Dynamic Systems and Applications 15 (2006) 415-428

# POSITIVE SOLUTIONS OF THE NONLINEAR FOURTH-ORDER BEAM EQUATION WITH THREE PARAMETERS (II)

XI-LAN LIU[1,2] AND WAN-TONG LI[1]

[1]School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, People's Republic of China
E-mail: liuchli03@st.lzu.edu.cn wtli@lzu.edu.cn
[2]Department of Mathematics, Yanbei Normal College
Datong, Shanxi 037000, People's Republic of China

**ABSTRACT.** In this paper, we study the existence and uniqueness of positive solutions of the nonlinear fourth-order beam equation $u^{(4)}(t) + \eta u''(t) - \zeta u(t) = \lambda f(t, u(t))$, $0 < t < 1$, $u(0) = u(1) = u''(0) = u''(1) = 0$, where $f(t, u) : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and $\eta, \xi$ and $\lambda$ are parameters. By using the semiorder method on cones in a Banach space, we show that there exists a $\lambda^* > 0$ such that the boundary value problem (BVP) has a unique positive solution for $0 < \lambda < \lambda^*$ and no positive solutions for $\lambda \geq \lambda^*$. In particular, we also give an estimate for $\lambda^*$. Furthermore, we show that such a positive solution $u^\lambda(t)$ depends continuously on the parameter $\lambda$. More precisely, we prove that $u^\lambda(t)$ is increasing and continuous in $\lambda$ for $\lambda \in [0, \lambda^*)$, $\lim_{\lambda\to\lambda_0} \|u^\lambda - u^{\lambda_0}\| = 0$ for $\lambda_0 \in [0, \lambda^*)$, and $\lim_{\lambda\to\lambda^*-0} \|u_\lambda\| = +\infty$.

**AMS (MOS) Subject Classification:** 34B15.

## 1. INTRODUCTION

In this paper, we continue our work in [7] to study the fourth-order ordinary differential equation

$$u^{(4)}(t) + \eta u''(t) - \zeta u(t) = \lambda f(t, u(t)),\ 0 < t < 1, \tag{1.1}$$

with the boundary condition

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $\zeta, \eta$ and $\lambda > 0$ are parameters. This equation is often used to describe the deformation of an elastic beam with both endpoints fixed [1]. Recently, many researchers have paid attention to the BVP (1.1) and (1.2). For example, Bai and Wang [1], Liu [6], Ma [8, 10] and Yao [11] considered the BVP (1.1) and (1.2) for the case $\eta = \zeta = 0$, and Li [4] and Liu and Li [7] for the case $\eta > 0$ and $\zeta > 0$.

For the sake of convenience, we list some conditions:

Received March 16, 2005 

(H1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

(H2) $\zeta, \eta \in \mathbb{R}$ and $\eta < 2\pi^2, \zeta \geq -\frac{\eta^2}{4}, \zeta/\pi^4 + \eta/\pi^2 < 1$.

(H3) $f(t,u)$ is nondecreasing in $u$ for $t \in [0,1]$.

(H4) $f(t,0) \geq h > 0$, for all $t \in [0,1]$, where $h$ is a constant.

(H5) $f_\infty = \lim_{u\to\infty} \frac{f(t,u)}{u} = \infty$ for any $t \in [0,1]$.

(H6) $f(t,\rho u) \geq \rho^\alpha f(t,u)$, for any $0 < \rho < 1$, where $0 < \alpha < 1$ and $\alpha$ is independent of $\rho$ and $u$, and $t \in [0,1]$.

By a positive solution $u(t)$ of the BVP (1.1) and (1.2), we mean that $u(t)$ satisfies (1.1) and (1.2), and $u(t) > 0$ for $0 < t < 1$.

In 2003, Li [4] studied the problem (1.1) and (1.2) when $\lambda = 1$ and obtained that (1.1) and (1.2) has at least one positive solution by using the fixed point index in cones. Recently, Liu and Li [7] further considered the BVP (1.1) and (1.2) when $\lambda$ is a positive parameter and obtained the following main results.

**Theorem 1.1** (Liu and Li [7]). Assume that (H1)-(H5) hold. Then there exists a $\lambda^* > 0$ such that (1.1) and (1.2) has at least two, one and no positive solutions for $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$ and for $\lambda > \lambda^*$, respectively.

**Theorem 1.2** (Liu and Li [7]). Assume that (H1)-(H4) and (H6) hold. Then (1.1) and (1.2) has a unique positive solution $u_\lambda(t)$ for any $\lambda > 0$. In addition, such a solution $u_\lambda(t)$ satisfies the following properties:

(i) $u_\lambda(t)$ is nondecreasing in $\lambda$. Furthermore, $\lambda_1 > \lambda_2 > 0$ implies $u_{\lambda_1}(t) \gg u_{\lambda_2}(t)$ for $t \in [0,1]$.

(ii) $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0, \lim_{\lambda\to+\infty} \|u_\lambda\| = +\infty$.

(iii) $u_\lambda(t)$ is continuous with respect to $\lambda$, i.e., $\lambda \to \lambda_0 > 0$, implies $\|u_\lambda - u_{\lambda_0}\| \to 0$.

Note that if the function $f(t,u)$ satisfies all conditions (H1)-(H6), then Theorems 1.1 and 1.2 hold simultaneously. That is to say, if (H1)-(H6) hold, then there exists a $0 < \lambda^* < \infty$ such that (1.1) and (1.2) has a unique positive solution $u_\lambda(t)$ for any $\lambda \in (0, \lambda^*]$, and has no positive solutions for $\lambda > \lambda^*$. Furthermore, such a unique positive solution $u_\lambda(t)$ satisfies (i), (ii) and (iii) of Theorem 1.2.

However, the conditions (H5) and (H6) on the function $f(t,u)$ can not be satisfied simultaneously. For example, let $u > 1$ and $\rho = 1/u$. Then by (H6) we have

$$f(t,1) = f\left(t, \frac{1}{u}u\right) \geq \frac{f(t,u)}{u^\alpha} \geq \frac{f(t,u)}{u}.$$

However, (H5) implies that

$$\infty = \lim_{u\to\infty} \frac{f(t,u)}{u} \leq \lim_{u\to\infty} f(t,1),$$

which contradicts (H1).

Now it is natural to ask if there exist some appropriate sufficient conditions to ensure that there exists a $\lambda^* > 0$ such that the BVP (1.1) and (1.2) has a unique positive solution and has no positive solutions for $0 < \lambda < \lambda^*$ and $\lambda \geq \lambda^*$, respectively. In this paper we shall consider this problem and find such some sufficient conditions (see (H3) and (H4) in this Section, (C1) and (3.13) in Section 3). In particular, we also give an estimate for $\lambda^*$.

## 2. PRELIMINARIES

In this section, we introduce some definitions and lemmas which are important to prove our main results.

Let $E$ be a real Banach space which is partially ordered by a cone $P$ of $E$, i.e., for $x, y \in E$, $x \leq y$ if and only if $y - x \in P$. Recall that a cone $P$ is said to be normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where $\theta$ is the zero element of $E$. Let, for $e > \theta$,

$$P_e = \{x \in E | \text{there exist } \alpha, \beta > 0 \text{ such that } \alpha e \leq x \leq \beta e\},$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are real numbers. Then we have

(P1) $P_e \subset P$;

(P2) for all $x, y \in P_e$ there exists $\epsilon_0 \in (0,1)$ such that $x \geq \epsilon_0 y$.

**Definition 2.1** [3]. The operator $A : P \to P$ is concave if

$$A(\ell x + (1-\ell)y) \geq \ell Ax + (1-\ell)Ay$$

for $x, y \in P$ and $\ell \in (0,1)$.

**Definition 2.2** [3]. The operator $A : P \to P$ is increasing if $Ax \geq Ay$ for $x, y \in P$ and $x \geq y$.

**Definition 2.3** [2]. Let $P$ be a cone in the real Banach space $E$, $A : P \to P$ be an operator, and $e > \theta$. $A$ is called an $e$-concave operator if the following conditions hold:

(a1) For any $x > \theta$ there exist $\alpha = \alpha(x) > 0$ and $\beta = \beta(x) > 0$ such that

$$\alpha e \leq Ax \leq \beta e;$$

(a2) for any $s \in (0,1)$ and any $x \in P$ satisfying $\alpha_1 e \leq x \leq \beta_1 e$ where $\alpha_1 = \alpha_1(x) > 0$ and $\beta_1 = \beta_1(x) > 0$, there exists $\eta = \eta(x,s) > 0$ such that

$$A(sx) \geq s(1+\eta)Ax.$$

**Lemma 2.1** [3]. Suppose that $P$ is a normal cone of the Banach space $E$, $\vartheta > \theta$, and $A : [\theta, \vartheta] \to E$ is a concave increasing operator. Suppose further that there exist $\varepsilon, \lambda_0 > 0$ satisfying $\varepsilon\lambda_0 < 1$ such that $A\theta \geq \varepsilon\vartheta$ and $\lambda_0 A\vartheta \leq \vartheta$. Then, for any

$\lambda \in (0, \lambda_0]$, the equation $x = \lambda Ax$ has a minimal solution $x^* > \theta$ in $[\theta, \vartheta]$. In addition, set $x_0 = \theta,\ x_n = \lambda Ax_{n-1}, n = 1, 2, \cdots$ . Then

$$\|x_n - x^*\| \leq \lambda N \|A\theta\| (\varepsilon\lambda_0)^{-2}(1 - \varepsilon\lambda_0)^n, \tag{2.1}$$

where $N$ is the normal constant of $P$, $n = 1, 2, \cdots$ .

**Lemma 2.2** [2]. If $A : P \to P$ is an increasing and $e$-concave operator, then $A$ has at most one positive fixed point.

We let $C[0, 1]$ be the Banach space of all continuous functions defined on $[0, 1]$ with the sup-norm $\|\cdot\|$ and $C^+[0, 1] = \{x | x \geq 0,\ x \in C[0, 1]\ \}$. It is clear that $C^+[0, 1]$ is a normal cone in $C[0, 1]$.

Throughout this paper, we still assume the hypotheses (H1) and (H2) hold.

Let $\gamma_1, \gamma_2$ be the roots of the polynomial $P(\gamma) = \gamma^2 - \eta\gamma - \zeta$, i.e.,

$$\gamma_1, \gamma_2 = \frac{-\eta \pm \sqrt{\eta^2 + 4\zeta}}{2}.$$

In view of (H2), it is easy to see that $\gamma_1 \geq \gamma_2 > -\pi^2$.

It is well known that the BVP (1.1) and (1.2) has a solution $u := u^\lambda(t)$ if and only if $u$ solves the equation

$$u(t) = \lambda(\Phi u)(t) := \lambda \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) f(s, u(s)) ds d\tau, \tag{2.2}$$

where $G_i(t, s)$ are Green's functions of the linear boundary value problems

$$-u''(t) + \gamma_i u(t) = 0,\ u(0) = u(1) = 0, \tag{2.3}$$

and furthermore

$$G_i(t,s) = \begin{cases} \begin{cases} \frac{\sinh \nu_i t \cdot \sinh \nu_i (1-s)}{\nu_i \sinh \nu_i},\ 0 \leq t \leq s \leq 1 \\ \frac{\sinh \nu_i s \cdot \sinh \nu_i (1-t)}{\nu_i \sinh \nu_i},\ 0 \leq s \leq t \leq 1 \end{cases} & \text{for } \gamma_i > 0, \\ \begin{cases} t(1-s),\ 0 \leq t \leq s \leq 1 \\ s(1-t),\ 0 \leq s \leq t \leq 1 \end{cases} & \text{for } \gamma_i = 0, \\ \begin{cases} \frac{\sin \nu_i t \cdot \sin \nu_i (1-s)}{\nu_i \sin \nu_i},\ 0 \leq t \leq s \leq 1 \\ \frac{\sin \nu_i s \cdot \sin \nu_i (1-t)}{\nu_i \sin \nu_i},\ 0 \leq s \leq t \leq 1 \end{cases} & \text{for } -\pi^2 < \gamma_i < 0, \end{cases}$$

where $\nu_i = \sqrt{|\gamma_i|} (i = 1, 2)$. For more information regarding to the Green's function of (2.3), we refer to [4] and [7].

## 3. MAIN RESULTS

Before stating our results, we list the condition which will be needed in the sequel.

(C1) $f(t,u):[0,1]\times[0,\infty)\to[0,\infty)$ is concave in $u$ for $t\in[0,1]$.

If no confusion arises, in some situations we write $(\Phi u)(t)$ as $\Phi u$ and $u(t)$ as $u$.

**Lemma 3.1.** Assume that (C1), (H3) and (H4) hold. Then $\Phi : C^+[0,1]\to P_e$ is a continuous, increasing and concave operator, where $\Phi$ is defined by (2.2) and $e(t) = \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau$.

**Proof**. We first prove that $\Phi$ maps $C^+[0,1]$ into $P_e$. For any $u\in C^+[0,1]$, we consider three cases: (i) $0<\|u\|<1$; (ii) $\|u\|>1$; (iii) $\|u\|=0$ or $\|u\|=1$.

Assume that (i) holds. Then, for $t\in[0,1]$, we have

$$\begin{aligned} f(t,u) &= f\left(t,\|u\|\left(\frac{u}{\|u\|}\right)\right)\geq \|u\|\, f\left(t,\left(\frac{u}{\|u\|}\right)\right)+(1-\|u\|)\,f(t,\theta)\\ &\geq (1-\|u\|)\,f(t,\theta), \end{aligned}$$

and

$$\begin{aligned} (\Phi u)(t) &= \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau.\\ &\geq (1-\|u\|)\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau\\ &\geq h\,(1-\|u\|)\,e(t). \end{aligned}$$

Assume that (ii) holds. Then, for $t\in[0,1]$, we have

$$\begin{aligned} f(t,u) &= f\left(t,\left(\frac{1}{\|u\|}\right)\|u\|\,u\right)\geq \frac{1}{\|u\|}f\left(t,\|u\|\,u\right)+\left(1-\frac{1}{\|u\|}\right)f(t,\theta)\\ &\geq \left(1-\frac{1}{\|u\|}\right)f(t,\theta), \end{aligned}$$

and

$$(\Phi u)(t)=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))d\tau ds\geq h\left(1-\frac{1}{\|u\|}\right)e(t).$$

Assume $\|u\|=0$ or $\|u\|=1$. Then

$$(\Phi u)(t)\geq\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)d\tau ds\geq he(t).$$

Thus, $(\Phi u)(t)\geq\mu_2 e(t)$, where

$$\mu_2=\begin{cases} h(1-\|u\|), 0<\|u\|<1,\\[2em] h(1-1/\|u\|), \quad \|u\|>1,\\[2em] h, \qquad\qquad\quad \|u\|=0,1.\end{cases}$$

On the other hand, for any $u \in C^+[0,1]$, we have

$$\begin{aligned}(\Phi u)(t) &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &\leq \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f\left(s,\|u\|\right)dsd\tau \\ &\leq \mu_1 e(t),\end{aligned}$$

where $\mu_1 = \max_{\tau\in[0,1]} f(\tau,\|u\|) > 0$. That is to say $\Phi u \in P_e$ for any $u \in C^+[0,1]$.

It is easy to obtain that $\Phi$ is concave, increasing and continuous by virtue of (C1), (H3) and (H1). The proof is complete.

**Theorem 3.1.** Suppose that (C1), (H3) and (H4) hold. Then there exists a $0 < \lambda^* \leq \infty$ such that (1.1) and (1.2) has a unique solution $u^\lambda(t) \in P_e \cup \{\theta\}$ for $\lambda \in [0,\lambda^*)$ and has no solutions in $P_e$ for $\lambda \geq \lambda^*$. Furthermore, for any $u_0 \in C^+[0,1]$, we have

$$\lim_{n\to\infty} \left\|u_n^\lambda - u^\lambda\right\| = 0,\ \ \lambda \in [0,\lambda^*), \tag{3.1}$$

where

$$u_n^\lambda(t) = \lambda \Phi u_{n-1}^\lambda(t),\ \ n = 1,2,\cdots. \tag{3.2}$$

**Proof**. It is clear that the nonnegative solutions for (1.1) and (1.2) are equivalent to the solutions of the equation $u = \lambda\Phi u$ in $C^+[0,1]$. Let

$$\Theta = \{\lambda \geq 0 | \text{there exist } u \in C^+[0,1] \text{ such that } u = \lambda\Phi u\},$$

and $M_{f1} = \max_{t\in[0,1]} f(t,1)$. Then $\lambda = 0 \in \Theta$ and $M_{f1} > 0$. Set $w_0(t) = \frac{e(t)}{\|e\|} \in P_e \subset C^+[0,1]$. We can choose two positive numbers $\lambda_0$ and $\varepsilon_0$ satisfying $\lambda_0 M_{f1}\|e\| \leq 1$ and $0 < \varepsilon_0 < \min\left\{h\|e\|, \frac{1}{\lambda_0}, 1\right\}$ so that $\varepsilon_0\lambda_0 < 1$,

$$\begin{aligned}\Phi\theta &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau \\ &\geq he(t) = h\|e\| \cdot \frac{e(t)}{\|e\|} \geq \varepsilon_0 w_0(t), \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}\lambda_0(\Phi w_0)(t) &= \lambda_0 \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,w_0(s))dsd\tau \\ &\leq \lambda_0 \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f\left(s,\|w_0\|\right)dsd\tau \\ &\leq \lambda_0 M_{f1} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau. \\ &= \lambda_0 M_{f1}\|e\| \times \frac{e(t)}{\|e\|} \leq w_0(t) \qquad (3.4)\end{aligned}$$

for $t \in [0,1]$. By (3.3) and (3.4), it follows from Lemma 2.1 that $[0, \lambda_0] \subset \Theta$. Set $\lambda^* = \sup \Theta$. Then $\lambda^* \geq \lambda_0 > 0$. We assert that $\Theta = [0, \lambda^*)$. By the above discussion, we only need to prove $\lambda^* \notin \Theta$. If $\lambda^* = \infty$, then $\infty \notin \Theta = [0, \infty)$ is obvious. Suppose, to the contrary, that $\lambda^* < \infty$ and $\lambda^* \in \Theta$. Then, by Lemma 3.1, there exists an $u \in P_e$ such that $u = \lambda^* \Phi u$. Since (C1) holds, then

$$\begin{aligned}
\lambda^*(\Phi u)(t) &= \lambda^* \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s,u(s)) ds d\tau \\
&= \lambda^* \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f\left(s, \frac{1}{2} \times 2u(s) + \frac{1}{2}\theta\right) ds d\tau \\
&\geq \frac{1}{2}\lambda^* \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s, 2u(s)) ds d\tau \\
&\quad + \frac{1}{2}\lambda^* \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s,\theta) ds d\tau \\
&= \frac{1}{2}\lambda^* \left(\Phi(2u)\right)(t) + \frac{1}{2}\lambda^* \Phi\theta \qquad (3.5)
\end{aligned}$$

for $t \in [0,1]$, which leads to

$$\lambda^* \Phi(2u) \leq 2\lambda^* \Phi u - \lambda^* \Phi\theta = 2u - \lambda^* \Phi\theta < 2u. \qquad (3.6)$$

This implies that there exists a $\hat{\lambda} > \lambda^*$ such that $\hat{\lambda}\Phi(2u) \leq 2u$. By (P2), similar to (3.3), we can choose $\varepsilon_1 > 0$ satisfying $\varepsilon_1 \hat{\lambda} < 1$ such that $\Phi\theta \geq \varepsilon_1 \cdot (2u)$. In view of Lemma 2.1, we have $[0, \hat{\lambda}] \subset \Theta$, which is a contradiction to the definition of $\lambda^*$.

Next, we show that for any fixed $\lambda \in [0, \lambda^*)$, there exists a unique $u \in P_e \cup \{\theta\}$ that solves the equation $u = \lambda \Phi u$. It is easy to see that $u = \theta$ is the unique solution of (1.1) and (1.2) if $\lambda = 0$. We assert that for any $\lambda \in (0, \lambda^*)$, $\lambda\Phi$ is an $e$-concave operator. Indeed, by Lemma 3.1, for any $u \in P_e$ we have $\Phi u \in P_e$, and so there exists an $\hat{\varepsilon}_u \in (0,1)$ such that $\Phi\theta \geq \hat{\varepsilon}_u \Phi u$ by (P2). Hence, by virtue of the concavity of the operator $\Phi$, for any $s \in (0,1)$, we have

$$\begin{aligned}
\Phi(su) &\geq s\Phi(u) + (1-s)\Phi(\theta) \geq s\Phi(u) + (1-s)\hat{\varepsilon}_u \Phi(u) \\
&= s(1 + (\frac{1}{s} - 1)\hat{\varepsilon}_u)\Phi(u) = s(1 + \eta(s,u))\Phi(u), \qquad (3.7)
\end{aligned}$$

where $\eta(s,u) = (\frac{1}{s} - 1)\hat{\varepsilon}_u$. Hence $\Phi$ and $\lambda\Phi$ ($\lambda \in (0, \lambda^*)$) are $e$-concave operators. So by Lemma 2.1, Lemma 2.2, and Lemma 3.1, the problem (1.1) and (1.2) has a unique positive solution in $P_e$ for $\lambda \in (0, \lambda^*)$.

We now show that (3.1) holds. It is true by (2.1) if $u_0 = \theta$. If $u_0 \in C^+[0,1] \backslash \{\theta\}$, then $\Phi u_0 := x_0 \in P_e$ according to Lemma 3.1. For fixed $\lambda \in (0, \lambda^*)$, we assume that $\bar{w}_0 \in P_e$ is the unique solution of (1.1) and (1.2). Then, there exists $\bar{t} \in (0,1)$ such that $\bar{w}_0 \geq \bar{t} x_0$. Set $\bar{v}_0 = \bar{t}^{-1} \bar{w}_0$, $\bar{u}_0 = \theta$, $\bar{u}_n = \lambda \Phi \bar{u}_{n-1}$, $\bar{v}_n = \lambda \Phi \bar{v}_{n-1}$ and $x_n = \lambda \Phi x_{n-1}$ $(n = 1, 2, \cdots)$. Then, $\bar{v}_0, \Phi\bar{v}_0 \in P_e$, and we can choose a number $\sigma \in (0,1)$ such that

$\Phi\theta \geq \sigma\Phi\bar{v}_0$, i.e., $\bar{u}_1 \geq \sigma\bar{v}_1$. Similar to (3.7), we have

$$\bar{w}_0 = \lambda\Phi\bar{w}_0 = \lambda\Phi(\bar{t}\bar{v}_0) \geq \lambda\bar{t}\left(1+\bar{\eta}(\bar{t},\bar{v}_0)\right)\Phi\bar{v}_0,$$

where $\bar{\eta}(\bar{t},\bar{v}_0) = \left(\frac{1}{\bar{t}}-1\right)\sigma$, and furthermore

$$\lambda\Phi(\bar{v}_0) \leq \left(\bar{t}(1+\bar{\eta}(\bar{t},\bar{v}_0))\right)^{-1}\bar{w}_0 = \left(1+\bar{\eta}(\bar{t},\bar{v}_0)\right)^{-1}\bar{v}_0 \leq \bar{v}_0,$$

which implies that $\bar{v}_1 = \lambda\Phi(\bar{v}_0) \leq \bar{v}_0$. Thus, by the increasing property of $\Phi$, we obtain that

$$\bar{u}_0 \leq \bar{u}_1 \leq \cdots \leq \bar{u}_n \leq \cdots \leq \bar{v}_n \leq \cdots \leq \bar{v}_1 \leq \bar{v}_0. \tag{3.8}$$

Since $\bar{u}_n$ and $\bar{v}_n$ $(n = 1, 2, ....)$ belong to $P_e$, by (3.8), we can obtain the numbers

$$c_n^* = \sup\{c^* > 0 | \bar{u}_n \geq c^*\bar{v}_n\},\ n = 1, 2, \cdots.$$

Furthermore, we have

$$\|\bar{u}_{n+m} - \bar{u}_n\| \leq |1-c_n^*|\,\|\bar{v}_n - \bar{u}_n\| \leq |1-c_n^*|\,\|\bar{v}_0\|,\ m, n = 1, 2, \cdots, \tag{3.9}$$

and $\lim\limits_{n\to\infty} c_n^* = \hat{c} = 1$. It is clear that the sequence $\{c_n^*\}$ is bounded above $(c_n^* \leq 1)$ and increasing. We only verify that $\hat{c} = 1$. If this is not true, i,e., $c_n^* \leq \hat{c} < 1$, then there exists a subset $[\sigma,\tau] \subset (0,1)$ such that $c_n^* \leq \hat{c} \in [\sigma,\tau]$. By the concavity and increasing property of $\Phi$, we have

$$\begin{aligned} \bar{u}_{n+1} &= \lambda\Phi\bar{u}_n \geq \lambda\Phi(c_n^*\bar{v}_n) \geq c_n^*\lambda\Phi(\bar{v}_n) + (1-\tau)\lambda\Phi\theta \\ &\geq c_n^*\lambda\Phi(\bar{v}_n) + (1-\tau)\sigma\lambda\Phi\bar{v}_0 \geq c_n^*\left(1+\frac{(1-\tau)\,\sigma}{\tau}\right)\lambda\Phi(\bar{v}_n) \\ &= c_n^*\left(1+\frac{(1-\tau)\,\sigma}{\tau}\right)\bar{v}_{n+1} \end{aligned}$$

which leads to

$$c_{n+1}^* \geq c_n^*\left(1+\frac{(1-\tau)\,\sigma}{\tau}\right),$$

and

$$c_{n+1}^* \geq \sigma\left(1+\frac{(1-\tau)\,\sigma}{\tau}\right)^n n = 1, 2, \cdots. \tag{3.10}$$

It is evident that (3.10) can imply the contradiction

$$\infty = \lim_{n\to\infty} c_n^* = \hat{c} \leq 1.$$

So $\hat{c} = 1$. From (3.9) and the following inequalities

$$\begin{aligned} \|\bar{v}_{n+m} - \bar{v}_n\| &\leq \|\bar{v}_{n+m} - \bar{u}_{n+m}\| \leq \left|\frac{1}{c_{n+m}^*} - 1\right|\|\bar{u}_{n+m}\| \\ &\leq \left|\frac{1}{c_{n+m}^*} - 1\right|\|\bar{v}_0\|,\ n, m = 1, 2, \cdots, \end{aligned}$$

we know that both $\{\bar{u}_n\}$ and $\{\bar{v}_n\}$ are Cauchy sequences. Let $x^* = \lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \bar{v}_n$. Then, by the inequality $\bar{u}_0 \le \bar{x}_0 \le \bar{w}_0 \le \bar{v}_0$, we have

$$\bar{u}_{n+1} = \lambda\Phi\bar{u}_n \le \lambda\Phi x_n \le \lambda\Phi\bar{w}_0 \le \lambda\Phi\bar{v}_n = \bar{v}_{n+1},$$

and

$$\bar{u}_n \le x_n \le \bar{w}_0 \le \bar{v}_n$$

$(n = 1, 2, \cdots)$ by induction, and so $\lim_{n\to\infty} x_n = x^* = \bar{w}_0$. The proof is complete.

**Theorem 3.2.** Suppose that (C1), (H3) and (H4) hold and that $u^\lambda(t) \in P_e \cup \{\theta\}$ is the unique solution of (1.1) and (1.2). Then we have

(i) $u^\lambda(t)$ is continuous and increasing in $\lambda$ for $\lambda \in [0, \lambda^*)$, and

(ii) $\lim_{\lambda\to\lambda^*-0} \|u^\lambda\| = \infty$.

**Proof**. By Theorem 3.1, (1.1) and (1.2) has a unique solution $u^\lambda(t) \in P_e \cup \{\theta\}$.

(i) It is easy to see that $u^\lambda(t) = \theta$ is the unique solution of (1.1) and (1.2) if $\lambda = 0$. We first show that $u^\lambda(t)$ is increasing in $\lambda$. Indeed, if we let $u_0(t) = \theta$ and $u_n^\lambda(t) = \lambda\Phi u_{n-1}^\lambda(t)$ $(n = 1, 2, \cdots)$, then

$$\begin{aligned} u_1^{\lambda_1}(t) &= \lambda_1 \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau \\ &\ge \lambda_2 \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau = u_1^{\lambda_2}(t) \end{aligned}$$

for any $\lambda_1, \lambda_2 \in [0, \lambda^*)$, $\lambda_1 \ge \lambda_2$ and $t \in [0, 1]$. By induction, together with Lemma 2.1, we obtain that $u_n^\lambda(t)$ $(n = 1, 2, \cdots)$ is increasing in $\lambda$, and $u^\lambda(t)$ is also increasing in $\lambda$ by (3.1). We next show that $u^\lambda(t)$ is continuous in $\lambda \in (0, \lambda^*)$ for $t \in [0, 1]$. Note that for any $\lambda_0 \in (0, \lambda^*)$

$$\begin{aligned} \left\|u_1^\lambda - u_1^{\lambda_0}\right\| &= |\lambda - \lambda_0| \left\| \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau \right\| \\ &\le |\lambda - \lambda_0| \cdot \|e\| \max_{s\in[0,1]} |f(s,0)|, \end{aligned}$$

which implies that $\lim_{\lambda\to\lambda_0} u_1^\lambda(t) = u_1^{\lambda_0}(t)$ for $t \in [0, 1]$. By induction, $u_n^\lambda(t)$ $(n = 1, 2, \cdots)$ is continuous in $\lambda \in (0, \lambda^*)$ for $t \in [0, 1]$. Let $[\lambda_1, \lambda_2]$ be a closed subset of $(0, \lambda^*)$. Then $u_n^\lambda(t)$ $(n = 1, 2, \cdots)$ is continuous uniformly in $\lambda \in [\lambda_1, \lambda_2]$, and so $u^\lambda(t)$ is continuous in $\lambda \in [\lambda_1, \lambda_2]$ by (3.1). Finally, we show that $u^\lambda(t)$ is continuous in $\lambda = 0$. Since $u^\lambda(t)$ is increasing in $\lambda \in [0, \lambda^*)$, then for any $\lambda_0 \in (0, \lambda^*)$ and $0 \le \lambda \le \lambda_0$, we have for $t \in [0, 1]$

$$\theta \le u^\lambda(t) \le u^{\lambda_0}(t),$$

and

$$\theta < \Phi\theta \le (\Phi u^\lambda)(t) \le (\Phi u^{\lambda_0})(t).$$

Thus,

$$\left\|\Phi u^\lambda\right\| \le \left\|\Phi u^{\lambda_0}\right\|$$

together with

$$\left\|u^{\lambda}\right\| = \left\|\lambda \Phi u^{\lambda}\right\| = \lambda \left\|\Phi u^{\lambda}\right\| \tag{3.11}$$

leads to

$$\lim_{\lambda \to 0+} \left\|u^{\lambda}\right\| = 0 = \left\|u^{0}\right\|$$

for $t \in [0,1]$. This implies the first part of the proof.

(ii) In order to finish the second part of proof, we consider two cases. The first case is $\lambda^* = \infty$. In this case, it is easy to see that $\left\|u^{\lambda}\right\| \to \infty$ as $\lambda \to \lambda^*$ by (3.11) and $\lambda(\Phi u^{\lambda})(t) \geq \lambda \Phi\theta \geq \lambda h e(t)$ for $t \in [0,1]$. The second case is $0 < \lambda^* < \infty$. Under this assumption, we claim that (ii) of Theorem 3.2 is still true. Suppose, to the contrary, that $\lim_{\lambda \to \lambda^*-0} \left\|u^{\lambda}\right\| < \infty$. Then, there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ $(\lambda_n \in (0,\lambda^*))$ and a constant $\varpi > 0$ such that the solutions $u^{\lambda_n}(t) \in P_e$ of (1.1) and (1.2) satisfying $\left\|u^{\lambda_n}(t)\right\| \leq \varpi$ $(n = 1, 2, \cdots)$ for $t \in [0,1]$. Thus, we can take $y_0 \in P_e$ such that $u^{\lambda_n}(t) \leq y_0$ $(n = 1, 2, \cdots)$ for $t \in [0,1]$. Also, by (P2) we can take $\varepsilon^* > 0$ satisfying $2\varepsilon^*\lambda^* < 1$ such that $\Phi\theta \geq 2\varepsilon^* y_0$. As in (3.5) and (3.6), for $t \in [0,1]$, we have

$$\begin{aligned}
\lambda_n \left(\Phi(2u^{\lambda_n})\right)(t) &\leq 2\lambda_n(\Phi u^{\lambda_n})(t) - \lambda_n \Phi\theta \leq 2\lambda_n(\Phi u^{\lambda_n})(t) - 2\lambda_n \varepsilon^* y_0 \\
&\leq 2u^{\lambda_n}(t) - 2\lambda_n \varepsilon^* u^{\lambda_n}(t) = 2u^{\lambda_n}(t)(1 - \lambda_n \varepsilon^*).
\end{aligned}$$

That is,

$$\frac{\lambda_n}{1 - \lambda_n \varepsilon^*} \left(\Phi(2u^{\lambda_n})\right)(t) \leq 2u^{\lambda_n}(t).$$

We know that

$$\Phi\theta \geq \varepsilon^* \cdot (2y_0) \geq \varepsilon^* \left(2u^{\lambda_n}(t)\right)$$

and

$$\varepsilon^* \frac{\lambda_n}{1 - \lambda_n \varepsilon^*} < \varepsilon^* \frac{\lambda^*}{1 - \lambda^* \varepsilon^*} < 2\lambda^* \varepsilon^* < 1.$$

It follows from Lemma 2.1 that

$$\frac{\lambda_n}{1 - \lambda_n \varepsilon^*} < \lambda^*, n = 1, 2, \cdots. \tag{3.12}$$

By passing to the limit $n \to \infty$ in (3.12), we have $\frac{\lambda^*}{1-\lambda^*\varepsilon^*} < \lambda^*$, which contradicts the fact that $\lambda^*$, $\varepsilon^* \in (0,1)$. The proof is complete.

We now give an estimate for $\lambda^*$. Define

$$f_{\infty} = \lim_{u \to \infty} \frac{f(t,u)}{u} \text{ for any } t \in [0,1]. \tag{3.13}$$

Then, $0 \leq f_{\infty} \leq \infty$. Furthermore, if (C1) holds, then $f_{\infty} < \infty$. Indeed, if $u > 1$, in view of the concavity of $f$, we see that

$$\begin{aligned}
f(t,1) &= f\left(t, \frac{1}{u} \times u + \left(1 - \frac{1}{u}\right) \times 0\right) \\
&\geq \frac{1}{u} f(t,u) + \left(1 - \frac{1}{u}\right) f(t,0),
\end{aligned}$$

and so

$$\begin{aligned} f(t,u) &\le uf(t,1)-(u-1)f(t,0) \\ &= [f(t,1)-f(t,0)]u+f(t,0), \end{aligned}$$

i,e.,

$$\frac{f(t,u)}{u} \le [f(t,1)-f(t,0)]+\frac{f(t,0)}{u},$$

which implies that $f_\infty < \infty$.

**Theorem 3.3.** Suppose that (C1), (H3), (H4) and (3.13) hold. If $f_\infty > 0$, then

$$\lambda^* > \frac{1}{f_\infty \|e\|},$$

where $\lambda^*$ is defined in Theorem 3.1.

**Proof**. In view of (3.13), for any $\varepsilon > 0$ and any $t \in [0,1]$, there exists $r > 0$ such that

$$f(t,r) \le (f_\infty+\varepsilon)r.$$

Set $z_0(t) = \frac{r}{\|e\|}e(t)$ for $t \in [0,1]$, and take $\tilde{\lambda}_\varepsilon = 1/[(f_\infty+\varepsilon)\|e\|]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned} \tilde{\lambda}_\varepsilon(\Phi z_0)(t) &= \tilde{\lambda}_\varepsilon \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,z_0(s))dsd\tau \\ &= \tilde{\lambda}_\varepsilon \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f\left(s,\frac{r}{\|e\|}e(s)\right)dsd\tau \\ &\le \tilde{\lambda}_\varepsilon\left((f_\infty+\varepsilon)\|e\|\right)\times\frac{r}{\|e\|}\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &\le z_0(t), \end{aligned} \tag{3.14}$$

and so $\tilde{\lambda}\Phi z_0 \le z_0$ by passing to the limit $\varepsilon \to 0$ in (3.14), where $\tilde{\lambda} = 1/(f_\infty\|e\|)$. Similarly, we can choose an $\breve{\varepsilon} > 0$ satisfying $\breve{\varepsilon} < \min\left\{(h\|e\|)/r, 1/\tilde{\lambda}, 1\right\}$ such that

$$\begin{aligned} \Phi\theta &= \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,0)dsd\tau \\ &\ge h\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &= \frac{h}{r}\|e\|\cdot\frac{r}{\|e\|}\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau \\ &= \frac{h}{r}\|e\|\cdot z_0(t) \ge \breve{\varepsilon}z_0(t) \end{aligned} \tag{3.15}$$

for $t \in [0,1]$. We know that $C^+[0,1]$ is a normal cone in $C[0,1]$ and then $\lambda^* > \tilde{\lambda}$ holds by (3.14), (3.15) and Lemma 2.1. This completes our proof.

We can easily obtain the following results from the proof of Theorem 3.3.

**Corollary 3.1.** Suppose that (C1), (H3) and (H4) hold. Suppose further that there exist $\tilde{R}$, $f_{\tilde{R}} > 0$ such that

$$f(t, \tilde{R}) \le f_{\tilde{R}} \cdot \tilde{R}$$

for any $t \in [0, 1]$. Then,

$$\lambda^* > \frac{1}{f_{\tilde{R}} \|e\|}.$$

**Corollary 3.2.** Suppose that (C1), (H3), (H4) and (3.13) hold. If $f_\infty = 0$, then $\lambda^* = \infty$.

**Remark 3.1.** From Corollary 3.2, it is easy to see that (1.1) and (1.2) has a unique positive solution for any $\lambda > 0$. Hence, Corollary 3.2 is similar to Theorem 1.2 in Section 1 (see Theorem 4.2 of [7]).

**Remark 3.2.** It is not difficult to find some functions that satisfy the conditions (C1), (H3) and (H4). For example, the function

$$f(t, u) = 1 + 2u + \sqrt{u} + \left(\frac{3}{2} - t\right)^2, t \in [0, 1]$$

satisfies the conditions (C1), (H3), (H4) and $f_\infty = 2$.

**Acknowledgment**. The authors would like to thank Professor John R. Graef for his valuable comments and suggestions. This work was partially supported by the NNSF of China, the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of Ministry of Education of China (Li) and the Foundation of Yanbei Normal College (Liu).

## REFERENCES

[1] Z. Bai and H. Wang, On positive solutions of some nonlinear fourth-order beam equations, *J. Math. Anal. Appl.*, 270: 357-368, 2002.

[2] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, 1988.

[3] D. Guo, *The Semiorder Methods in Nonlinear Analysis*, Shandong Science and Technology Press, Shandong, China, 2000 (in Chinese).

[4] Y. Li, Positive solutions of fourth-order boundary value problems with two parameters, *J. Math. Anal. Appl.*, 281:477-484, 2003.

[5] Y. Li, Existence and method of lower and upper solutions for fourth-order boundary value problems, *Acta Mathematica Scientia*, 23A: 245-252, 2003 (in Chinese).

[6] B. Liu, Positive solutions of fourth-order two point boundary value problems, *Appl. Math. Comput.*, 148: 407-420, 2004.

[7] X. L. Liu and W. T. Li, Positive solutions of the nonlinear fourth-order beam equation with three parameters, *J. Math. Anal. Appl.*, 303: 150-163, 2005.

[8] R. Ma, Multiple positive solutions for a semipositone fourth-order boundary value problem, *Hiroshima Math. J.*, 33: 217-227, 2003.

[9] R. Ma, J. Hui and S. Fu, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.*, 215: 415-422, 1997.

[10] R. Ma and H. Wang, On the existence of positive solutions of fourth order ordinary differential equation, *Appl. Anal.*, 59: 225-231, 1995.

[11] Q. Yao and Z. Bai, Existence of positive solutions for the boundary problems of $u^{(4)}(t) - \lambda h(t)f(u(t)) = 0$, *Chinese Ann. Math.*, A5: 575-578, 1999 (in Chinese).

[12] L. Zhu and X. Weng, Multiple positive solutions for a fourth-order boundary value problem, *Bol. Soc. Paran. Mat. (3s)*, 21: 1-10, 2003.