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INFINITE DIMENSIONAL STOCHASTIC VOLTERRA EQUATIONS WITH DISSIPATIVE NONLINEARITY

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ABSTRACT. We investigate a class of integro-differential stochastic evolution equation with additive noise and dissipative nonlinearity. We find the existence and uniqueness of a generalized solution in the space of pathwise continuous adapted processes with values in a Banach space. We also establish a large deviation principle for the law of the solution with explicit rate functional.

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1. INTRODUCTION

Let H be a real separable Hilbert space. In this paper we are concerned with the following class of integral Volterra equations perturbed by a additive Wiener noise

$$u(t) = x - \int_0^t a(t-s)Au(s)\,\mathrm{d}s + \int_0^t a(t-s)F(u(s))\,\mathrm{d}s + B\,W(t), \tag{1.1}$$

for $t \in [0,T]$; W is a cylindrical Wiener noise and we assume that F is a nonlinear operator defined on a subset of the Hilbert space H. Following Da Prato & Zabczyk [10], we shall consider (1.1) in a smaller state space $X \subset H$, on which the operator F is well defined and continuous. This method requires also that the initial condition x takes values in the smaller space X.

Our main result, Theorem 3.4, asserts the existence and uniqueness of the solution (in a suitable sense) for (1.1). Notice that the solution actually exists for all time, since T is arbitrary; however, many of the bounds involved in the proofs depend on T, so it is not possible to give the result directly on $\mathbb{R}_+ = [0,\infty)$.

The study of stochastic evolutionary integral equations, of which (1.1) is a special case, is mainly motivated by applications in mathematical physics, such as viscoelasticity, heat conduction and electro-dynamic with memory. For a description of such models we refer to the monograph of Prüss [17].

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Recently, Clement *et al.* [5, 6, 7] proposed to introduce white noise perturbations for these models to account for the presence of rapidly varying forces. We extended, in the authors [2], this approach to cover the case non-linear equations in case of a Lipschitz non-linearity. Our techniques applied there do not extend to cover the case of dissipative perturbations; an extension of known existence results for deterministic Volterra equations was considered separately in the authors [1], which is, therefore, a basic reference for us. However, this result requires stronger assumptions on the scalar kernel a with respect to the authors [2].

In the second part of paper we consider, under the same assumptions, further regularity of the solution and we establish, in Theorem 5.3, a large deviation principle (LDP for short) for (1.1), with reduced noise $\sqrt{\varepsilon}B$, $\varepsilon \downarrow 0$.

Large deviation estimates have gained an increasing interest in literature since the foundational works of Varadhan [19] and Freĭdlin and Wentzell[14]. One reason for this importance is the existence of a general reference framework, from which one can adapt the results in diverse cases; on the other hand, large deviations proved to be crucial in several applications in applied probability: see for instance the bibliographical discussion in Dembo and Zeitouni [11, Section 5.9].

2. SETTING OF THE PROBLEM

Notation. Throughout the paper H is a separable Hilbert space with norm $|\cdot|$. We denote $\mathfrak{L}(H)$ the space of linear bounded operators from H into itself and $\mathfrak{L}_2(H)$ the subspace of Hilbert-Schmidt operators.

In this section we discuss the existence and uniqueness result for (1.1); but, before, we shall introduce some relevant background material as well as our assumptions.

2.1. Volterra equations.

Let us consider the linear integral Volterra equation

$$u(t) = x - \int_0^t a(t-\vartheta) A u(\vartheta)\,\mathrm{d}\vartheta, \qquad t \in [0,T]. \tag{2.1}$$

In this section we state our conditions on the coefficients of (2.1). They necessarily follow those in the authors [1, Theorem 1.1], which is a basic tool for our construction. As a general reference for the material quoted here we refer to Prüss [17].

Let us recall that a kernel $a : (0,\infty) \to \mathbb{R}$ is said to be *completely monotonic* if, for any $n \geq 0$ it holds $\frac{\mathrm{d}^n}{\mathrm{d}t^n} a(t) \geq 0$ for $t \in (0,\infty)$.

Hypothesis 2.1. *The kernel a is completely monotonic, $a \in L^1_{loc}(0,\infty)$, and there exists a function $k(t) = k_0 + \int_0^t k_1(s)\,\mathrm{d}s$ associated to a, the relation between a and k being given by*

$$k_0 a(t) + \int_0^t k_1(t-s) a(s)\,\mathrm{d}s = 1, \qquad t \in (0,\infty). \tag{2.2}$$

The function k is called a *Bernstein function*, compare Prüss [17, Proposition 4.4]. It holds that k_1 is completely monotonic, and k_0 is a non-negative constant, related to $a(0+)$ by

$$k_0 = 0 \iff a(0+) = +\infty.$$

Hypothesis 2.2. *We assume the following conditions on the operator A.*

2.2a: *$-A$ is a linear operator in H, generates a strongly continuous semigroup of type ω; the eigenvalues $\{\mu_k\}_{k\geq 1}$ of A form a non-decreasing sequence with $\lim_{k\to\infty} \mu_k = \infty$ and the corresponding eigenvectors $\{e_k\}_{k\geq 1}$ form a complete orthonormal system in H;*

2.2b: *X is a Banach space, densely, continuously and as Borel subspace embedded in H, with norm $\|\cdot\|$; the part on X of $-(A+\omega I)$ generates a strongly continuous contractions semigroup on X.*

Now we recall the basic definition of resolvent operator. A family $\{S(t)\}_{t\in\mathbb{R}_+}$ of bounded linear operators in a Banach space X is called a resolvent of (2.1) if the following conditions are satisfied:

1. $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
2. $S(t)$ commutes with A for all $t \in \mathbb{R}_+$;
3. the resolvent equation holds: for all $x \in D(A)$, $t \in \mathbb{R}_+$:

$$S(t)x = x - \int_0^t a(t-s)AS(s)x\,\mathrm{d}s. \tag{2.3}$$

Proposition 2.3. *Under Hypotheses 2.1 and 2.2, Volterra equation* (2.1) *is well posed,* i.e., *it defines a family of resolvent operators $\{S(t)\}_{t\in\mathbb{R}_+}$.*

It is possible to show that the resolvent admits a diagonal decomposition in the basis $\{e_k\}_{k\geq 1}$ of H. We introduce, then, the solution s_α of the scalar integral equation

$$s_\alpha(t) + \alpha \int_0^t a(t-\vartheta)s_\alpha(\vartheta)\,\mathrm{d}\vartheta = 1, \qquad t \in \mathbb{R}_+. \tag{2.4}$$

Let μ_k be an eigenvalue of A with eigenvector e_k. Then

$$S(t)e_k = s_{\mu_k}(t)e_k, \qquad t \in \mathbb{R}_+. \tag{2.5}$$

We investigate further the properties of s_α. Under Hypothesis 2.1, it is proved in Clément & Da Prato [5] that

$$\int_0^T |s_\alpha(t)|^2\,\mathrm{d}t \leq C\frac{1}{\alpha}$$

for any $\alpha > 0$. For similar estimates we refer also to Bonaccorsi & Tubaro [3].

We can express s_α in terms of another kernel r_α, defined as the solution to the integral equation

$$r_\alpha(t) + \alpha \int_0^t r_\alpha(t-s) a(s)\, \mathrm{d}s = \alpha a(t), \qquad t \in \mathbb{R}_+. \tag{2.6}$$

By Prüss [17, Lemma 4.1], since a is completely monotonic, we know that for any $\alpha > 0$, r_α belongs to $L^1(\mathbb{R}_+) \cap C(0,\infty)$, it is completely monotonic, $0 \le r_\alpha(t) \le \alpha a(t)$ and

$$\int_0^\infty r_\alpha(s)\, \mathrm{d}s \le 1.$$

Moreover, if $\alpha < 0$, then r_α belongs to $L^1_{loc}(\mathbb{R}_+) \cap C(0,\infty)$ and $r_\alpha(t) \le \alpha a(t) < 0$, compare also Friedman [15]. The relation between s_α and r_α is clarified in the following formula:

$$s_\alpha(t) = \left(1 - \int_0^t r_\alpha(\tau)\, \mathrm{d}\tau\right), \qquad t \in \mathbb{R}_+. \tag{2.7}$$

A Gronwall-type lemma. We state, here, a useful tool to prove estimates for the solution of Volterra equations. The proof of this lemma is given in the authors [1].

Lemma 2.4. *Let v be a continuous, non negative function which satisfies the estimate*

$$v(t) \le s_\lambda(t) x + \tfrac{1}{\lambda} f(t) + \tfrac{\omega}{\lambda} v(t) + r_\lambda * v(t),$$

where $\lambda > \omega$, while s_λ and r_λ are defined in (2.4) *and* (2.6) *respectively. Then*

$$v(t) \le \frac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{\omega_\lambda}{\omega}(x + \tfrac{1}{\lambda} f + a * f) * s_{-\omega_\lambda}\right)(t), \tag{2.8}$$

where $s_{-\omega_\lambda}$ is defined as in (2.4) *with $\omega_\lambda = \frac{\lambda\omega}{\lambda-\omega}$.*

2.2. **Stochastic convolution.** The material in this section is standard; for further background material on stochastic differential equations we refer to Da Prato & Zabczyk [10], our approach to stochastic Volterra equations follows the lines of Clément & Da Prato [5, 6], Clément *et al.* [7].

We are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a right-continuous filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$. Every stochastic process in the sequel will be given with respect to this basis.

We shall take a cylindrical Wiener process $W(t)$, $t \in \mathbb{R}_+$, of the form

$$\langle W(t), h \rangle = \sum_{k=1}^\infty \langle e_k, h \rangle\, \beta_k(t), \qquad t \in \mathbb{R}_+,$$

where $\{\beta_k\}_{k \ge 1}$ is a sequence of independent real standard Brownian motions.

Hypothesis 2.5. *The operator $B \in \mathfrak{L}(H)$ satisfies the following condition*

$$\int_0^T \|S(t)B\|^2_{\mathfrak{L}_2(H)}\, \mathrm{d}\vartheta < +\infty. \tag{2.9}$$

We define the *stochastic convolution process* W_S as the solution of the linear equation

$$z(t) = \int_0^t a(t-s)Az(s)\,\mathrm{d}s + B\,W(t), \quad t \in [0,T];$$

then W_S is defined by the formula

$$W_S(t) = \int_0^t S(t-\sigma)B\,\mathrm{d}W(\sigma), \qquad t \in [0,T],$$

where S is the resolvent operator introduced above. In this paper it is necessary to assume a space-time regularity for the noise which is more stringent than the case treated in [2].

Hypothesis 2.6. *The process W_S has a X-continuous version.*

Remark 2.7. Assume that the operator B is a non-negative operator which commutes with A, *i.e.*, there exists a sequence $\{\lambda_k \in \mathbb{R}_+\}_{k\geq 1}$ such that

$$\langle B\,W(t), h\rangle = \sum_{k=1}^{\infty} \lambda_k \beta_k(t)\,\langle e_k, h\rangle, \qquad t \in \mathbb{R}_+.$$

then it is possible to express condition (2.9) in terms of the eigenvalues λ_k and μ_k, since the operators $S(t)$ and B, diagonalize on the same basis, as

$$\sum_{k=k_0}^{\infty} \frac{\lambda_k}{\mu_k} < +\infty$$

where μ_{k_0} is the first positive eigenvalue.

In some cases, we can state also Hypothesis 2.6 in terms of s_{μ_k}, compare for instance the next example, which is taken from Clément & Da Prato [5, Theorem 4.1].

Proposition 2.8. *Assume that H is the space $L^2(\mathcal{O})$ of square integrable functions on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. Suppose that for some $\gamma > 0$ there exists $\delta < 1$ with*

$$\int_0^u |s_\mu(t-\vartheta) - s_\mu(\tau-\vartheta)|^2\,\mathrm{d}\vartheta + \int_\tau^t |s_\mu(t-\vartheta)|^2\,\mathrm{d}\vartheta \leq C\frac{1}{\mu^\delta}|t-\tau|^{2\gamma}, \tag{2.10}$$

and that:

$$\sum_{k=1}^{\infty} \lambda_k^2 \frac{1}{\mu_k^\delta} < \infty. \tag{2.11}$$

Assume further that there exists $M > 0$ such that

$$\begin{cases} |e_k(x)| \leq M, & k \in \mathbb{N}, \quad x \in \mathcal{O}, \\ |\nabla e_k(x)| \leq M\mu_k^{1/2}, & k \in \mathbb{N}, \quad x \in \mathcal{O}. \end{cases} \tag{2.12}$$

Then the trajectories of $W_S(t,x)$ are almost surely continuous in (t,x).

Remark 2.9. Let $H = L^2(0,1)$ and $X = C_0([0,1])$, and set $-Au = D^2u$, for any $u \in D(A) = H^2(0,1) \cap H^1_0(0,1)$. Then a complete orthonormal system in H defined by eigenvectors of A is given by

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin(k\pi x), \qquad x \in [0,1], \quad k \in \mathbb{N},$$

with $\mu_k = \pi^2 k^2$, $k \in \mathbb{N}$. It is a simple computation to show that (2.12) holds.

Let a be a completely positive kernel: then inequality (2.10) holds for any $\delta > 0$. Now, from the definition of μ_k, the series in (2.11) converges for any $\delta > \frac{1}{2}$, hence we may apply Proposition 2.8 and in this case Hypothesis 2.6 is verified. □

2.3. **The solution of stochastic Volterra equation.** We say that a stochastic process u in X is a mild solution to the equation (1.1) if u is a X-continuous, adapted process and it verifies the integral equation

$$u(t) = S(t)x + \int_0^t S(t-\sigma)F(u(\sigma))\,\mathrm{d}\sigma + W_S(t) \tag{2.13}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. If, given a X-valued, adapted process u, there exists a sequence u_n of mild solutions for (a family of approximating equations for) (1.1) which converges to u uniformly in $[0,T]$ $\mathbb{P}$-a.s., then u is said a generalized mild solution of (1.1).

It remains to state our assumptions on the perturbation term F.

Hypothesis 2.10. *The perturbation term F maps X into X, it is uniformly continuous on bounded sets of X and F is m-dissipative on X.*

The theory of accretive (and dissipative) operators is well known in the literature; as a general reference we mention the monograph of Da Prato [9].

Dissipativity condition holds iff for any $x, y \in X$ and for all $\lambda > 0$: $\|x - y\| \le \|x - y - \lambda(F(x) - F(y))\|$. m-dissipativity means that F is dissipative and the Range$(I - \lambda F) = X$ for one (hence for all) $\lambda > 0$.

3. MAIN THEOREM

Theorem 3.1. *Let us assume Hypothesis 2.1, 2.2, 2.5, 2.6 and 2.10. Then for any $x \in X$, there exists a unique generalized mild solution of* (1.1).

3.1. **Volterra operators.** As stated in the introduction, the proof of Theorem 3.1 is based on the results in the authors [1] for existence of a solution to a class of deterministic non-linear Volterra equations. In order to make this paper self-contained, we give a survey of these results, later we finish the proof of Theorem 3.1.

Let us introduce the linear Volterra operator

$$Lu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[k_0 u(t) + \int_0^t k_1(t-s)u(s)\,\mathrm{d}s\right], \qquad t > 0, \tag{3.1}$$

with domain

$$D(L) = \{f \in L^1(\mathbb{R}_+; X) | k_0 f + (k_1 * f) \in W_0^{1,1}(\mathbb{R}_+; X)\}$$

The operator L is m-accretive and densely defined, see Clément [4], Proposition 3.2. There is a natural representation of its inverse operator L^{-1} in terms of the kernel a.

$$L^{-1}v(t) = \int_0^t a(t-s)v(s)\,\mathrm{d}s. \tag{3.2}$$

We shall consider also the Yosida approximation $L_\mu = L(I + \frac{1}{\mu}L)^{-1}$. The operator L_μ is given by the formula

$$L_\mu v(t) = \mu \frac{\mathrm{d}}{\mathrm{d}t}(v * s_\mu)(t). \tag{3.3}$$

Consider a non-linear Volterra equation

$$u(t) = x - \int_0^t a(t-s)Au(s)\,\mathrm{d}s + \int_0^t a(t-s)f(s)\,\mathrm{d}s, \qquad t \in [0,T]; \tag{3.4}$$

it is known that this problem is equivalent to

$$\begin{cases} L[u(\cdot) - x](t) + Au(t) = f(t), & t \in [0,T], \\ k_0 u(0) + (a * u)(0+) = k_0 x. \end{cases} \tag{3.5}$$

In order to define a generalized solution to (3.5), we shall consider an approximate equation, where the operator L is replaced by its Yosida approximation L_μ, $\mu > 0$. Let u_μ be the solution of the following equation

$$L_\mu[u_\mu(\cdot) - x](t) + Au_\mu(t) = f(t). \tag{3.6}$$

Once we establish the existence of a solution of (3.6) for any $\mu > 0$, we let μ go to ∞ and, provided that the sequence u_μ converges, define a generalized solution to (3.5) as the limit of such sequence.

Theorem 3.2. *Assume that Hypotheses 2.1 and 2.2 are satisfied and let $x \in X$ and $f \in C(\mathbb{R}_+; X)$. Then, for every $\mu > 0$ equation (3.6) has a unique solution $u_\mu \in C(\mathbb{R}_+; X)$.*

As $\mu \to \infty$, there exists a function $u = U(x,f)$ with $u \in C(\mathbb{R}_+; X)$ such that $u_\mu \to u$ in $L^\infty_{loc}(\mathbb{R}_+; X)$.

The function $u = U(x, f)$, that exists according to Theorem 3.2, is said the generalized solution for problem (3.5).

In this paper, we are interested to non-linear perturbation of the linear Volterra equation, i.e., with $f(t) = F(t, u(t))$. Then we shall define $u = U(x, F(\cdot, u))$ a generalized solution of

$$\begin{cases} L[u(\cdot) - x](t) + Au(t) = F(t, u(t)), \\ t \in (0, \infty), \qquad u(0+) = x. \end{cases} \tag{3.7}$$

The existence of a generalized solution to (3.7) depends on the assumptions on the non-linear term F.

Before we discuss the case of dissipative non-linearities, that is the object of Theorem 3.4, we shall consider the case of a Lipschitz non-linearity.

Theorem 3.3. *Let the assumptions of Theorem 3.2 be fulfilled and assume that the nonlinear term $F : [0, T] \times X \to X$ is a continuous function, and there exists a function $\eta(t) \in L^\infty_{loc}(\mathbb{R}_+)$ such that, for any $t \in \mathbb{R}_+$*

$$\|F(t, u) - F(t, v)\| \le \eta(t)\|u - v\|. \tag{3.8}$$

Then there exists a unique generalized solution to the problem (3.7)*.*

Then we arrive to the case of dissipative non-linearity needed in Theorem 3.1.

Theorem 3.4. *Assume X is a real Banach space and let the coefficients in* (3.7) *satisfy Hypotheses 2.1, 2.2, 2.10, 2.5 and 2.6. Then, for any $x \in X$, there exists a unique generalized solution v to the abstract non-linear Volterra equation* (3.7)*.*

Sketch of the proof. We introduce, for any $\alpha > 0$, the approximating equation

$$L(v_\alpha(\cdot) - x)(t) + Av_\alpha(t) = F_\alpha(t, v_\alpha(t)), \tag{3.9}$$

where F_α are the Yosida approximations of F. Since F_α is Lipschitz continuous and bounded in norm by $\|F\|$, we obtain an a priori estimate for the approximating solution v_α as follows:

$$\|v_\alpha(t)\| \le s_{-\omega}(t)\,\|x\| - \tfrac{1}{\omega}\left(r_{-\omega} * \|F(\cdot, 0)\|\right)(t).$$

This assures that the sequence $\{v_\alpha\}_{\alpha>0}$ is bounded uniformly in α, and there exists R_1 such that $\|v_\alpha(t) \le R_1\|$ for all α and $t \in [0, T]$.

To show the convergence of the sequence, we set, for any $\alpha, \beta > 0$,

$$g^{\alpha,\beta}(t) = v_\alpha(t) - v_\beta(t).$$

and we estimate the relevant norm with techniques –analogous to those used in the proof of [10, Theorem 7.13]– based on the previous estimate of $\|v_\alpha(t)\|$ and dissipativity of F. Define $R \ge R_1$ such that $\|F(t, v_\alpha(t) \le R\|$ for all α and $t \in [0, T]$. Standard calculations as in the authors [1] lead to

$$\|g^{\alpha,\beta}(t)\| \le [\rho_F(\tfrac{2}{\alpha}R) + \rho_F(\tfrac{2}{\beta}R)]\big(a * s_{-\omega}\big)(t), \tag{3.10}$$

where ρ_F is the modulus of continuity of F on the bounded set $B(0,2R)$ (*i.e.*, a function such that $\rho_F(s) = \sup\{\|F(x_1) - F(x_2)\| : x_1, x_2 \in B(0,2R), \|x_1 - x_2\| \le s\}$).

This yields the convergence of the sequence v_α in $C([0,T];X)$ to a function v, which is easily seen to be the unique generalized solution for problem (3.7). □

We conclude this section with a last remark about (3.7).

Remark 3.5. Notice that we are concerned with a continuous and m-dissipative operator F; however, since this term is non-autonomous, we cannot consider the sum $A - F$ as a unique operator, even if we assume that $A - F$ is m-accretive and use directly Gripenberg [16, Theorem 1].

3.2. **Proof of Theorem 3.1.** We search for a X-valued process $u(t)$, $t \in [0,T]$, which solves (1.1). We start by writing formally the mild integral equation, compare (2.13)

$$u(t) = S(t)x + \int_0^t S(t-\sigma)F(u(\sigma))\,\mathrm{d}\sigma + W_S(t).$$

Now, we define $v(t) = u(t) - W_S(t)$ and note that the X-valued process v shall be a solution of the Volterra equation

$$v(t) = x - \int_0^t a(t-s)Av(s)\,\mathrm{d}s + \int_0^t a(t-s)F(v(s) + z(s))\,\mathrm{d}s, \tag{3.11}$$

where $z(t) = W_S(t) \in C(\mathbb{R}_+;X)$ is a trajectory of the stochastic convolution process.

Hypothesis 2.6 and 2.10 on the non-linear term F and the X-continuity of W_S implies that $F(t,v) = F(z(t) + v)$ verifies the assumptions of Theorem 3.4; we have proved that there exists a unique generalized solution v of problem (3.11), and $u(t) = v(t) + W_S(t)$ is a generalized mild solution of (1.1) according to our definition in Section 2.3

4. TRANSFER FUNCTIONAL

In this section we focus on the functional $\Phi : C_0([0,T];X) \to C([0,T];X)$ that associates to any $z \in C_0([0,T];X)$ the solution v of problem (3.11). Our aim is to prove the continuity of this functional, which is the key point in the proof of the LDP. From now on we fix the initial condition $x \in X$.

Theorem 4.1. *Suppose the assumptions of Theorem 3.4 hold; then the transfer functional Φ is continuous.*

Proof. Our argument is divided in two steps. In the former step we suppose that the non linear term F is locally Lipschitz on X, while in the latter we prove the theorem in the general case. For the existence of the functional Φ in both cases we refer to Theorem 3.3 and Theorem 3.4 respectively.

To show continuity of Φ, fix a point z_1 of $C_0([0,T];X)$, and a bounded subset B around z_1. Then, there exists a bounded Borel subset $C \subset X$ such that $z(t) \in C$ for any $z \in B$ and $t \in [0,T]$. Since F is locally Lipschitz, we can suppose, without loss of generality, that F is Lipschitz on $C \subset X$, with Lipschitz constant equal to Λ.

Let $z_2 \in B$ and denote by v_1 and v_2 the solutions $\Phi(z_1)$ and $\Phi(z_2)$ respectively. By the definition of generalized solution, we have that there exist two sequences $v_{1,\mu}, v_{2,\mu}$ such that

$$v_{i,\mu} \to v_i \in C(\mathbb{R}_+;X),$$

and

$$L_\mu(v_{i,\mu}(\cdot) - x)(t) + Av_{i,\mu}(t) = F(v_i(t) + z_i(t)).$$

Then subtracting term to term we have

$$\begin{aligned} L_\mu(v_{1,\mu}(\cdot) - v_{2,\mu}(\cdot))(t) + A(v_{1,\mu}(t) - v_{2,\mu}(t)) \\ = F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)). \end{aligned} \tag{4.1}$$

Choose, now, an element y^* in the sub-differential $\partial\|v_{1,\mu}(t) - v_{2,\mu}(t)\|$: if we scalar multiply y^* both members in previous equation, we have

$$\begin{aligned} \langle L_\mu(v_{1,\mu}(\cdot) - v_{2,\mu}(\cdot))(t), y^*\rangle + \langle A(v_{1,\mu}(t) - v_{2,\mu}(t)), y^*\rangle \\ = \langle F(v_1(t) + z_1(t)) - F(v_2(t) + z_2(t)), y^*\rangle . \end{aligned} \tag{4.2}$$

Using the definition of L_μ, we get

$$\begin{aligned} \mu\Big(\|v_{1,\mu}(t) - v_{2,\mu}(t)\| - \big(\|v_{1,\mu} - v_{2,\mu}\| * r_\mu\big)(t)\Big) - \omega\|v_{1,\mu}(t) - v_{2,\mu}(t)\| \\ \le \Lambda(\|v_1(t) - v_2(t)\| + \|z_1(t) - z_2(t)\|). \end{aligned} \tag{4.3}$$

From this equation we obtain an estimate on the norm $\|v_{1,\mu}(t) - v_{2,\mu}(t)\|$ via Lemma 2.4:

$$\begin{aligned} \|v_{1,\mu}(t) - v_{2,\mu}(t)\| \le \Lambda\tfrac{\omega_\mu}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tfrac{1}{\mu}\big[\|v_1 - v_2\| + \|z_1 - z_2\|\big] * s_{-\omega_\mu} \\ + a * \big[(\|v_1 - v_2\| + \|z_1 - z_2\|)\big] * s_{-\omega_\mu}\Big)(t), \end{aligned}$$

and passing to the limit as $\mu \to \infty$ we obtain

$$\|v_1(t) - v_2(t)\| \le \Lambda\frac{\mathrm{d}}{\mathrm{d}t}\Big(a * \big[(\|v_1 - v_2\| + \|z_1 - z_2\|)\big] * s_{-\omega}\Big)(t)$$

that becomes

$$\begin{aligned}\|v_1(t) - v_2(t)\| \le \Lambda\Big(a * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ - \Lambda\Big(a * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ - \tfrac{1}{\omega}\Lambda\Big(r_{-\omega} * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t) \\ = -\tfrac{1}{\omega}\Lambda\Big(r_{-\omega} * \big[\|v_1 - v_2\| + \|z_1 - z_2\|\big]\Big)(t).\end{aligned}$$

Now since $-\frac{1}{\omega}r_{-\omega}$ is a completely monotone kernel, we can apply again Lemma 2.4 and we have

$$\|v_1(t) - v_2(t)\| \le \big(-\widetilde{r}_{-\Lambda} * \|z_1 - z_2\|\big)(t),$$

where $\widetilde{r}_{-\Lambda}$ satisfies

$$\widetilde{r}_{-\Lambda}(t) + \tfrac{\Lambda}{\omega}\big(\widetilde{r}_{-\Lambda} * r_{-\omega}\big)(t) = \tfrac{\Lambda}{\omega}r_{-\omega}(t).$$

Since $-\widetilde{r}_{-\Lambda}(t) \le -r_{-(\omega+\Lambda)}(t)$ for all $t \in \mathbb{R}_+$, we have

$$\|\Phi(z_1)(t) - \Phi(z_2)(t)\| \le \big(-r_{-(\omega+\Lambda)} * \|z_1 - z_2\|\big)(t). \tag{4.4}$$

from where we have that Φ is continuous in the Lipschitz case.

Let us proceed to step two: we can approximate F with its Yosida approximations F_α, so denoting with Φ_α the functional corresponding to Φ in the (3.11) with F_α in place of F, we have:

$$\begin{aligned}\|\Phi(z_1) - \Phi(z_2)\| \le \|\Phi(z_1) - \Phi_\alpha(z_1)\| \\ + \|\Phi_\alpha(z_1) - \Phi_\alpha(z_2)\| + \|\Phi_\alpha(z_2) - \Phi(z_2)\|.\end{aligned}$$

As in the estimate (3.10) in Theorem 3.4, possibly choosing $R' > R$, we have that for all ε there exists α small enough such that

$$\begin{aligned}\|\Phi(z_1) - \Phi_\alpha(z_1)\| \le \varepsilon \\ \|\Phi_\alpha(z_2) - \Phi(z_2)\| \le \varepsilon\end{aligned}$$

for all z_1, z_2 in the same bounded set of $C([0,T];X)$. Now continuity of Φ follows from continuity of Φ_α. □

Corollary 4.2. *Assume hypothesis of Theorem 4.1 hold. Let $\Psi = I + \Phi$, then Ψ is continuous.*

$\Psi : C_0([0;T];X) \to C([0;T];X)$ is the transfer functional related to (1.1), in the sense that Ψ associates to any trajectory of the stochastic convolution W_S the corresponding trajectory of the generalized mild solution u.

5. LARGE DEVIATIONS

In the authors [2], we started considering abstract Volterra equations with additive noise, in the framework of equations in Hilbert spaces. The main technique is the contraction principle; it requires the continuity of transfer functional Ψ, which is formally analogous to the one used for differential equations, see Fantozzi [13, 12].

Here, we are concerned with large deviations in a Banach subspace X of H. In the case of stochastic differential equations with additive Gaussian perturbation it was studied by Smoleński *et al.* [18], by applying Varadhan's contraction principle, see also Da Prato & Zabczyk [10, Theorem 12.15], and the problem was solved assuming that the semilinear part F is locally Lipschitz in X.

We consider (1.1) with B replaced by $\sqrt{\varepsilon}B$, bringing up a family of solutions u_ε. We denote by ν_ε the law of u_ε on the space $C([0,T];X)$, and we want to study the LDP for this laws. First of all, we recall same preliminary results. For any $\varepsilon > 0$, we consider the laws of the processes $\sqrt{\varepsilon}W_S(\cdot)$ on the space $L^2(0,T;H)$.

Theorem 5.1. *Suppose that Hypotheses 2.1, 2.2 and 2.5 hold, and let μ be the law of the stochastic convolution process $W_S(\cdot)$. Then the family μ_ε of laws of $\sqrt{\varepsilon}W_S(\cdot)$ satisfies a large deviation principle with respect to the rate functional I given by*

$$I(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}[f(\vartheta) + (a * Af)(\vartheta)]\right|^2 \mathrm{d}\vartheta \\ \qquad\qquad\qquad \textit{for } f \in R \\ +\infty \qquad\qquad \textit{otherwise.} \end{cases}$$

where R is the subspace of $L^2(0,T;H)$ defined as

$$R = \left\{ f \in L^2(0,T;H) \middle| \exists g \in L^2(0,T;H) : f(t) = -\int_0^t S(t-\vartheta)Bg(\vartheta)\,\mathrm{d}\vartheta \right\}.$$

For the proof of this result, based on the fact that $W_S(\cdot)$ is a centered Gaussian variable in $L^2(0,T;H)$, see the authors [2, Theorem 3.4].

Under Hypothesis 2.6, we denote as before μ_ε, for any $\varepsilon > 0$, the laws of the processes $\sqrt{\varepsilon}W_S(\cdot)$ on the space $C([0,T];X)$.

Theorem 5.2. *Assume that Hypotheses 2.1, 2.2, 2.5 and 2.6 hold. Let μ be the law of the stochastic convolution process $W_S(\cdot)$ on the space $C_0([0,T];X)$; then the family μ_ε of laws of $\sqrt{\varepsilon}W_S(\cdot)$ satisfies a large deviation principle with respect to the rate functional I given by*

$$I(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|B^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}[f(\vartheta) + (a * Af)(\vartheta)]\right|^2 \mathrm{d}\vartheta \\ \qquad\qquad\qquad \textit{for } f \in R \\ +\infty \qquad\qquad \textit{otherwise,} \end{cases} \tag{5.1}$$

where R is the subspace of $C_0([0,T];X)$ defined as

$$R = \left\{ f \in C([0,T];X) \middle| \exists g \in L^2(0,T;H) : f(t) = -\int_0^t S(t-\vartheta)Bg(\vartheta)\,\mathrm{d}\vartheta \right\}.$$

Proof. Since X is dense and continuously embedded in H, the same holds for $C([0,T];X)$ in $L^2(0,T;H)$.

We know that the Gaussian process $W_S(\cdot)$ has a Gaussian law on the space $L^2(0,T;H)$ but, since from hypotheses it has support on the space $C_0([0,T];X)$, we have that μ is a Gaussian variable also on $C_0([0,T];X)$. So a large deviation principle holds for the family μ_ε on the space $C_0([0,T];X)$; by uniqueness of the reproducing kernel, see Da Prato & Zabczyk [10, Proposition 2.8], the rate functional is the same as in Theorem 5.1. □

Theorem 5.3. *Under the assumptions of Theorem 3.4, the family of laws ν_ε satisfies the large deviation principle with respect to the following explicit functional $J : C([0,T];X) \to [0;+\infty]$*

$$J(f) = \begin{cases} \frac{1}{2}\int_0^T \left| B^{-1}\frac{\mathrm{d}}{\mathrm{d}t}[f(\vartheta)+(a*Af)(\vartheta) - (a*F(f))(\vartheta)]\right|^2 \mathrm{d}s & \text{for } f \in \widetilde{R} \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

where $\widetilde{R}$ is the subset of $C([0,T];X)$ defined as

$$\widetilde{R} = \Big\{ f \in C([0,T];X) \Big| \exists g \in L^2(0,T;H) : f(t) = S(t)x + \tfrac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t S(t-\vartheta)\,(a*F(f))(\vartheta)\,\mathrm{d}\vartheta\right] + \int_0^t S(t-\vartheta)Bg(\vartheta)\,\mathrm{d}\vartheta \Big\}. \tag{5.3}$$

Proof. We have that $\nu_\varepsilon = \Psi \circ \mu_\varepsilon$, where from Theorem 4.1 the functional Ψ is continuous. Thus, from Theorem 5.2 and [10, Proposition 12.3], the family of laws ν_ε has the large deviation property with respect to the functional $J = I \circ \Psi^{-1}$. Eventually the result follows since the definition of Ψ implies that J has the explicit formulation (5.2). □

Remark 5.4. The rate functional J is related to the control system

$$h^g(t) = x - \int_0^t a(t-\vartheta)[Ah^g(\vartheta) - F(h^g(\vartheta))]\,\mathrm{d}\vartheta + \int_0^t Bg(\vartheta)\,\mathrm{d}\vartheta, \quad t \in [0,T].$$

This equation has a unique solution, so it is possible to give the following definition for J in terms of g:

$$J(h^g) = \frac{1}{2}\int_0^T |g(\vartheta)|^2\,\mathrm{d}\vartheta.$$

this formula expresses the minimal *energy* given by the forcing term to stay out of the path of the deterministic system. To be more precise, it is possible to say that the probability for the system to remain in a given subset of trajectories, in the limit for $\varepsilon \to 0$, depends only on the smooth trajectory with minimal L^2-norm.

This is the reason why the rate functional (5.2) resembles the one in the authors [2]. $\square$

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