \tag{4.15}$$
$$x(t)=\phi(t),\quad -r\le t\le 0.$$

Upon making the substitution $v(t)=Bx(t)$ in (4.15), we arrive at the equivalent problem

$$v'(t)+AB^{-1}v(t)=f\left(t,B^{-1}v\left(\theta\left(t\right)\right),\mu\left(t\right)\right)dt$$
$$+\sigma\left(t,B^{-1}v\left(\theta\left(t\right)\right),\mu\left(t\right)\right)dW\left(t\right),\quad 0\le t\le T, \tag{4.16}$$
$$v(t)=B\phi(t),\quad -r\le t\le 0.$$

It is known that B is a bijective operator possessing a continuous inverse and that $-AB^{-1}$ is a bounded linear operator on $L^2(0,\pi)$ which generates a strongly continuous semigroup $\{T(t)\}$ on $L^2(0,\pi)$ satisfying (A1) with $M_T = \alpha = 1$ (see [22]). Further, f and σ are shown to satisfy (A2) as in Example 4.1. Consequently, we can again invoke Theorem 3.6 to conclude (4.16) has a unique mild solution $v \in C\left([-r,T]\,;L^2(\Omega, L^2(0,\pi))\right)$. Consequently, $x = B^{-1}v$ is the corresponding mild solution of (4.15) and hence, of (4.14). □

Remark 4.3. This example provides a generalization of the work in [8], [18], [24], [27] to the stochastic setting. Such initial-boundary value problems arise naturally in the mathematical modelling of various physical phenomena (e.g., thermodynamics [9], shear in second-order fluids [27], fluid flow through fissured rocks [4], and consolidation of clay [27]).

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