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STABILITY RESULTS FOR SET DIFFERENTIAL EQUATIONS WITH CAUSAL MAPS

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ABSTRACT. In this paper we obtain stability results in the general set-up of set differential equations with causal maps.

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1. INTRODUCTION

In this paper we employ the method of Lyapunov functions (MLF) to discuss the stability properties of set differential equations (SDEs) with causal operators. It is well-known that the MLF can be used to reduce the study of differential systems to that of scalar differential equations [3]. In [2, 4] this idea was exploited and the Lyapunov stability criteria for SDEs under suitable conditions were studied.

SDEs have recently been the subject of much attention and the theory of SDEs is rapidly becoming an independent discipline. One of the advantages of the theory of SDEs is its unifying approach. For example, under appropriate conditions, such theory reduces to either that of ordinary differential systems, differential equations in a Banach space, multivalued differential inclusions, or even fuzzy differential equations. On the other hand, causal operators present yet another form of unification since they include several types of differential and integro-differential maps. For a monograph on the subject see [1].

In this paper, we combine the unifying approach of both the theory of SDEs and that of causal operators and extend the MLF to obtain stability results in the general framework of SDEs with causal operators. First, we obtain comparison results in terms of Lyapunov-like functions. Next, we use the comparison results to prove several stability theorems.

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2. PRELIMINARIES

Let $K_c(\mathbb{R}^n)$ denote the set of all nonempty compact convex subsets of $\mathbb{R}^n$. We define the Hausdorff metric between two nonempty bounded sets A and B of $\mathbb{R}^n$ by

$$D(A,B) = max[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)] \tag{2.1}$$

where $d(x,A)$ =inf $[d(x,y) : y \in A]$. We observe that $(K_c(\mathbb{R}^n), D)$ is a complete metric space. When equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, $K_c(\mathbb{R}^n)$ becomes a semilinear metric space, which can be embedded as a complete cone into a corresponding Banach space.

The following properties of the Hausdorff metric (2.1) will be useful in the sequel:

$$D(A+C, B+C) = D(A,B) \text{ and } D(A,B) = D(B,A), \tag{2.2}$$

$$D(\lambda A, \lambda B) = \lambda D(A,B), \tag{2.3}$$

$$D(A,B) \le D(A,C) + D(C,B), \tag{2.4}$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$. Given two sets, A and $B \in K_c(\mathbb{R}^n)$ if there exists a set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$, then we say that the Hukuhara difference of the two sets A and B exists, and we denote it by $A - B$.

Let I be an interval of real numbers and let the mapping $U : I \to K_c(\mathbb{R}^n)$ be given. U is Hukuhara differentiable at a point $t_0 \in I$, if there exists $D_H U(t_0) \in K_c(\mathbb{R}^n)$ such that the limits

$$\lim_{h\to 0^+} \frac{U(t_0+h) - U(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{U(t_0) - U(t_0-h)}{h}$$

both exist in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H U(t_0)$. If $F : I \to K_c(\mathbb{R}^n)$ is a continuous function, then it is integrable and the integral

$$G(t) = G(t_0) + \int_{t_0}^{t} F(s)\, ds, \;\; t \in I,$$

is Hukuhara differentiable, and $D_{_H} G(t) = F(t)$.

3. COMPARISON RESULTS

In this section, we first prove some basic comparison results, which are used subsequently to establish stability properties of SDEs with causal operators. We begin with some definitions. Let $E = C[[t_0,\infty), K_c(\mathbb{R}^n)\,]$ with norm

$$\sup_{t\in[t_0,\infty)} \frac{D[U(t),\theta]}{h(t)} < \infty,$$

where θ is the zero element of $\mathbb{R}^n$, which is regarded as a point set and $h : [t_0,\infty) \to \mathbb{R}_+$ is a continuous map. E equipped with such a norm is a Banach space.

Definition 3.1. Let $Q \in C[E, E]$. Q is said to be a causal map or nonanticipative map if $U(s) = V(s)$, $t_0 \le s \le t < \infty$, and $U, V \in E$ then $(QU)(s) = (QV)(s)$, $t_0 \le s \le t < \infty$.

Consider the initial-value problem (IVP) for SDEs with causal map, defined using the Hukuhara derivative:

$$D_H U(t) = (QU)(t) \tag{3.1}$$
$$U(t_0) = U_0 \in K_c(\mathbb{R}^n).$$

In order to use the MLF, it is necessary to select minimal subsets of E over which the derivative of the Lyapunov function can be conveniently estimated. For that purpose, let $L \in C[\mathbb{R}_+ \text{ x } B, \mathbb{R}_+]$, where $B = B(\theta, b) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \le b\}$. Define the following sets:

$$\begin{aligned} E_\alpha &= \{U \in K_c(\mathbb{R}^n) : L(s, U(s))\, \alpha(s) \le L(t, U(t))\, \alpha(t),\ t_0 \le s \le t\}, \\ E_1 &= \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \le L(t, U(t)),\ t_0 \le s \le t\}, \\ E_0 &= \{U \in K_c(\mathbb{R}^n) : L(s, U(s)) \le f(L(t, U(t))),\ t_1 \le s \le t,\ t_1 \ge t_0\}, \end{aligned}$$

where

(i) α(t)> 0 is a continuous function on $\mathbb{R}_+$,
(ii) $f(r)$ is a continuous on $\mathbb{R}_+$, nondecreasing in r and $f(r) > r$ for $r > 0$.

We now prove the comparison results.

Theorem 3.2. *Let $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and let $L(t, U)$ be locally Lipschitzian in U, i.e., for $U, V \in B$, $t \in \mathbb{R}_+$, and $K > 0, |L(t, U) - L(t, V)| \le K\, D(U, V)$.*

(i) *Assume that for $t \ge t_0$ and $U \in E_1$*

$$D_- L(t, U(t)) \le g(t, L(t, U(t))) \tag{3.2}$$

where $D_- L(t, U(t)) = \lim inf_{h \to 0^-} \frac{1}{h}[L((t+h, U(t)+h(QU)(t)) - L(t, U(t))]$, and $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$.

(ii) *Let $r(t) = r(t, t_0, w_0)$ be the maximal solution of*

$$w' = g(t, w),\ w(t_0) = w_0 \ge 0, \tag{3.3}$$

existing on $t_0 \le t < \infty$.

Let $U(t, t_0, U_0)$ be any solution of (3.1) such that $U(t, t_0, U_0) \in B$ for $t \in [t_0, t_1]$ and let $L(t_0, U_0) \le w_0$. Then $L(t, U(t, t_0, U_0)) \le r(t)$ for all $t \in [t_0, t_1]$.

Proof. Let $U(t, t_0, U_0)$ be any solution of (3.1) such that $U(t, t_0, U_0) \in B$ for $t \in [t_0, t_1]$. Define $m(t) = L(t, U(t))$, $t \in [t_0, t_1]$. For sufficiently small $\epsilon > 0$, consider the differential equation

$$w' = g(t, w) + \epsilon = g_\epsilon(t, w),$$
$$w(t_0) = w_0 + \epsilon,$$

whose solutions $w(t,\epsilon) = w(t,t_0,w_0,\epsilon)$ exist as far as $r(t)$ exists to the right of t_0. Since the continuity of $w(t)$ implies that $\lim_{\epsilon\to 0} w(t,\epsilon) = r(t)$, it is sufficient to show that

$$m(t) < w(t,\epsilon),\ t \in [t_0,t_1]. \tag{3.4}$$

Suppose that (3.4) is not true. Then there exists $t_2 \in (t_0,t_1)$ such that

(a) $m(t) \leq w(t,\epsilon)$, $t_0 \leq t \leq t_2$, and
(b) $m(t_2) = w(t_2,\epsilon)$.

It then follows from (a) and (b) that

$$D_{-}m(t_2) \geq \liminf_{h\to 0^-} \frac{w(t_2+h,\epsilon) - w(t_2,\epsilon)}{h} = D_{-}w(t_2,\epsilon) = g(t_2,w(t_2,\epsilon)) + \epsilon. \tag{3.5}$$

From the assumption on g, the solutions $w(t,\epsilon)$ are increasing functions of t. Since, $m(t) = L(t,U(t))$ and using (a) and (b), we have

$$L(s,U(s)) \leq L(t_2,U(t_2)),\ \ t_0 \leq s \leq t_2.$$

Consequently, $U(t,t_0,U_0) \in E_1$, $t_0 \leq t \leq t_2$. Since $L(t,U)$ is Lipschitzian in U and satisfies condition (i), we have

$$\begin{aligned} m(t+h) - m(t) &= L(t+h,U(t+h)) - L(t,U(t)) \\ &= L(t+h,U(t+h)) - L(t+h,U(t)+h(QU)(t)) \\ &\quad + L(t+h,U(t)+h\,(QU)(t)) - L(t,U(t)) \\ &\geq -K\,D[U(t+h),U(t)+h(QU)(t)] \\ &\quad + L(t+h,U(t)+h(QU)(t)) - L(t,U(t)), \end{aligned}$$

which, upon taking the lim inf as $h \to 0^-$ and using the fact that $D_H U(t)$ exists and is equal to $(QU)(t)$ yields

$$D_{-}m(t) \leq D_{-}L(t,U(t)) \leq g(t,L(t,U(t)) = g(t,m(t)).$$

Therefore, it follows, for $t = t_2$, that

$$D_{-}m(t_2) \leq g(t_2,m(t_2)) = g(t_2,w(t_2,\epsilon)),$$

which is a contradiction to (3.5). Hence the proof of the theorem is complete.

Corollary 3.3. *Let $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and let $L(t,U)$ be locally Lipschitzian in U. Assume that*

$$D_{-}L(t,U(t)) \leq 0 \ \textit{ for } t \geq t_0 \ \textit{ and } U \in E_0.$$

Let $U(t) = U(t,t_0,U_0)$ be any solution of (3.1), then $L(t,U(t)) \leq L(t_0,U_0)$, $t \geq t_0$.

Proof. Proceeding as in the previous theorem with $g(t,w) = 0$, we have

$$L(s,U(s)) \leq L(t_2,U(t_2)),\ \ t_2 \in (t_0,t_1),\ \ t_0 \leq s \leq t_2.$$

Since $L(t_2, U(t_2)) = w(t_2, \epsilon) = L(t_0, U_0) + \epsilon(t_2 - t_0) + \epsilon > 0$, we have $L(s, U(s)) \leq f(L(t_2, U(t_2)))$ for $t_0 \leq s \leq t_2$. The rest of the proof is similar to that of Theorem 3.2.

Theorem 3.4. *Assume the hypotheses of Theorem 3.2 hold, except for inequality (3.2), which is replaced by*

$$\alpha(t)\, D_{-}L(t, U(t)) + L(t, U(t))\, D_{-}\alpha(t) \leq w(t, L(t, U(t))\ \alpha(t)), \tag{3.6}$$

for $t > t_0$, $U \in E_\alpha$, *where* $\alpha(t) > 0$ *is continuous on* $\mathbb{R}_+$ *and* $D_{-}\alpha(t) = \liminf_{h \to 0^-} \frac{\alpha(t+h)-\alpha(t)}{h}$. *Then* $\alpha(t_0)\, L(t_0, U_0) \leq w_0$ *implies that* $\alpha(t)\ L(t, U(t)) \leq r(t)$, $t \geq t_0$.

Proof. Let $P(t, U(t)) = L(t, U(t))\, \alpha(t)$. Let $t \geq t_0$ and $U \in E_\alpha$. For sufficiently small $h > 0$, we have

$$\begin{aligned} &P(t+h, U(t) + h(QU)(t)) - P(t, U(t)) \\ &= L(t+h, U(t) + h(QU)(t))\, \alpha(t+h) - L(t, U(t))\, \alpha(t) \\ &= L(t+h, U(t) + h(QU)(t))\, (\alpha(t+h) - \alpha(t)) \\ &+ [L(t+h, U(t) + h(QU)(t)) - L(t, U(t))]\, \alpha(t), \end{aligned}$$

from which it follows,

$$\begin{aligned} D_{-}P(t, U(t)) &= L(t, U(t))\, D_{-}\alpha(t) + \alpha(t)\, D_{-}L(t, U(t)) \\ &\leq w(t, L(t, U(t))\, \alpha(t)\,) = w(t, P(t, U(t))\,), \end{aligned}$$

for t$\in (t_0, t_1]$ and $U \in E_1$, where E_1, in this case, is to be defined with $P(t, U(t))$ replacing $L(t, U(t))$ in the definition of set E_1. Since $P(t, U)$ is locally Lipchitzian in U, then all the assumptions of Theorem 3.2 are satisfied with $P(t, U(t))$ replacing $L(t, U(t))$. Hence, the conclusion of the theorem follows from the proof of Theorem 3.2.

To prove a general comparison result in terms of Lyapunov-like functions, we need the following know result [5].

Lemma 3.5. *Let* g_0, $g \in C[\mathbb{R}^2_+, \mathbb{R}]$ *be such that*

$$g_0(t, w) \leq g(t, w), \ \ (t, w) \in \mathbb{R}^2_+. \tag{3.7}$$

Then the right maximal solution $r(t, t_0, w_0)$ *of (3.3) and the left maximal solution* $\eta(t, T_0, v_0)$ *of*

$$v' = g_0(t, v), \ \ v(T_0) = v_0, \tag{3.8}$$

satisfy the relation

$$r(t, t_0, w_0) \leq \eta(t, T_0, v_0), \ \ t \in [t_0, T_0],$$

whenever $r(T_0, t_0, w_0) \leq v_0$.

Theorem 3.6. *Assume that*

(i) *$L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and $L(t,U)$ is locally Lipschitzian in U,*

(ii) *$g_0,\ g \in C[\mathbb{R}_+^2,\ \mathbb{R}]$ are such that $g_0(t,w) \le g(t,w)$, $(t,w) \in \mathbb{R}_+^2$, and $\eta(t,T_0,v_0)$ is the left maximal solution of (3.8) existing on $t_0 \le t \le T_0$, and $r(t,t_0,w_0)$ the right maximal solution of (3.3) existing on $[t_0,\infty)$;*

(iii) *$D_-L(t,U(t)) \le g(t,L(t,U(t)))$ on Ω, where*

$$\Omega = \{U \in E : L(s,U(s)) \ \le\ \eta(s,t,L(t,U(t))),\ t_0 \le s \le t\}.$$

Then we have

$$L(t,U(t,t_0,U_0)) \le r(t,t_0,w_0)\,,\ t \ge t_0, \tag{3.9}$$

whenever $L(t_0,U_0) \le w_0$.

Proof. Set $m(t) = L(t,U(t,t_0,U_0))$, $t \ge t_0$, so that $m(t_0) = L(t_0,U_0) \le w_0$. Let $w(t,\epsilon)$ be any solution of

$$w' = g(t,w) + \epsilon\,,\ \ w(t_0) = w_0 + \epsilon,$$

for sufficiently small $\epsilon > 0$. Then since $r(t,t_0,w_0)\ = \lim_{\epsilon\to 0^+} w(t,\epsilon)$, it is enough to prove that $m(t) < w(t,\epsilon)$ for $t \ge t_0$. If this is not true, there exists a $t_1 > t_0$ such that $m(t_1) = w(t_1,\epsilon)$ and $m(t) < w(t,\epsilon)$ for $t_0 < t < t_1$. This implies that

$$D_-m(t_1) \ge w'(t,\epsilon) = g(t_1,m(t_1)) + \epsilon. \tag{3.10}$$

Now consider the left maximal solution $\eta(s,t_1,m(t_1))$ of (3.8) with $v(t_1) = m(t_1)$ on the interval $t_0 < t < t_1$. By Lemma 3.5, we have

$$r(s,t_0,w_0)\ \le \eta(s,t_1,m(t_1)),\ s \in [t_0,t_1].$$

Since

$$r(t_1,t_0,w_0)\ =\ \lim_{\epsilon\to 0^+} w(t,\epsilon) = m(t_1) = \eta(t_1,t_1,m(t_1))$$

and $m(s) \le w(s,\epsilon)$ for $t_0 < s \le t_1$, it follows that

$$m(s) \le r(s,t_0,w_0)\ \le \eta(s,t_1,m(t_1)),\ s \in [t_0,t_1].$$

This inequality implies that hypothesis (iii) holds for $U(s,t_0,U_0)$ on $t_0 < s \le t_1$, and hence, standard computation yields

$$D_-m(t_1) \le g(t_1,m(t_1)),$$

which contradicts (3.10). Thus $m(t) \le r(t,t_0,w_0)$, $t \ge t_0$, and the proof is complete.

4. STABILITY CRITERIA

In order to discuss the stability properties of (3.1), let us assume that the solutions of (3.1) exist and are unique for all $t \geq t_0$. In addition, in order to match the behavior of solutions of (3.1) with those of the corresponding ordinary differential equation with causal map, we assume that $U_0 = V_0 + W_0$, so the Hukuhara difference $U_0 - V_0 = W_0$ exists. Consequently, in what follows, we consider the solutions $U(t) = U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$. Thus we have the initial-value problem

$$D_H U(t) = (QU)(t), \qquad (4.1)$$
$$U(t_0) = W_0.$$

To illustrate the idea mentioned above, we present a simple example in $K_c(\mathbb{R}^n)$.

Consider

$$D_H U(t) = -\int_0^t U(s)\,ds,\ U(0) = U_0 \in K_c(\mathbb{R}^n).$$

Then using interval methods, we get

$$u_1' = -\int_0^t u_2(s)\,ds,$$
$$u_2' = -\int_0^t u_1(s)\,ds,$$

where $U(t) = [u_1(t), u_2(t)]$ and $U_0 = [u_{10}, u_{20}]$. Clearly, this yields

$$u_1^{(4)} = u_1,\ u_1(0) = u_{10},$$
$$u_2^{(4)} = u_2,\quad u_2(0) = u_{20},$$

whose solutions are given by

$$u_1(t) = \left(\frac{u_{10} - u_{20}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{u_{10} + u_{20}}{2}\right) cos(t),$$
$$u_2(t) = \left(\frac{u_{20} - u_{10}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{u_{10} + u_{20}}{2}\right) cos(t).$$

That is, for $t \geq 0$,

$$\begin{aligned} U(t, t_0, U_0) &= \left[-\tfrac{1}{2}(u_{20} - u_{10}), \tfrac{1}{2}(u_{20} - u_{10})\right] \left(\tfrac{e^t + e^{-t}}{2}\right) \\ &+ \left[\tfrac{1}{2}(u_{10} + u_{20}), \tfrac{1}{2}(u_{10} + u_{20})\right] cos(t),\quad t \geq 0. \end{aligned}$$

Then choosing

$$V_0 = \left[-\frac{1}{2}(u_{20} - u_{10}), \frac{1}{2}(u_{20} - u_{10})\right],$$

we obtain

$$U(t, t_0, W_0) = \left[\frac{1}{2}(u_{10} + u_{20})\ , \frac{1}{2}(u_{10} + u_{20})\right] cos(t),\quad t \geq 0,$$

which implies the stability of the trivial solution of the initial value problem.

Next, we give an example which illustrates that one can get asymptotic stability as well in SDE with causal maps. Consider the following differential equation

$$D_H U(t) = -a\, U - b \int_0^t U(s)\, ds,\ U(0) = U_0 \in K_c(\mathbb{R}^n), \tag{4.2}$$

$a, b > 0$. As before we take $U(t) = [u_1(t), u_2(t)]$ and $U_0 = [u_{10}, u_{20}]$. Then equation (4.2) reduces to

$$\begin{aligned} u_1' &= -a u_2 - b \textstyle\int_0^t u_2(s)\, ds, \\ u_2' &= -a\, u_1 - b \textstyle\int_0^t u_1(s)\, ds, \end{aligned}$$

and

$$\begin{aligned} u_1^{(4)} &= a^2\, u_1'' + 2\, ab\, u_1' + b^2\, u_1\,, \quad u_1(0) = u_{10}, \\ u_2^{(4)} &= a^2\, u_2'' + 2\, ab\, u_2' + b^2\, u_2\,, \quad u_2(0) = u_{20}, \end{aligned}$$

from which, by choosing $a = 1$ and $b = 2$, we obtain

$$\begin{aligned} u_1(t) &= \tfrac{1}{6}(u_{10} - u_{20})\, e^{-t} + \tfrac{1}{3}\,(u_{10} - u_{20})\, e^{2t} \\ &\quad + e^{-\frac{1}{2}t}\Big[\tfrac{1}{2}(u_{10} + u_{20})\, cos(\tfrac{\sqrt{7}}{2}\, t) - \tfrac{1}{2\sqrt{7}}\,(u_{10} + u_{20})\, sin(\tfrac{\sqrt{7}}{2}\, t)\Big], \\ u_2(t) &= \tfrac{1}{6}(u_{20} - u_{10})\, e^{-t} + \tfrac{1}{3}\,(u_{20} - u_{10})\, e^{2t} \\ &\quad + e^{-\frac{1}{2}t}\Big[\tfrac{1}{2}(u_{10} + u_{20})\, cos(\tfrac{\sqrt{7}}{2}\, t) - \tfrac{1}{2\sqrt{7}}\,(u_{10} + u_{20})\, sin(\tfrac{\sqrt{7}}{2}\, t)\Big]. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} U(t, t_0, U_0) = (u_{20} - u_{10}\Big)\Big[-\frac{1}{6}, \frac{1}{6}\Big]\, e^{-t} + \ (u_{20} - u_{10})\Big[-\frac{1}{3}, \frac{1}{3}\Big]\, e^{2t} \\ +(u_{20} + u_{10}\Big)\Big[\frac{1}{2}, \frac{1}{2}\Big]\, e^{-\frac{1}{2}t}\, cos(\frac{\sqrt{7}}{2}\, t) \\ -(u_{20} + u_{10}\Big)\Big[\frac{1}{2\sqrt{7}}, \frac{1}{2\sqrt{7}}\Big]\, e^{-\frac{1}{2}t}\, sin(\frac{\sqrt{7}}{2}\, t), \quad t \geq 0. \end{aligned}$$

Now, choosing $u_{10} = u_{20}$, we eliminate the undesirable terms and, therefore, we get asymptotic stability of the zero solution of (4.2).

We are now in a position to give sufficient conditions for the stability, and the asymptotic and uniform asymptotic stability of the zero solution of (4.1). First, we state some basic definitions.

Definition 4.1. σ is said to be a $\mathcal{K}$-class function, or $\sigma \in \mathcal{K}$, if $\sigma \in C[[0, \infty), \mathbb{R}_+]$, $\sigma(0) = 0$ and $\sigma(w)$ is increasing in w.

Definition 4.2. The trivial solution $U = \theta$ of (4.1) is said to be stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta = \delta(\, t_0, \epsilon)$ such that $D[W_0, \theta] < \delta$ implies $D[U(t), \theta] < \epsilon,\ t \geq t_0$, where $U(t) = U(t, t_0, W_0)$ is the solution of (4.1).

Other notions of Lyapunov stability can be formulated in a similar way following the standard stability definitions given in [3]. We start by proving a stability result.

Theorem 4.3. *Assume that there exist functions $L(t, U(t))$ and $g(t, w)$ satisfying the following conditions*

(i) $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t, 0) \equiv 0$;

(ii) *$L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$, $L(t, \theta) \equiv 0$, and $L(t, U)$ is positive definite and locally Lipschitzian in U;*

(iii) *for $t > t_0$ and $U \in E_1$, $D_-L(t, U(t)) \leq g(t, L(t, U(t))$.*

Then the stability of the zero solution of (3.3) implies the stability of the zero solution of (4.1).

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Since $L(t, U)$ is positive definite, it follows that there exists a function $b \in \mathcal{K}$ such that

$$b(D[U, \theta]\,) \leq L(t, U) \text{ for } (t, U) \in \mathbb{R}_+ \times B. \tag{4.3}$$

Suppose that the zero solution of (3.3) is stable. Then given $b(\epsilon) > 0,\ t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that whenever $w_0 < \delta$, we have

$$w(t) < b(\epsilon),\, t \geq\, t_0, \tag{4.4}$$

where $w(t, t_0,\, w_0)$ is any solution of (3.3). Choose $w_0 = L(\,t_0,\, W_0)$. Since $L(t, U(t))$ is continuous and $L(t, \theta) \equiv 0$, there exists a positive function $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $D[W_0, \theta] \leq\, \delta_1$ and $L(\,t_0,\, W_0) \leq \delta$ hold simultaneously.

We claim that if $D[W_0, \theta] \leq \delta_1$, then $D[U(t), \theta] \,< \epsilon$ for all $t \geq t_0$. Suppose this not true. Then there exists a solution $U(t) \,=\, U(t,\, t_0,\, W_0)$ of (4.1) satisfying the properties $D[U(t_2), \theta] \,= \epsilon$ and $D[U(t), \theta] \,< \epsilon$ for $t_0 < t < t_2, t_2 \in (t_0, t_1)$. Together with (4.3), this implies that

$$L(t_2, U(t_2)) \geq b(\epsilon). \tag{4.5}$$

Furthermore, $U(t) \in B$ for $t \in [t_0, t_2]$. Hence, the choice of $w_0 = L(t_0, W_0)$ and condition (iii) give, as a consequence of Theorem 3.2, the estimate

$$L(t, U(t)) \leq r(t),\, t \in [t_0, t_2],$$

where $r(t) \,=\, r(t, t_0, w_0)$ is the maximal solution of the comparison problem (3.3). Now from equations (4.3)-(4.5), we have

$$b(\epsilon) \leq L(t_2, U(t_2)) \leq r(t_2) < b(\epsilon),$$

which is a contradiction. Therefore the proof of the theorem is complete.

The following theorem provides sufficient conditions for asymptotic stability of (4.1).

Theorem 4.4. *Assume that*

(i) *there exist functions $L(t, U)$, $g(t, w)$ satisfying the conditions of Theorem 4.3;*

(ii) *there exits a function $\alpha(t)$ such that $\alpha(t) > 0$ is continuous for $t \in \mathbb{R}_+$ and $\alpha(t) \to \infty$ as $t \to \infty$.*

Further, assume that relation (3.6) holds for $t > t_0$, $U \in E_\alpha$. Then, if the zero solution of (3.3) is stable, then the zero solution of (4.1) is asymptotically stable.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Set $\alpha_0 = \min_{t\in\mathbb{R}_+} \alpha(t)$, then $\alpha_0 > 0$ follows from assumption (ii). Since $L(t, U)$ is positive definite, there exists a $b \in \mathcal{K}$ such that (4.3) holds. Define

$$\epsilon_1 = \alpha_0 b(\epsilon). \tag{4.6}$$

Then, the stability of the zero solution of (3.3) implies that, given $\epsilon_1 > 0$ and a $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\epsilon_1, t_0)$ such that $w_0 < \delta$ implies that

$$w(t, t_0, w_0) < \epsilon_1,\ t \geq t_0, \tag{4.7}$$

where $w(t, t_0, w_0)$ is any solution of (3.3). Choose $w_0 = L(t_0, W_0)$. Then proceeding as in the proof of Theorem 3.7 with ϵ_1 instead of $b(\epsilon)$, we can prove that the zero solution of (4.1) is stable.

Let $U(t, t_0, W_0)$ be any solution of (4.1) such that $D[W_0, \theta] \leq \delta_0$, where $\delta_0 = \delta(t_0, \frac{1}{2}\rho)$. Since the zero solution of (4.1) is stable, it follows that $D[U(t), \theta] < \frac{1}{2}\rho,\ t \geq t_0$. Since $\alpha(t) \to \infty$ as $t \to \infty$, there exists a number $T = T(t_0, \epsilon) > 0$ such that

$$b(\epsilon)\, \alpha(t) > \epsilon_1,\ \ t \geq t_0 + T. \tag{4.8}$$

Now from Theorem 4.3 and relation (4.3), we get

$$\alpha(t)\ b(D[U(t), \theta])$$

$$\leq \alpha(t)\ L(t, U(t)) \leq r(t),\ \ t \geq t_0, \tag{4.9}$$

where $U(t) = U(t, t_0, W_0)$ is any solution of (4.1) such that $D[W_0, \theta] \leq \delta_0$.

If the zero solution of (4.1) is not asymptotically stable, then there exists a sequence $\{t_k\}$, $t_k \geq t_0 + T$ and $t_k \to \infty$ as $k \to \infty$ such that $D[U(t_k), \theta] \geq \epsilon$ for some solution $U(t)$ satisfying $D[W_0, \theta] \leq \delta_0$. The relations (4.7) and (4.9) yield that $b(\epsilon)\, \alpha(t_k) \geq \epsilon_1$,a contradiction to (4.8). Thus, the zero solution of (4.1) is asymptotically stable.

The next theorem gives sufficient conditions for the uniform asymptotic stability of (4.1).

Theorem 4.5. *Assume there exists a function $L(t, U)$ satisfying the following properties:*

(i) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$, *where* $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$, $L(t, U)$ *is positive definite, decrescent and locally Lipschitzian in* U*;*

(ii) $D_- L(t, U(t)) \leq -c(D[U(t), \theta])$ *for* $t > t_0$, $U \in E_0$,*and* $c \in \mathcal{K}$.

Then the zero solution of (4.1) is uniformly asymptotically stable.

Proof. Since $L(t, U)$ is positive definite and decrescent, there exist $a, b \in \mathcal{K}$ such that

$$b(D[U,\theta]\,) \le L(t,U)\ \le a(D[U,\theta]) \tag{4.10}$$

for $(t, U) \in \mathbb{R}_+ \times B$. Let $0 < \epsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\epsilon) > 0$ such that

$$a(\delta) < b(\epsilon). \tag{4.11}$$

We claim that if $D[W_0, \theta] \le \delta$, $D[U(t), \theta] < \epsilon$ for all $t \ge t_0$, where $U(t) = U(t, t_0, W_0)$ is any solution of (4.1). Suppose this is not true. Then there exists a solution $U(t)$ of (4.1) with $D[W_0, \theta] \le \delta$ and $t_2 > t_0$, such that $D[U(t_2, t_0, W_0), \theta] = \epsilon$ and $D[U(t, t_0, W_0), \theta] \le \epsilon$ for $t \in [t_0, t_2]$. Thus, in view of (4.10), we have

$$L(t_2, U(t_2)) \ge b(\epsilon). \tag{4.12}$$

It is clear that, since $\epsilon < \rho$, $U(t) \in B$. By our choice of $w_0 = L(t_0, W_0)$ and by the condition that $D_{-}L(t, U(t)) \le 0$ for $t > t_0$, $U \in E_0$, and by Corollary 3.3, we have the estimate

$$L(t, U(t)) \le L(t_0, W_0),\ t \in [t_0, t_2]. \tag{4.13}$$

Now the relations (4.10)-(4.13) lead to the contradiction $b(\epsilon) \le L(t_2, U(t_2)) \le a(D[W_0, \theta]\,) \le a(\delta) < b(\epsilon)$.

This proves uniform stability. Now let $U(t) = U(t, t_0, W_0)$ be any solution of (4.1) such that $D[W_0, \theta] \le \delta_0$, where $\delta_0 = \delta(\frac{\rho}{2})$, δ being the same as before. It then follows from uniform stability that $D[U(t), \theta]\,] \le \frac{\rho}{2}$ for $t \ge t_0$, and hence $U(t) \in B$ for all $t > t_0$. Let $0 < \eta < \delta_0$ be given. Clearly, we have $b(\eta) \le a(\ \delta_0)$. In view of the assumptions on $f(r)$, there exists a $\beta = \beta(\eta) > 0$ such that

$$f(r) > r + \beta \quad \text{if} \quad b(\eta) \le r \le a(\delta_0). \tag{4.14}$$

Furthermore, there exists a positive integer $N = N(\eta)$ such that

$$b(\eta) + N\beta > a(\delta_0). \tag{4.15}$$

If we have, for some $t \ge t_0$, $L(t, U(t)) \ge b(\eta)$, it follows from (4.10) that there exists a $\delta_2 = \delta(\eta) > 0$, such that $D[U(t), \theta] \ge \delta_2$. This in turn implies that

$$c(D[U(t), \theta]) \ge c(\delta_2) = \delta_3, \tag{4.16}$$

where $\delta_3 = \delta_3(\eta)$. We construct $N + 1$ numbers $t_k = t_k(t_0, \eta)$ such that $t_0(t_0, \eta) = t_0$ and $t_{k+1}(t_0, \eta) = t_k(t_0, \eta) + \beta/\ \delta_3$. By letting $T(\eta) = N\beta/\ \delta_3$, we have $t_k(t_0, \eta) = t_0 + T(\eta)$.

Now to prove uniform asymptotic stability, we still have to prove $D[U(t),\theta] < \eta$ for all $t \geq t_0 + T(\eta)$. It is therefore sufficient to show that

$$L(t,U(t)) < b(\eta) + (N-k)\,\beta,\ \ t \geq t_k,\ k = 0,1,2,\ldots,N. \tag{4.17}$$

Now we prove (4.17) by induction. For $k = 0$, $t \geq t_0$, we have, using (4.10) and (4.11),

$$L(t,U(t)) \leq L(t_0,U_0) \leq a(\delta_0) < b(\eta) + N\beta \tag{4.18}$$

Suppose we have, for some k,

$$L(s,U(s)) < b(\eta) + (N-k)\,\beta,\ \ s \geq t_k,$$

and, if possible, assume that for $t \in [t_k, t_{k+1}]$,

$$L(t,U(t)) \geq b(\eta) + (N-k-1)\,\beta.$$

It then follows that

$$a(\delta_0) \geq a(D[U(s),\theta]) \geq L(s,U(s)) \geq b(\eta) + N\beta - (k+1)\,\beta \geq b(\eta).$$

Therefore from (4.14), we conclude that

$$f(L(s,U(s))) \geq L(s,U(s)) + \beta > b(\eta) + (N-k)\beta > L(s,U(s))$$

for $t_k < s < t$, $t \in [t_k, t_{k+1}]$. In turn, this implies that $U(t) \in E_0$ for $t_k < s < t$, $t \in [t_k, t_{k+1}]$. Hence, we obtain from assumption (ii) and (4.18) that

$$\begin{aligned} L(t_{k+1},U(t_{k+1})) &\leq L(t_k,U(t_k)) - \int_{t_k}^{t_{k+1}} c(D[U(s),\theta])\,ds \\ &< b(\eta) + (N-k)\beta - \delta_3\,(t_{k+1}-t_k) \\ &< b(\eta) + (N-k)\beta. \end{aligned}$$

This contradiction shows that there exists $t^* \in [t_k, t_{k+1}]$ such that

$$L(t^*,U(t^*)) < b(\eta) + (N-k-1)\,\beta. \tag{4.19}$$

Now we show that (4.19) implies that

$$L(t,U(t)) < b(\eta) + (N-k-1)\,\beta,\ t \geq t^*.$$

If not true, then there exists $t_1 > t^*$ such that $L(\,t_1,U(\,t_1)) = b(\eta)+(N-k-1)\,\beta$, or for small $h < 0$, $L(\,t_1+h,U(\,t_1+h)) < b(\eta) + (N-k-1)\,\beta$, which implies that

$$D_-L(t_1,U(t_1)) \geq 0. \tag{4.20}$$

As we did before, we can show that $U(t) \in B$, for $t^* \leq s \leq t_1$, and $D_-L(t_1,U(t_1)) \leq -\delta_3 < 0$. This contradicts (4.20), and hence

$$L(t,U(t)) < b(\eta) + (N-k-1)\,\beta,\ t \geq t_{k+1}.$$

This completes the proof of the theorem.

Our final stability result is a general result, which offers various stabillity criteria in a single set-up. The proof of this theorem, which can be obtained using the comparison result given in Theorem 3.6, is omitted.

Theorem 4.6. *Assume that there exists a function $L(t, U)$ satisfying properties (i), (ii), and (iii) of Theorem 3.6. Then the stability properties of the zero solution of (3.3) imply the corresponding properties of the zero solution of (4.1).*

We now show that Theorem 4.6 unifies the various stability results discussed earlier. To that end, consider the following special cases:

(a) Suppose $g_0(t, w) \equiv 0$. Then $\eta(s, T_0, v_0) = v_0$, and hence Ω reduces to E_1.

(b) Suppose $g_0(t, w) = -[\alpha'(t)/\alpha(t)]\, w$, where $\alpha(t) > 0$ is continuously differentiable on $\mathbb{R}_+$ and $\alpha(t) \to \infty$ as $t \to \infty$. Let $g(t, w) = g_0(t, w) + [1/\alpha(t)]\ g_1(t, \alpha(t)w)]$ with $g_1 \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, then $\eta(s, T_0, v_0) = v_0[\alpha(T_0)\,/\alpha(s)]$. Thus $\Omega = E_\alpha$.

(c) Let $g_0 = g = -c(w)$, $c \in \mathcal{K}$. Then it is easy to show that $\eta(s, T_0, v_0) = \phi^{-1}[\phi(v_0) -(s - T_0)]$, $t_0 \le s \le T_0$ where $\phi(w) = \phi(w_0) + \int_{w_0}^{w} \frac{ds}{c(s)}$ and ϕ^{-1} is the inverse function of ϕ. Since $\eta(s, T_0, v_0)$ is increasing in s to the left of T_0, on choosing a fixed $s_0 \le T_0$ and defining $f(r) = \eta(s_0, T_0, v_0)$, it is clear that $f(r) > r$ for $r > 0$. Thus $f(r)$ is continuous and increasing in r. Hence, $\Omega = E_0$.

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