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# A BOUNDARY VALUE PROBLEM ON THE HALF-LINE VIA CRITICAL POINT METHODS

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**ABSTRACT.** The aim of this paper is to establish some results on the existence of multiple solutions for a second order boundary value problem on the half-line. Our technique is based on critical point arguments.

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## 1. INTRODUCTION

Boundary value problems on the half-line arise quite naturally in various branches of applied mathematics. However, the general theory on the semi-infinite interval is not well developed and most of the known results require rather technical assumptions, which are difficult to verify in concrete applications. The main tools used in the literature to guarantee existence or multiplicity of solutions for such problems are fixed point arguments together with the lower and upper solution method. In this paper using variational methods we obtain multiple solutions for a second order boundary value problem on the half line under very simple assumptions (see Theorem 3.3).

Consider the boundary value problem

$$(P_\lambda) \qquad \begin{cases} -y'' + m^2 y = \lambda f(t,y) \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

where $m$ is a non zero constant, $\lambda$ is a real parameter, and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is an $L^2-$Carathéodory function, that is

(a) $t \to f(t,y)$ is measurable for every $y \in \mathbb{R}$;
(b) $y \to f(t,y)$ is continuous for almost every $t \in [0,\infty)$;

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(c) for every $\rho > 0$ there exists a function $l_\rho \in L^2([0,\infty))$ such that

$$\sup_{|y|\leq\rho} |f(t,y)| \leq l_\rho(t)$$

for almost every $t \in [0,\infty)$.

We refer to [5, Chapter 13] and the references therein, for more details on problem $(P_\lambda)$. Our aim is to establish a precise open interval $\Lambda \subseteq (0,\infty)$ such that, for each $\lambda \in \Lambda$, problem $(P_\lambda)$ admits at least three generalized solutions.

The present paper is arranged as follows. Section 2 is devoted to preliminary results and basic properties on the functionals which are useful for our ends. In Section 3 we establish multiplicity results for nonlinear boundary problems on the semi-infinite interval.

## 2. PRELIMINARIES

Let $W^{1,2}([0,\infty))$ be the Sobolev space endowed with the norm

$$\|u\| := \left(\int_0^\infty |u'(t)|^2 dt + m^2 \int_0^\infty |u(t)|^2 dt\right)^{1/2},$$

which is equivalent to the usual one. As is usual, we denote by $W_0^{1,2}([0,\infty))$ the closure of $C_0^1([0,\infty))$ in $W^{1,2}([0,\infty))$. Now, we recall some properties on the previous Sobolev spaces that can be deduced easily from [3, Chapter VIII].

**Proposition 2.1.** *Let $u \in W^{1,2}([0,\infty))$. Then,*

$$u \in W_0^{1,2}([0,\infty)) \text{ if and only if } u(0) = 0.$$

**Proposition 2.2.** *Let $u \in W^{1,2}([0,\infty))$. Then, $\lim_{t\to\infty} u(t) = 0$.*

**Proposition 2.3.** *One has*

$$W^{1,2}([0,\infty)) \subseteq BC([0,\infty)),$$

*where $BC([0,\infty))$ is the set of all functions $u : [0,\infty) \to \mathbb{R}$ which are bounded and continuous. Moreover,*

$$\max_{t\in[0,\infty)} |u(t)| \leq \max\left\{1; \frac{1}{|m|}\right\} \|u\| \tag{2.1}$$

*for all $u \in W^{1,2}([0,\infty))$.*

A function $u : [0,\infty) \to \mathbb{R}$ is said a generalized solution to $(P_\lambda)$ if $u \in C^1([0,\infty))$, $u' \in AC_{loc}([0,\infty))$, $-u''(t)+m^2u(t) = \lambda f(t,u(t))$ for almost every $t \in [0,\infty)$, $u(0) = 0$ and $\lim_{t\to\infty} u(t) = 0$. Also a function $u : [0,\infty) \to \mathbb{R}$ is said a weak solution to $(P_\lambda)$ if $u \in W_0^{1,2}([0,\infty))$ and

$$\int_0^\infty u'(t)v'(t)\,dt + m^2 \int_0^\infty u(t)v(t)\,dt = \lambda \int_0^\infty f(t,u(t))v(t)\,dt$$

for every $v \in W_0^{1,2}([0,\infty))$.

We explicitly observe that the weak solution is well-defined.

**Proposition 2.4.** *Generalized and weak solutions to $(P_\lambda)$ coincide.*

*Proof.* Assume that $u$ is a weak solution to $(P_\lambda)$. Proposition 2.1 and Proposition 2.2 guarantee that the function $u$ satisfies the boundary conditions. Moreover, from definition of weak solution it follows that $u'$ is weakly differentiable and its derivative is $m^2u(t) - \lambda f(t,u(t)) \in L^2([0,\infty))$. Hence, $u' \in W^{1,2}([0,\infty))$ and the conclusion follows.

Now assume that $u$ is a generalized solution to $(P_\lambda)$. Arguing as in [3, Chapter VIII, Example 8] we see that $u \in W_0^{1,2}([0,\infty))$ and a standard argument guarantees that $u$ is a weak solution to $(P_\lambda)$. ☐

**Remark 2.5.** *Every weak solution $u$ for problem $(P_\lambda)$ satisfies*

$$\lim_{t\to\infty} u'(t) = 0;$$

*this can be deduced from the proof of Proposition 2.4, taking into account Proposition 2.2.*

*We also observe that if the $L^2-$Carathéodory function $f$ is continuous in $[0,\infty)\times\mathbb{R}$, then the weak solutions are also classical solutions, namely $u \in C^2([0,\infty))$ and $-u''(t) + m^2u(t) = \lambda f(t,u(t))$ for every $t \in [0,\infty)$.*

Here and in the sequel, $F : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is the function defined by

$$F(t,z) = \int_0^z f(t,y)\,dy.$$

Clearly, one has

(a) $t \to F(t,z)$ is measurable for every $z \in \mathbb{R}$;
(b) $z \to F(t,z) \in C^1(\mathbb{R})$ and $F'_z(t,y) = f(t,y)$, for almost every $t \in [0,\infty)$.

Moreover, one also has

(c) for every $\rho > 0$ there exists a function $L_\rho \in L^2([0,\infty))$ such that

$$\sup_{|z|\le\rho} |F(t,z)| \le L_\rho(t)$$

for almost every $t \in [0,\infty)$.

In fact, $|F(t,z)| = |\int_0^z f(t,y)\,dy| \le |z| \sup_{|y|\le\rho} |f(t,y)| \le \rho l_\rho(t) := L_\rho(t)$ for almost every $t \in [0,\infty)$ and for all $|z| \le \rho$.

We now put

$$\Psi(u) = \int_0^\infty F(t,u(t))\,dt$$

for all $u \in W_0^{1,2}([0,\infty))$.

By standard methods, we have the following result.

**Proposition 2.6.** *The functional* $\Psi : W_0^{1,2}([0,\infty)) \to \mathbb{R}$ *is well-defined. Moreover, it is Gâteaux differentiable and its Gâteaux derivative is*

$$\Psi'(u)(v) = \int_0^\infty f(t,u(t))v(t)\,dt$$

*for all* $u, v \in W_0^{1,2}([0,\infty))$.

If we assume that the function $f$ satisfies the further condition

(c') there exists a function $l \in L^2([0,\infty))$ such that

$$\sup_{y\in\mathbb{R}} |f(t,y)| \le l(t) \tag{2.2}$$

for almost every $t \in [0,\infty)$,

we have the following result.

**Proposition 2.7.** *Assume that condition* (2.2) *holds. Then* $\Psi' : W_0^{1,2}([0,\infty)) \to \left(W_0^{1,2}([0,\infty))\right)^*$ *is a compact operator. In particular,* $\Psi : W_0^{1,2}([0,\infty)) \to \mathbb{R}$ *is a weakly sequentially continuous functional.*

*Proof.* Let $X$ be a bounded set in $W_0^{1,2}([0,\infty))$ and let $\{\alpha_n\}$ be a sequence in $\overline{\Psi'(X)}$. Then there is a sequence $\{u_n\}$ in $X$ such that $\beta_n = \Psi'(u_n)$ and $\|\alpha_n - \beta_n\|_{\left(W_0^{1,2}([0,\infty))\right)^*} < \frac{1}{n}$ for all $n \in \mathbb{R}$. Since $W_0^{1,2}([0,\infty))$ is reflexive, there is a subsequence $u_{n_k}$ converging weakly to $u \in W_0^{1,2}([0,\infty))$. Therefore, arguing as in [4, Lemma 1.1], $\{u_n\}$ has a subsequence, which without loss of generality we again call $\{u_{n_k}\}$, which converges everywhere in $[0,\infty)$ to the function $u$. Hence, $\{f(t,u_{n_k}(t))\}$ converges to $\{f(t,u(t))\}$ a.e. on $[0,\infty)$. Now, taking into account (2.1), one has
$|\Psi'(u_{n_k})(v) - \Psi'(u)(v)| \le \int_0^\infty |f(t,u_{n_k}(t)) - f(t,u(t))||v(t)|\,dt \le$
$\left(\int_0^\infty |f(t,u_{n_k}(t)) - f(t,u(t))|^2\,dt\right)^{\frac{1}{2}} \left(\int_0^\infty |v(t)|^2\,dt\right)^{\frac{1}{2}} \le$
$\le \frac{1}{|m|}\left(\int_0^\infty |f(t,u_{n_k}(t)) - f(t,u(t))|^2\,dt\right)^{\frac{1}{2}}$ for all $v \in W_0^{1,2}([0,\infty))$ such that $\|v\| \le 1$. Hence, from (2.2) and Lebesgue Dominated Convergence Theorem, the sequence $\{\Psi'(u_{n_k})\}$ converges to $\Psi'(u)$ in $\left(W_0^{1,2}([0,\infty))\right)^*$.

Therefore, taking into account that

$$\|\alpha_{n_k} - \Psi'(u)\|_{\left(W_0^{1,2}([0,\infty))\right)^*} \le \|\alpha_{n_k} - \beta_{n_k}\|_{\left(W_0^{1,2}([0,\infty))\right)^*} + \|\beta_{n_k} - \Psi'(u)\|_{\left(W_0^{1,2}([0,\infty))\right)^*},$$

the sequence $\{\alpha_{n_k}\}$ converges in $\overline{\Psi'(X)}$ and the compactness is proved.

Finally, from [8, Corollary 41.9, page 236] we obtain the other conclusion. ☐

**Remark 2.8.** *Clearly, under the condition* (2.2), $\Psi : W_0^{1,2}([0,\infty)) \to \mathbb{R}$ *is a continuously Gâteaux differentiable functional. In fact, arguing in a similar way as in the proof of Proposition 2.7, we see that* $\Psi' : W_0^{1,2}([0,\infty)) \to \big(W_0^{1,2}([0,\infty))\big)^*$ *is a continuous operator.*

Now, we define the functional $\Phi : W_0^{1,2}([0,\infty)) \to \mathbb{R}$ by putting for every $u \in W_0^{1,2}([0,\infty))$

$$\Phi(u) := \frac{1}{2}\|u\|^2.$$

Clearly, $\Phi$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in W_0^{1,2}([0,\infty))$ is the functional $\Phi'(u) \in \big(W_0^{1,2}([0,\infty))\big)^*$ given by

$$\Phi'(u)(v) = \int_0^\infty u'(t)v'(t)\,dt + m^2\int_0^\infty u(t)v(t)\,dt$$

for every $v \in W_0^{1,2}([0,\infty))$, and $\Phi' : W_0^{1,2}([0,\infty)) \to \big(W_0^{1,2}([0,\infty))\big)^*$ is continuous. Moreover, since $\Phi$ is convex, from [7, Proposition 25.20 (i), page 514] we see that $\Phi$ is a sequentially weakly lower semicontinuous functional. Finally, since $\Phi'$ is uniformly monotone, from [7, Theorem 26.A (d), page 557] it admits a continuous inverse on $\big(W_0^{1,2}([0,\infty))\big)^*$.

The main tool to prove our results in Section 3 is the three critical points theorem below.

**Theorem A.** ([1, Theorem B]) *Let $X$ be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, and $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0,+\infty[$;

(ii) *there is $r \in \mathbb{R}$ such that:*

$$\inf_X \Phi < r,$$

*and*

$$\varphi_1(r) < \varphi_2(r),$$

*where*

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi}{r - \Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

*and* $\overline{\Phi^{-1}(]-\infty,r[)}^w$ *is the closure of* $\Phi^{-1}(]-\infty,r[)$ *in the weak topology.*

*Then, for each* $\lambda \in ]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$*, the functional* $\Phi + \lambda\Psi$ *has at least three critical points in* $X$*.*

Note $\varphi_1(r)$ in Theorem A could be 0. In this and similar cases, here and in the sequel, we agree to read $\frac{1}{0}$ as $+\infty$.

We also use the following two critical points theorem.

**Theorem B.** ([2, Theorem 1.1]) *Let* $X$ *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* $\Phi$ *is (strongly) continuous and satisfies* $\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$*. Assume also that there exist two constants* $r_1$ *and* $r_2$ *such that*

(j) $\inf_X \Phi < r_1 < r_2$;
(jj) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;
(jjj) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$,

*where* $\varphi_1$ *is defined as in Theorem A and*

$$\varphi_2^*(r_1, r_2) := \inf_{x\in\Phi^{-1}(]-\infty,r_1[)} \sup_{y\in\Phi^{-1}([r_1,r_2[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

*Then, for each* $\lambda \in \left]\frac{1}{\varphi_2^*(r_1,r_2)}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\right\}\right[$*, the functional* $\Phi + \lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(]-\infty, r_1[)$ *and* $\Phi^{-1}([r_1, r_2[)$ *respectively.*

We recall that Theorem A and Theorem B are based on the variational principle of B.Ricceri in [6].

## 3. RESULTS

Our main result is the following theorem.

**Theorem 3.1.** *Let* $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *be an* $L^2-$*Carathéodory function such that (2.2) holds. Put* $F(t, z) = \int_0^z f(t, y)\, dy$ *for all* $(t, z) \in [0, \infty) \times \mathbb{R}$, $\overline{m} = \min\{1; |m|\}$*, and assume that there exist two positive constants* $c$ *and* $d$ *such that:*

(1) $c^2 < \frac{2}{\overline{m}^2}(d + \frac{4}{3}m^2 d^3)$;
(2) $\int_0^d F(t, t)\, dt \geq 0$ *and* $\int_{3d}^{4d} F(t, 4d - t)\, dt \geq 0$;
(3) $\frac{2}{\overline{m}^2}\frac{\left(\int_0^\infty \left|\sup_{|z|\leq c} |f(t, z)|\right|^2 dt\right)^{\frac{1}{2}}}{c} < \frac{\left(\int_d^{3d} F(t,d)\, dt - \left(\int_0^\infty \left|\sup_{|z|\leq c} |f(t, z)|\right|^2 dt\right)^{\frac{1}{2}} c\right)}{d+\frac{4}{3}m^2 d^3}$.

*Then, for each*

$$\lambda \in \left] \frac{d+\frac{4}{3}m^2d^3}{\int_d^{3d} F(t,d)\,dt - \left(\int_0^\infty \left|\sup_{|z|\le c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c}, \frac{\overline{m}^2 c}{2\left(\int_0^\infty \left|\sup_{|z|\le c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}} \right[,$$

*the problem $(P_\lambda)$ admits at least three generalized solutions.*

*Proof.* Let $X$ be the Sobolev space $W_0^{1,2}([0,\infty))$ endowed with the norm $\|u\| := \left(\int_0^\infty |u'(t)|^2 dt + m^2 \int_0^\infty |u(t)|^2 dt\right)^{1/2}$. For each $u \in X$, put:

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := -\int_0^\infty F(t,u(t))dt.$$

As seen in Section 2, the critical points in $X$ of the functional $\Phi+\lambda\Psi$ are precisely the generalized solutions of problem $(P_\lambda)$. Our goal is to apply Theorem A to $\Phi$ and $\Psi$. We see from Section 2 that it is enough to show $\Phi$ and $\Psi$ satisfy (i) and (ii) in Theorem A. We note that (2.2) implies

$$\lim_{\|u\|\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$.

Now it remains to show (ii) of Theorem A. Let

$$r := \frac{\overline{m}^2 c^2}{2}$$

and

$$\overline{y}(t) := \begin{cases} t & \text{if } t \in [0,d[ \\ d & \text{if } t \in [d,3d[ \\ 4d-t & \text{if } t \in [3d,4d[ \\ 0 & \text{if } t \in [4d,\infty). \end{cases}$$

Clearly, one has $\overline{y} \in X$, $\Phi(\overline{y}) = d + \frac{4}{3}m^2d^3$ and, taking into account (2),

$\Psi(\overline{y}) \le -\int_d^{3d} F(t,d)\,dt$.

Therefore, from (1) one has $\Phi(\overline{y}) > r$. Moreover, taking into account (2.1), one has

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \int_0^\infty F(t,u(t))\,dt \le \sup_{u\in\Phi^{-1}((-\infty,r])} \int_0^\infty \sup_{|z|\le c}|f(t,z)||u(t)|\,dt \le$$

$$\le \sup_{u\in\Phi^{-1}((-\infty,r])} \left(\int_0^\infty \left|\sup_{|z|\le c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty |u(t)|^2\,dt\right)^{\frac{1}{2}} \le$$

$$\le \left(\int_0^\infty \left|\sup_{|z|\le c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c.$$

Hence, taking into account that $\Phi(0) = \Psi(0) = 0$, $\overline{\Phi^{-1}(]-\infty, r[)}^{w} = \Phi^{-1}((-\infty, r])$ and that the previous inequality holds, one has

$$\varphi_1(r) = \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi}{r-\Phi(x)} \leq \frac{-\inf_{\overline{\Phi^{-1}(]-\infty,r[)}^{w}} \Psi}{r}$$

$$\leq \frac{\sup_{u\in\Phi^{-1}((-\infty,r])} \int_0^\infty F(t,u(t))\,dt}{r} \leq \frac{2}{\overline{m}^2} \frac{\left(\int_0^\infty \left|\sup_{|z|\leq c} |f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}{c}.$$

On the other hand, one has

$$\varphi_2(r) = \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x)-\Psi(y)}{\Phi(y)-\Phi(x)} \geq \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x)-\Psi(\overline{y})}{\Phi(\overline{y})-\Phi(x)} \geq$$

$$\frac{\inf_{x\in\Phi^{-1}(]-\infty,r[)} \Psi(x)-\Psi(\overline{y})}{\Phi(\overline{y})-\Phi(x)} \geq \frac{-\sup_{x\in\Phi^{-1}(]-\infty,r[)} \int_0^\infty F(t,x(t))\,dt + \int_d^{3d} F(t,d)\,dt}{\Phi(\overline{y})-\Phi(x)} \geq$$

$$\frac{\int_d^{3d} F(t,d)\,dt - \left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c}{\Phi(\overline{y})-\Phi(x)} \geq \frac{\int_d^{3d} F(t,d)\,dt - \left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c}{\frac{1}{2}\|\overline{y}\|^2}$$

$$= \frac{1}{d+\frac{4}{3}m^2d^3} \left(\int_d^{3d} F(t,d)\,dt - \left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c\right).$$

Hence from (3) one has

$$\varphi_1(r) < \varphi_2(r).$$

Therefore, from Theorem A, taking also into account that

$$\frac{1}{\varphi_2(r)} \leq \frac{d+\frac{4}{3}m^2d^3}{\int_d^{3d} F(t,d)\,dt - \left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}} c} \text{ and}$$

$$\frac{1}{\varphi_1(r)} \geq \frac{\overline{m}^2 c}{2\left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}},$$ we obtain the desired conclusion. □

**Remark 3.2.** *When (1) of Theorem 3.1 holds, simple calculations show that the condition*

$$\text{(3')} \qquad \frac{2}{\overline{m}^2} \frac{\left(\int_0^\infty \left|\sup_{|z|\leq c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}{c} < \frac{\int_d^{3d} F(t,d)\,dt}{2\left(d+\frac{4}{3}m^2d^3\right)}$$

*implies (3) of Theorem 3.1. Hence, if (1), (2) and (3') hold, then for each* $\lambda \in$

$$\Lambda = \left] \frac{2\left(d+\frac{4}{3}m^2d^3\right)}{\int_d^{3d} F(t,d)\,dt}, \frac{\overline{m}^2 c}{2\left(\int_0^\infty \left|\sup_{|z|\le c}|f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}} \right[,$$

*the problem* $(P_\lambda)$ *admits three generalized solutions. Moreover, if* $f(t,x) = \alpha(t)g(x)$*, (3') becomes*

$$(3")\qquad \frac{2\|\alpha\|_2}{\overline{m}^2}\frac{\sup_{|x|\le c}|g(x)|}{c} < \left(\int_d^{3d}\alpha(t)\,dt\right)\frac{\int_0^d g(x)\,dx}{2\left(d+\frac{4}{3}m^2d^3\right)}$$

*and* $\Lambda = \left] \frac{2\left(d+\frac{4}{3}m^2d^3\right)}{\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx}, \frac{\overline{m}^2 c}{2\|\alpha\|_2 \sup_{|x|\le c}|g(x)|}\right[.$

*We also observe that the condition*

(2') $\quad F(t,z) \ge 0$ *for all* $(t,z) \in ([0,d]\cup[3d,4d])\times[0,d]$

*implies (2). In particular, when* $f(t,x) = \alpha(t)g(x)$ *the condition*

(2") $\quad \alpha(t) \ge 0$ *for all* $t \in [0,d]\cup[3d,4d]$ *and* $\int_0^z g(x)\,dx \ge 0$ *for all* $z \in [0,d]$

*implies (2').*

*Finally, we observe that*

(1') $\quad \max\{1,c\} < d$

*implies (1).*

Now, we point out a special case of Theorem 3.1.

**Theorem 3.3.** *Let* $\alpha \in C([0,\infty)) \cap L^2([0,\infty))$ *and* $g \in BC([0,\infty))$ *be two nonnegative functions. Assume that*

$$\lim_{x\to 0^+}\frac{g(x)}{x} = 0 \tag{3.1}$$

*and assume that there exists a positive constant d such that:*

$$\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx > 0. \tag{3.2}$$

*Then, for each* $\lambda > \lambda^*$*, where*

$$\lambda^* = \frac{d+\frac{4}{3}m^2d^3}{\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx},$$

*the problem*

$$(P'_\lambda)\qquad \begin{cases} -y'' + m^2 y = \lambda\alpha(t)g(y) \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

*admits at least two nonnegative and non zero classical solutions.*

*Proof.* Of course from (3.3) we are assuming that g(0)=0. Then, without loss of generality, we can assume $g(x) = 0$ for all $x < 0$. Put $f(t,x) = \alpha(t)g(x)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$. Clearly, one has $F(t,z) = \int_0^z f(t,y)\,dy = \alpha(t)\int_0^z g(y)\,dy \equiv \alpha(t)G(z)$ for all $(t,z) \in [0,\infty)\times\mathbb{R}$, $G$ is a non-increasing function and (2) of Theorem 3.1 is clearly satisfied. Now, fix $\lambda > \lambda^*$. From (3.1) we have $\lim_{x\to 0^+} \frac{\sup_{|\xi|\le x}|g(\xi)|}{x} = 0$, so taking into account (3.2), there is $\delta > 0$ such that

$$\frac{\sup_{|\xi|\le x}|g(\xi)|}{x} < \frac{1}{\|\alpha\|_2}\frac{\overline{m}^2}{4}\frac{1}{d+\frac{4}{3}m^2d^3}\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx \tag{3.3}$$

for all $x \in ]0,\delta[$. By choosing $c \in ]0,\delta[$ such that $c^2 < \frac{2}{\overline{m}^2}(d+\frac{4}{3}m^2d^3)$,

$\frac{d+\frac{4}{3}m^2d^3}{\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx - \|\alpha\|_2 \sup_{|x|\le c}|g(x)|c} < \lambda$ and $\frac{\overline{m}^2 c}{2\|\alpha\|_2\sup_{|x|\le c}|g(x)|} > \lambda$, from (3.3) one has

$\frac{4}{\overline{m}^2}\|\alpha\|_2\frac{\sup_{|x|\le c}|g(x)|}{c} < \frac{1}{d+\frac{4}{3}m^2d^3}\int_d^{3d}\alpha(t)\,dt\int_0^d g(x)\,dx,$

$\frac{2}{\overline{m}^2}\|\alpha\|_2\frac{\sup_{|x|\le c}|g(x)|}{c} + \|\alpha\|_2\frac{1}{d+\frac{4}{3}m^2d^3}c\sup_{|x|\le c}|g(x)| < \frac{4}{\overline{m}^2}\|\alpha\|_2\frac{\sup_{|x|\le c}|g(x)|}{c} <$

$< \frac{1}{d+\frac{4}{3}m^2d^3}\int_d^{3d}\alpha(t)\,dtG(d),$

and $\frac{2}{\overline{m}^2}\|\alpha\|_2\frac{\sup_{|x|\le c}|g(x)|}{c} < \frac{1}{d+\frac{4}{3}m^2d^3}\left(\int_d^{3d}\alpha(t)\,dtG(d) - \|\alpha\|_2\sup_{|x|\le c}|g(x)|c\right).$

Hence, all the assumptions of Theorem 3.1 are satisfied. Therefore, problem $(P'_\lambda)$ admits at least two nonzero classical solutions. We claim that these solutions are nonnegative. Assume that there exists $t_0 \in ]0,\infty)$ such that $u_{1,\lambda}(t_0) < 0$. Then there exists $a \in [0,t_0[$ and $b \in ]t_0,\infty]$ such that $-u''_{1,\lambda}(t) + m^2u_{1,\lambda}(t) = 0$ for all $t \in ]a,b[$, $u_{1,\lambda}(a) = 0$, $u_{1,\lambda}(b) = 0$, so $u_{1,\lambda} = 0$ for all $t \in [a,b]$ and this is a contradiction. Hence, our claim is proved and the conclusion follows. □

**Example 3.4.** *Let* $\alpha(t) = \frac{1}{1+t}$ *and*

$$g(x) = \begin{cases} e^x & \text{if} \quad x \le 10 \\ e^{10} & \text{if} \quad x > 10. \end{cases}$$

*Theorem 3.1 (see also Remark 3.2) guarantees that, for each* $\lambda \in ]\frac{1}{8},\frac{1}{6}[$, *the problem*

$$(P'_\lambda)\qquad \begin{cases} -y'' + y = \lambda\frac{g(y)}{1+t} \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

*admits at least three nonnegative and non zero classical solutions. To see this it is enough to pick* $c = 1$ *and* $d = 10$.

**Example 3.5.** *Theorem 3.3 guarantees that the following problem*

$$(P')\qquad \begin{cases} -y'' + y = \frac{32e^{-y}y^2}{1+t^2} \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

*admits at least two nonnegative and non zero classical solutions. To see this it is enough to pick* $d = 1$, *and observe that* $32 > \lambda^*$.

Now, we present a result, where (2.2) is not required, and we also point out some consequences.

**Theorem 3.6.** *Let* $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *be an* $L^2-$*Carathéodory function. Put* $F(t,z) = \int_0^z f(t,y)\, dy$ *for all* $(t,z) \in [0,\infty) \times \mathbb{R}$, $\overline{m} = \min\{1; |m|\}$, *and assume that there exist three positive constants* $c_1$, $d$ *and* $c_2$ *such that:*

(1) $c_1^2 < \frac{2}{\overline{m}^2}(d + \frac{4}{3}m^2d^3) < c_2^2$;

(2) $F(t,z)\, dt \geq 0$ *for all* $(t,z) \in ([0,d] \cup [3d,4d]) \times [0,d]$;

(3) $\frac{2}{\overline{m}^2}\dfrac{\left(\int_0^\infty \left|\sup\limits_{|z|\leq c_1} |f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}{c_1} < \dfrac{\int_d^{3d} F(t,d)\, dt}{2\left(d + \frac{4}{3}m^2d^3\right)}$;

(4) $\frac{2}{\overline{m}^2}\dfrac{\left(\int_0^\infty \left|\sup\limits_{|z|\leq c_2} |f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}{c_2} < \dfrac{\int_d^{3d} F(t,d)\, dt}{2\left(d + \frac{4}{3}m^2d^3\right)}$.

*Then, for each* $\lambda \in$

$$\left]\frac{2\left(d + \frac{4}{3}m^2d^3\right)}{\int_d^{3d} F(t,d)\, dt}, \min\left\{\frac{\overline{m}^2 c_1}{2\left(\int_0^\infty \left|\sup\limits_{|z|\leq c_1} |f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}, \frac{\overline{m}^2 c_2}{2\left(\int_0^\infty \left|\sup\limits_{|z|\leq c_2} |f(t,z)|\right|^2 dt\right)^{\frac{1}{2}}}\right\}\right[,$$

*the problem* $(P_\lambda)$ *admits at least two generalized solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\max_{t\in[0,\infty)} |u_{1,\lambda}(t)| \leq c_1$ *and* $\max_{t\in[0,\infty)} |u_{2,\lambda}(t)| \leq c_2$.

*Proof.* Put

$$\overline{f}(t,x) := \begin{cases} f(t,-c_2) & \text{if } (t,x) \in [0,\infty) \times (-\infty, -c_2[ \\ f(t,x) & \text{if } (t,x) \in [0,\infty) \times [-c_2, c_2] \\ f(t,c_2) & \text{if } (t,x) \in [0,\infty) \times ]c_2, \infty). \end{cases}$$

Clearly, $\overline{f} : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies (2.2). Now, put $\overline{F}(t,z) = \int_0^z \overline{f}(t,y)\, dy$ for all $(t,z) \in [0,\infty) \times \mathbb{R}$ and take $X$ and $\Phi$ as in the proof of Theorem 3.1, and

$$\Psi(u) := -\int_0^{\infty} \overline{F}(t,u(t))dt$$

for all $u \in X$. Our goal is to apply Theorem B to $\Phi$ and $\Psi$. We explicitly observe that, from Proposition 2.7, $\Psi : X \to \mathbb{R}$ is a weakly sequentially continuous functional. Moreover, we see again from Section 2 that it is enough to show $\Phi$ and $\Psi$ satisfy (j)-(jjj) in Theorem B. Let

$$r_1 := \frac{\overline{m}^2 c_1^2}{2}, \qquad r_2 := \frac{\overline{m}^2 c_2^2}{2},$$

and $\overline{y} \in X$ as in the proof of Theorem 3.1. Clearly, $\inf_X \Phi < r_1 < r_2$ and $r_1 < \Phi(\overline{y}) < r_2$. Moreover, arguing as in the proof of Theorem 3.1 and taking also into account Remark 3.2 we obtain $\varphi_1(r_1) \leq \frac{2}{\overline{m}^2} \frac{\left(\int_0^{\infty} \left| \sup_{|z|\leq c_1} |f(t,z)| \right|^2 dt\right)^{\frac{1}{2}}}{c_1}$, $\varphi_1(r_2) \leq \frac{2}{\overline{m}^2} \frac{\left(\int_0^{\infty} \left| \sup_{|z|\leq c_2} |f(t,z)| \right|^2 dt\right)^{\frac{1}{2}}}{c_2}$ and $\varphi_2^*(r_1,r_2) \geq \frac{\int_d^{3d} F(t,d)\, dt}{2\left(d + \frac{4}{3}m^2 d^3\right)}$. Hence, from (3) and (4) we have (jj) and (jjj) of Theorem B. Therefore, from Theorem B we obtain that, for each $\lambda \in \Lambda$, the problem

$$(\overline{P}_\lambda) \qquad \begin{cases} -y'' + m^2 y = \lambda \overline{f}(t,y) \\ y(0) = 0 \\ \lim_{t\to\infty} y(t) = 0, \end{cases}$$

admits at least two generalized solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{t\in[0,\infty)} |u_{1,\lambda}(t)| \leq c_1$ and $\max_{t\in[0,\infty)} |u_{2,\lambda}(t)| \leq c_2$. Observing that these solutions are also solutions for $(P_\lambda)$, the conclusion follows. $\square$

**Theorem 3.7.** *Let $\alpha \in C([0,\infty)) \cap L^2([0,\infty))$ be a positive function and let $g \in C([0,\infty))$ be a nonnegative and nonzero functions. Assume that*

$$\lim_{x\to 0^+} \frac{g(x)}{x} = \lim_{x\to\infty} \frac{g(x)}{x} = 0 \tag{3.4}$$

*Then, for each $\lambda > \overline{\lambda}$, where*

$$\overline{\lambda} = \inf\left\{ \frac{2\left(d + \frac{4}{3}m^2 d^3\right)}{\int_d^{3d} \alpha(t)\, dt \int_0^d g(x)\, dx} : d > 0 \quad \text{and} \quad \int_d^{3d} \alpha(t)\, dt \int_0^d g(x)\, dx > 0 \right\},$$

*the problem $(P'_\lambda)$ admits at least one nonnegative and non zero classical solution.*

*Proof.* Let $\lambda > \overline{\lambda}$. Then, there is $d > 0$ such that $\int_d^{3d} \alpha(t)\, dt \int_0^d g(x)\, dx > 0$ and $\lambda > \frac{2\left(d+\frac{4}{3}m^2d^3\right)}{\int_d^{3d} \alpha(t)\, dt \int_0^d g(x)\, dx}$. From (3.4) we obtain $\lim_{x\to 0^+} \frac{\sup_{|\xi|\le x} |g(\xi)|}{x} = \lim_{x\to\infty} \frac{\sup_{|\xi|\le x} |g(\xi)|}{x} = 0$, so we can pick $c_1 > 0$ and $c_2 > 0$ such that $c_1^2 < \frac{2}{\overline{m}^2}(d + \frac{4}{3}m^2d^3) < c_2^2$, $\frac{\sup_{|x|\le c_1} |g(x)|}{c_1} < \frac{\overline{m}^2}{2}\frac{1}{\|\alpha\|_2}\frac{1}{\lambda}$, and $\frac{\sup_{|x|\le c_2} |g(x)|}{c_2} < \frac{\overline{m}^2}{2}\frac{1}{\|\alpha\|_2}\frac{1}{\lambda}$. Hence, from Theorem 3.6 we obtain the conclusion. □

**Example 3.8.** *Let* $\alpha(t) = \frac{1}{1+t}$ *and*

$$g(x) = \begin{cases} e^x & \text{if} \quad x \le 10 \\ e^{10} & \text{if} \quad 10 < x \le 10.000 \\ e^{(x-9.990)} & \text{if} \quad x > 10.000. \end{cases}$$

*Theorem 3.6 guarantees that, for each* $\lambda \in ]\frac{1}{8}, \frac{1}{6}[$, *the problem*

$$(P''_\lambda) \qquad \begin{cases} -y'' + y = \lambda\frac{g(y)}{1+t} \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

*admits at least two nonnegative and non zero classical solutions. To see this it is enough to pick* $c_1 = 1$, $d = 10$, *and* $c_2 = 10.000$.

**Example 3.9.** *Theorem 3.7 guarantees that the following problem*

$$(P'') \qquad \begin{cases} -y'' + y = 30\frac{\sqrt{y^3}}{(1+y)(1+t)} \\ y(0) = 0 \\ \lim\limits_{t\to\infty} y(t) = 0, \end{cases}$$

*admits at least one nonnegative and non zero classical solutions. To see this it is enough to observe that* $\overline{\lambda} < 29$ *(by choosing, for instance,* $d = 1$*).*

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