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A NOTE ON KWONG AND WONG'S PAPER

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ABSTRACT. Oscillation criteria are established for the second order nonlinear differential equation with a nonlinear periodic damping. Our results generalize earlier oscillation result of Kwong and Wong [8]. Furthermore, we are also able to answer the question in recent paper [8] for the oscillation of solutions of the special case.

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1. INTRODUCTION

In this paper, we consider the oscillation behaviour of solutions of the second order nonlinear differential equation with a nonlinear damping term,

$$(k(x,x'))' + p(t)k(x,x') + q(t)f(x) = 0\ ,\ t \in [0,\infty), \tag{1.1}$$

where k, p, q and f are continuous functions; p and q are also of period T and restrict our attention to those solutions $x(t)$ of (1.1) which exist on $[t_0,\infty)$ and satisfy $\sup\{|x(t)| : \ t \geq t_x\} > 0$ for any $t_x \geq t_0$. When $k(x,x') = x'$, $f(x) = x$ and $p(t) \equiv 0$, it is well known that if $q(t)$ is of mean value zero, i.e., $\int_0^T q(t)dt = 0$, and $q(t) \not\equiv 0$, then equation (1.1) is oscillatory, i.e., a nontrivial solution $x(t)$ has arbitrarily large zeros, i.e., for every $t_0 \in [0,\infty)$, there exists $t_1 > t_0$ such that $x(t_1) = 0$. Otherwise, equation (1.1) is said to be nonoscillatory if it has no oscillatory solutions, or alternatively a nontrivial solution of (1.1) has only finitely many zeros (see, e.g., [3, p. 25]).

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for different classes of second order differential equations [1-17]. Many results are obtained for the particular case of (1.1) such as the second order nonlinear differential equation with damped term,

$$x'' + p(t)x' + q(t)f(x) = 0, \tag{1.2}$$

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and when $f(x) = |x|^\lambda \, sgnx$, $\lambda > 0$, equation (1.2) takes the form

$$x'' + p(t)x' + q(t)\,|x(t)|^\lambda \, sgnx = 0, \tag{1.3}$$

which is the damped nonlinear Emden-Fowler equation, and when $f(x) = x$, equation (1.2) takes also the form

$$x'' + p(t)x' + q(t)x = 0, \tag{1.4}$$

which is the damped linear equation.

There are two main techniques used for proving the oscillatory character of a given class of equations. One of the important tool in the study of oscillatory behaviour of solutions for equations (1.1)-(1.4) is the integral averaging technique. This method has been used by many authors (see, for instance, [2, 4, 9-12, 16, 17]). The other one is interval technique which uses information on the behaviour of coefficients of the equation only on a sequence of subintervals $[s_i, t_i]$ of the half-line $[t_0, \infty)$, where $t_0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots$, and $s_i, t_i \to \infty$ as $i \to \infty$. This method has also been considered by numerous authors (see, for instance, [1, 5, 15]). However, most of the oscillation criteria obtained in the literature involves the nonnegativity restrictions on the coefficients $p(t)$ and/or $q(t)$.

Very recently, Kwong and Wong [8] studied the oscillation and nonoscillation of equation (1.4) where $p(t)$ and $q(t)$ are continuous functions and of period T, and obtained another kind of oscillation and nonoscillation result for equation (1.4). This technique is somewhat different from that of previous techniques and allows not only $p(t)$ but also $q(t)$ to change sign.

Motivated by the idea of Kwong and Wong [8], in this paper, by using the same way given in [8], we prove to study equation (1.1), and establish oscillation criteria, which contain earlier oscillation result of Kwong and Wong [8]. Furthermore, we are also able to answer the question in recent paper [8] for the oscillation of solutions of the special case.

The following two lemmas for the equation (1.4), are well known Wintner's lemmas, will be need in the proofs of our results.

Lemma 1. [14] *Equation (1.4) is nonoscillatory on* $[0, \infty)$ *if and only if there exist* $t_0 \in [0, \infty)$ *and a continuous differentiable function* $r(t)$ *such that*

$$r'(t) \geq r^2(t) - p(t)r(t) + q(t) \tag{1.5}$$

for all $t \geq t_0$.

Lemma 1 can be found in [7, p. 362, Theorem 7.2].

Lemma 2. [8, 13] *Suppose that $p(t)$ satisfies*

$$\int^{\infty} \exp\left(-\int^{t} p(s)ds\right) dt = \infty, \tag{1.6}$$

then equation (1.4) is oscillatory if

$$\lim_{X\to\infty} \int^{X} \exp\left(\int^{t} p(s)ds\right) q(t)dt = \infty. \tag{1.7}$$

2. MAIN RESULTS

In this section, we obtain theorems analogous to the results given in [8]. We shall impose the following conditions :

(a) $xf(x) > 0$ and $f'(x) \geq K > 0$ for all $x \in \mathbb{R}\backslash\{0\}$,

(b) $k^2(u,v) \leq \alpha\ v\ k(u,v)$ for some $K \geq \alpha > 0$ and all $(u,v) \in \mathbb{R}^2$.

Theorem 3. *Let (a) and (b) hold, and $Q(t)$ be an indefinite integral of $q(t)$, namely, $Q'(t) = q(t)$, where $q(t)$ is periodic of mean value zero, i.e., $\int_0^T q(t)dt = 0$. If $q(t) \neq 0$, $p(t)$, $Q(t)$ are also periodic with mean value zero and satisfy*

$$(p(t) - Q(t))\, Q(t) \leq 0, \qquad 0 \leq t \leq T, \tag{2.1}$$

and furthermore

$$measure\{t \in [0,T] \ : (p(t) - Q(t))\, Q(t) < 0 \ \} > 0, \tag{2.2}$$

then equation (1.1) is oscillatory.

Proof. Assume on the contrary that equation (1.1) is nonoscillatory, then $x(t)$ be a nonoscillatory solution of equation (1.1). We may assume that $x(t) \neq 0$ on $[t_0, \infty)$ for some $t_0 \geq 0$ depending on the solution $x(t)$. Denote $r(t) = -k(x,x')/f(x)$ on $t \geq t_0$. Then differentiating $r(t)$, in view of (1.1) and using (a) and (b), we easily get

$$r'(t) \geq r^2(t) - p(t)r(t) + q(t). \tag{2.3}$$

Define $R(t) = r(t) - Q(t)$. It is easy to verify from (2.3) that $R(t)$ satisfies on account of (2.1) the following Riccati inequality :

$$\begin{aligned} R'(t) &= R^2(t) + (2Q(t) - p(t))\, R(t) + Q^2(t) - p(t)Q(t) \\ &\geq R^2(t) + (2Q(t) - p(t))\, R(t). \end{aligned} \tag{2.4}$$

Since $R(t)$ is continuously differentiable and satisfies (2.4), we can now apply the sufficiency part of Lemma 1 to deduce that the second order equation

$$z''(t) + (p(t) - 2Q(t))\, z'(t) + \left(Q^2(t) - p(t)Q(t)\right) z(t) = 0 \tag{2.5}$$

is nonoscillatory. Since $p(t)$, $Q(t)$ are periodic in T with mean value zero, the function $E(t) = \exp\left(\int_0^t (p(s) - 2Q(s))\, ds\right)$ is bounded below by a positive constant. Then by condition (2.2),

$$\int_0^T E(t)\left(Q^2(t) - p(t)Q(t)\right) dt = m_0 > 0, \tag{2.6}$$

which implies that condition (1.7) is satisfied. Now apply Lemma 2 to equation (2.5) and conclude that it is oscillatory. This contradiction proves the theorem. □

Corollary 4. *When $k(x, x') = k_1(x')$, Theorem 3 remains valid if condition (b) is replaced by*

(b_1) $k_1^2(u) \le \alpha\ u\ k_1(u)$ for some $K \ge \alpha > 0$ and all $u \ne 0$.

Remark 5. *When $k(x, x') = x'$ and $f(x) = x$, it is easy to see that Theorem 3 reduces to Theorem 2 of Kwong and Wong* [8].

Now, we present a special case of equation (1.1), and let $f(x) = |x|^\lambda\, sgnx$, $\lambda > 0$, then $f'(x) = \lambda\ |x|^{\lambda-1}$. Next we consider the following two cases :

(i) If $x(t)$ is an unbounded nonoscillatory solution of equation (1.1) with $\lambda > 1$, then there exists a constant $c_1 > 0$ such that $|x(t)| \ge c_1$. Therefore, $f'(x) = \lambda\ |x|^{\lambda-1} \ge \lambda\ c_1^{\lambda-1} = K$, where $K > 0$ is a constant.

(ii) If $x(t)$ is a bounded nonoscillatory solution of equation (1.1) with $0 < \lambda < 1$, then there exists a constant $c_2 > 0$ such that $|x(t)| \le c_2$, and hence $f'(x) = \lambda\ |x|^{\lambda-1} \ge \lambda\ c_2^{\lambda-1} = K$, where $K > 0$ is a constant.

Now, we have the following result whose proof is similar to that of Theorem 3.

Theorem 6. *Let $f(x) = |x|^\lambda\, sgnx$, $\lambda > 0$, and all conditions of Theorem 3 be satisfied, then*

(i) every unbounded solution of equation (1.1) with $\lambda > 1$ is oscillatory,

(ii) every bounded solution of equation (1.1) with $0 < \lambda < 1$ is oscillatory.

Remark 7. *When $k(x, x') = x'$ and $f(x) = |x|^\lambda\, sgnx$, $\lambda > 0$, Theorem 6 answers to the open problem* (i) *given by Kwong and Wong* [8].

We now consider a special case of equation (1.1) in the form

$$(k(x, x'))' + a\ p(t)k(x, x') + q(t)f(x) = 0, \tag{2.7}$$

where $p'(t) = q(t)$, $p(0) = 0$, and a is a constant. As an application of Theorem 3 and 6, respectively, we have

Corollary 8. *Equation (2.7) is oscillatory for all $a < 1$.*

Corollary 9. *Let* $f(x) = |x|^{\lambda}\, sgnx$, $\lambda > 0$, *then*

(i) every unbounded solution of equation (2.7) with $\lambda > 1$ *is oscillatory for all* $a < 1$,

(ii) every bounded solution of equation (2.7) with $0 < \lambda < 1$ *is oscillatory for all* $a < 1$.

Now, let us consider two examples to illustrate our results.

Example 10. Consider

$$(k(x, x'))' + (\cos t)k(x, x') + (\sin t)f(x) = 0\ , \qquad t \geq 0, \tag{2.8}$$

where $k(u, v)$ can be taken any of the following functions:

$$\frac{v}{1+v^2},\ \frac{u^2\ v}{1+u^2},\ \frac{u^2\ v^3}{1+u^2\ v^2},\ \frac{v}{1+u^2},\ \frac{v\ \cos^2 u}{1+u^2},\ (\alpha = 1)\,, \tag{2.9}$$

and $f(x)$ is any function which satisfies (a) with $K \geq 1$. We may take $f(x) = x + x^3$, $(K = 1)$. It is easy to verify that all conditions of Theorem 3 are satisfied. Therefore, equation (2.8) is oscillatory.

Example 11. Consider

$$(k(x, x'))' + a\ (\sin t)k(x, x') + (\cos t)f(x) = 0\ , \qquad t \geq 0, \tag{2.10}$$

where a is a constant; $k(u, v)$ are as in (2.9), and $f(x)$ is any function which satisfies (a) with $K \geq 1$. We may take $f(x) = x + x^3$, $(K = 1)$. Therefore, by Corollary 8, equation (2.10) is oscillatory for all $a < 1$.

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