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A GENERIC TURNPIKE RESULT FOR A CLASS OF ONE-DIMENSIONAL VARIATIONAL PROBLEMS ARISING IN CONTINUUM MECHANICS

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ABSTRACT. In this paper we study the structure of optimal solutions of one-dimensional second order variational problems arising in continuum mechanics. We are interested in a turnpike property of the optimal solutions which is independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions. We establish that a generic integrand possess the turnpike property.

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1. INTRODUCTION

In this paper we study the structure of optimal solutions of variational problems

$$\text{(P)} \qquad \int_0^T f(w(t), w'(t), w''(t))dt \to \min,$$

$$w \in W^{2,1}([0,T]),\ (w(0), w'(0)) = x,\ (w(T), w'(T)) = y,$$

where $T > 0$, $x, y \in R^2$, $W^{2,1}([0,T]) \subset C^1$ is the Sobolev space of functions possessing an integrable second derivative and f belongs to a space of functions to be described below.

The interest in variational problems of the form (P) stems from the theory of thermodynamical equilibrium for second-order materials developed in (Coleman et al., 1992; Leizarowitz & Mizel, 1989; Marcus, 1993; Marcus, 1998; Marcus & Zaslavski, 1999a; Marcus & Zaslavski, 1999b; Marcus & Zaslavski, 2002; Zaslavski, 1995a; Zaslavski, 1995b; Zaslavski, 1996).

We are interested in a turnpike property of the optimal solutions which is independent of the length of the interval, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of

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the choice of interval and endpoint conditions. We establish that a generic integrand possess the turnpike property.

Denote by $\mathfrak{A}$ the set of all continuous functions $f : R^3 \to R$ such that for each $N > 0$ the function $|f(x, y, z)| \to \infty$ as $|z| \to \infty$ uniformly on the set $\{(x, y) \in R^2 : |x|, |y| \le N\}$. For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base:

$$E(N, \epsilon, \Gamma) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A} : \tag{1.1}$$

$$|f(x_1, x_2, x_3) - g(x_1, x_2, x_3)| \le \epsilon$$

$$\text{for each } (x_1, x_2, x_3) \in R^3 \text{ such that } |x_i| \le N,\ i = 1, 2, 3$$

$$\text{and } (|f(x_1, x_2, x_3)| + 1)(|g(x_1, x_2, x_3)| + 1)^{-1} \in [\Gamma^{-1}, \Gamma]$$

$$\text{for each } (x_1, x_2, x_3) \in R^3 \text{ such that } |x_1|, |x_2| \le N\},$$

where $N > 0$, $\epsilon > 0$ and $\Gamma > 1$ (Zaslavski, 1996). Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable (by a metric ρ). It is easy to verify that the uniform space $\mathfrak{A}$ is complete.

Let $a = (a_1, a_2, a_3, a_4) \in R^4$, $a_i > 0$, $i = 1, 2, 3, 4$ and let α, β, γ be positive numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha, \beta, \gamma, a)$ the set of all functions $f \in \mathfrak{A}$ such that:

$$f(w, p, r) \ge a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w, p, r) \in R^3; \tag{1.2}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w, p, r) > 0 \text{ for all } (w, p, r) \in R^3; \tag{1.3}$$

there is a monotone increasing function $M_f : [0, \infty) \to [0, \infty)$ such that for every $(w, p, r) \in R^3$

$$\max\{f(w, p, r),\ |\partial f/\partial w(w, p, r)|,\ |\partial f/\partial p(w, p, r)|,\ |\partial f/\partial r(w, p, r)|\} \le$$

$$M_f(|w| + |p|)(1 + |r|^\gamma). \tag{1.4}$$

Denote by $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ the closure of $\mathfrak{M}(\alpha, \beta, \gamma, a)$ in $\mathfrak{A}$. Note that the full description of integrands belonging to the space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ was obtained in (Zaslavski, 2004). We consider the topological subspace $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a) \subset \mathfrak{A}$ with the relative topology. Leizarowitz and Mizel (Leizarowitz & Mizel, 1989) and Coleman, Marcus and Mizel (Coleman et al., 1992) considered problems of type (P) with integrands $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ in order to study certain models in in the theory of thermodynamical equilibrium for materials. A typical example is an integrand

$$f(w, p, r) = \psi(w) - bp^2 + cr^2,\ (w, p, r) \in R^3,$$

where b, c are positive constants and $\psi(\cdot)$ is a smooth function satisfying

$$\psi(w) \ge a|w|^\alpha - d,\ w \in R$$

for some $\alpha > 2$, $a, d > 0$ (Leizarowitz & Mizel, 1989; Marcus, 1998). In (Marcus & Zaslavski, 1999b; Zaslavski, 1995b; Zaslavski, 1996) we considered problems of type (P) with integrands $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

Consider any $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{\liminf_{T \to +\infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t))dt :$$

$$w \in W^{2,1}_{loc}([0, \infty)) \text{ and } (w(0), w'(0)) = x\}, \tag{1.5}$$

where $x \in R^2$. Here $W^{2,1}_{loc}([0, \infty)) \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative. It was shown in (Leizarowitz & Mizel, 1989) that $\mu(f)$ is well defined and is independent of the initial vector x.

A function $w \in W^{2,1}_{loc}([0, \infty))$ is called an (f)-good function if the function

$$\phi^f_w \colon T \to \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt, \; T \in (0, \infty)$$

is bounded. For every $w \in W^{2,1}_{loc}([0, \infty))$ the function ϕ^f_w is either bounded or diverges to $+\infty$ as $T \to +\infty$ and moreover, if ϕ^f_w is a bounded function, then

$$\sup\{|(w(t), w'(t))| \colon t \in [0, \infty)\} < \infty$$

(see Proposition 3.5 of (Zaslavski, 1995b)). This fact is a continuous version of a result of (Leizarowitz, 1985) established for discrete time control systems. Its proof is based on the result of (Leizarowitz, 1985) applied to a function U^f_T which is defined below.

Leizarowitz and Mizel (Leizarowitz & Mizel, 1989) established that for every function $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfying $\mu(f) < \inf\{f(w, 0, s) \colon (w, s) \in R^2\}$ there exists a periodic (f)-good function. In (Zaslavski, 1995a) it was shown that this resut is valid for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. Recently the existence of a periodic (f)-good function was established for every $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ (Zaslavski, 2005). Namely, it was established the following result (see Theorem 1.1 of (Zaslavski, 2005)).

Theorem 1.1. *Let $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Then there exist an (f)-good function $v_f \in W^{2,\gamma}_{loc}([0, \infty))$ and $T_f > 0$ such that $v_f(t + T_f) = v_f(t)$ for each $t \geq 0$. Moreover, if*

$$\mu(f) < \inf\{f(t, 0, 0) : \; t \in R\},$$

then there is $T_{f,0} \in (0, T_f)$ such that v_f is strictly increasing in $[0, T_{f,0}]$ and strictly decreasing in $[T_{f,0}, T_f]$.

This existence result also describes the structure of a periodic (f)-good function. It was shown in (Marcus & Zaslavski, 1999b) that if $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, then all periodic (f)-good functions which are not constant have this structure.

For each $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ denote by $\mathcal{P}(f)$ the set of all periodic (f)-good functions. By Theorem 1.1 $\mathcal{P}(f) \neq \emptyset$ for all $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

The following result was established in (Marcus & Zaslavski, 1999b).

Proposition 1.1. *Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ and let $w \in \mathcal{P}(f)$. Assume that*

$$w(0) = \inf\{w(t) : \ t \in R\} \text{ and } w'(t) \neq 0 \text{ for some } t \in R.$$

Then there exist $\tau_1 > 0$, $\tau_2 > \tau_1$ such that the function w is strictly increasing in $[0, \tau_1]$, w is strictly decreasing in $[\tau_1, \tau_2]$, and

$$w(\tau_1) = \sup\{w(t) : \ t \in [0, \infty)\}, \ w(t + \tau_2) = w(t), \ t \in R.$$

The description of the set $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ was given in (Zaslavski, 2004) where we establish the following result.

Theorem 1.2. *The space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ is the set of all continuous functions $f : R^3 \to R$ which satisfy the following assumptions:*

$$f(w, p, r) \geq a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4, \ (w, p, r) \in R^3;$$

there is a monotone increasing function $M_f : [0, \infty) \to [0, \infty)$ such that for every $(w, p, r) \in R^3$

$$f(w, p, r) \leq M_f(|w| + |p|)(1 + |r|^\gamma);$$

for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1, x_2, x_3) - f(y_1, y_2, y_3)| \leq \epsilon \max\{f(x_1, x_2, x_3), f(y_1, y_2, y_3)\}$$

for each $x_i, y_i \in R$, $i = 1, 2, 3$ which satisfy

$$|x_i|, |y_i| \leq M, \ i = 1, 2, \ |y_3|, |x_3| \geq \Gamma, \ |x_i - y_i| \leq \delta, \ i = 1, 2, 3;$$

for any $(w, p) \in R^2$, the function $f(w, p, \cdot) : R \to R$ is convex.

In this paper we study the structure of (f)-good functions and the structure of approximate solutions of the problem (P) with $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. The paper is organized as follows. In Section 2 we discuss the results of (Zaslavski, 1996) and state the main results of the paper. Some useful properties of periodic good functions are considered in Section 3. In Section 4 we discuss turnpike results obtained in (Marcus & Zaslavski, 1999b). The main results of the paper are proved in Section 5.

2. MAIN RESULTS

In the sequel we use the following notation and definitions. We denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau > 0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v : [0,\tau] \to R^2$ as follows:

$$X_v(t) = (v(t), v'(t)),\ t \in [0,\tau]. \tag{2.1}$$

We also use this definition for $v \in W^{2,1}_{loc}([0,\infty))$. Sometimes $(v(t), v'(t))$ is also denoted as $(v, v')(t)$.

We consider the functionals of the form

$$I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt \tag{2.2}$$

where $-\infty < T_1 < T_2 < \infty$, $w \in W^{2,1}([T_1, T_2])$ and $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

For $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and $T > 0$ we consider the function $U^f_T : R^2 \times R^2 \to R$ which is defined by

$$U^f_T(x, y) = \inf\{I^f(0, T, w) :\ w \in W^{2,1}([0,T]) :$$

$$(w(0), w'(0)) = x \text{ and } (w(T), w'(T)) = y\}. \tag{2.3}$$

Let $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. In (Leizarowitz & Mizel, 1989) Leizarowitz and Mizel studied the function U^f_T: $R^2 \times R^2 \to R$, $T > 0$ and established the following representation formula

$$U^f_T(x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x, y),\ x, y \in R^2,\ T > 0, \tag{2.4}$$

where π^f: $R^2 \to R$ and $(T, x, y) \to \theta^f_T(x, y)$, $x, y \in R^2$, $T > 0$ are continuous functions,

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0, T, w) - T\mu(f)] :$$

$$w \in W^{2,1}_{loc}([0,\infty)) \text{ and } (w(0), w'(0)) = x\},\ x \in R^2, \tag{2.5}$$

$\theta^f_T(x, y) \geq 0$ for each $T > 0$, and each $x, y \in R^2$, and for every $T > 0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta^f_T(x, y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$, but their result also holds for every $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ without change in the proofs.

For a function $w \in W^{2,1}_{loc}([0,\infty))$ we denote by $\Omega(w)$ the set of all points $z \in R^2$ such that $X_w(t_j) \to z$ as $j \to \infty$ for some sequence of numbers $t_j \to \infty$.

For each $x \in R^n$ and each $A \subset R^n$ set

$$d(x, A) = \inf\{|x - y| :\ y \in A\}$$

and denote by $\text{dist}(A, B)$ the Hausdorff metric for two sets $A \subset R^n$ and $B \subset R^n$.

The main results in this paper deal with the so-called turnpike properties of the variational problems (P). To have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see Samuelson, 1965) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, (Makarov & Rubinov, 1977; McKenzie, 2002) and the references mentioned there). The turnpike properties of problem (P) were studied in (Marcus & Zaslavski, 1999b; Zaslavski, 1995b; Zaslavski, 1996).

In (Zaslavski, 1995b; Zaslavski, 1996) we studied the structure of (f)-good functions and the structure of approximate solutions of the problem (P) with $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. In these papers we established the existence of a set $\mathcal{F}_0 \subset \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and such that for every $f \in \mathcal{F}_0$ the following properties hold:

for every (f)-good function w the equality $\Omega(w) = H(f)$ holds, where $H(f) \subset R^2$ is a compact set depending only on the function f;

for any $\epsilon > 0$ there exist constants $L_1, L_2 > 0$ which depend only on $|x|, |y|$ and ϵ such that for each optimal solution v of problem (P) and each $\tau \in [L_1, T - L_1]$ the set $\{(v(t), v'(t)) : \ t \in [\tau, \tau + L_2]\}$ is equal to the set $H(f)$ up to ϵ in the Hausdorff metric.

Namely, in (Zaslavski, 1996) we established the existence of a set $\mathcal{F}_0 \subset \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which is a countable intersection of open everywhere dense subsets of the space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and for which the following theorems are valid.

Theorem 2.1. *Let $f \in \mathcal{F}_0$. Then there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for any (f)-good function w.*

Theorem 2.2. *Let $f \in \mathcal{F}_0$ and let $\epsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and numbers $l_0 > l > 0$, $K_* > K$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $\tau \geq 2l_0$ and each $v \in W^{2,1}([0, \tau])$ which satisfies*

$$|(v(0), v'(0))|, \ |(v(\tau), v'(\tau))| \leq K,$$

$$I^g(0, \tau, v) \leq U^g_\tau((v(0), v'(0)), (v(\tau), v'(\tau))) + \delta$$

the relation $|(v(t), v'(t))| \leq K_*$ holds for all $t \in [0, \tau]$ and

$$dist(H(f), \{(v(t), v'(t)) : \ t \in [T, T + l]\}) \leq \epsilon$$

for each $T \in [l_0, \tau - l_0]$.

In (Marcus & Zaslavski, 1999b; Zaslavski, 1996) we considered certain important subspaces of the space $\mathfrak{M}(\alpha, \beta, \gamma, a)$ equipped with natural uniformities and showed that each of them contains an everywhere dense G_δ subset such that each its element f has the following two properties:

there exists a unique up to translation periodic (f)-good function w;

let $T_w > 0$ be a period of w. For any $\epsilon > 0$ there exists a constant $L > 0$ which depends only on $|x|, |y|$ and ϵ such that for each optimal solution v of problem (P) and each $\tau \in [L, T - L - T_w]$ there exists $s \in [0, T_w)$ such that

$$|(v(\tau + t), v'(\tau + t)) - (w(s + t), w'(s + t))| \leq \epsilon \text{ for each } t \in [0, T_w].$$

Clearly, this turnpike property established for the subspaces of $\mathfrak{M}(\alpha, \beta, \gamma, a)$ is essentially stronger than the turnpike property established in (Zaslavski, 1996) for the space $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. In this paper we strengthen the main result of (Zaslavski, 1996) by showing that the turnpike property established in (Marcus & Zaslavski, 1999b; Zaslavski, 1996) for most integrands of the subspaces of $\mathfrak{M}(\alpha, \beta, \gamma, a)$ also holds for a generic integrand $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$.

First of all note that by Theorems 1.1 and Theorem 2.1 for each $f \in \mathcal{F}_0$

$$H(f) = \{(w(t), w'(t)) :\ t \in [0, \infty)\}$$

where w is a periodic (f)-good function.

In this paper we establish the existence of a set $\mathcal{F} \subset \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and for which the following theorems are valid.

Theorem 2.3. *Let $f \in \mathcal{F}$. Then there exist a function $w_f \in W^{2,1}_{loc}([0, \infty))$ and a number $T_f > 0$ such that the following assertions hold:*

1. w_f is a periodic (f)-good function and $w_f(t + T_f) = w_f(t)$ for all $t \in [0, \infty)$.

2. If $\mu(f) < \inf\{f(z, 0, 0) :\ z \in R^1\}$, then

$$(w_f(t_1), w_f'(t_1)) \neq (w_f(t_2), w_f'(t_2))$$

for each t_1, t_2 satisfying $0 \leq t_1 < t_2 < T_f$ and there is $T_{f,0} \in (0, T_f)$ such that w_f is strictly increasing in $[0, T_{f,0}]$ and strictly decreasing in $[T_{f,0}, T_f]$. Otherwise $w_f(t) = w_f(0)$ for all $t \in [0, \infty)$.

3. For any $w \in \mathcal{P}(f)$ there exists a number τ such that $w(t) = w_f(t + \tau)$ for all $t \geq 0$.

4. For every $\epsilon > 0$ and every natural number n there exists a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ such that for every $g \in \mathcal{U}$, every (g)-good function v and every large enough τ there is $h \geq 0$ for which

$$\sup\{|(v(t), v'(t)) - (w_f(t + h), w_f'(t + h))| :\ t \in [\tau, \tau + T_f n]\} \leq \epsilon. \tag{2.6}$$

Theorem 2.4. *Let $f \in \mathcal{F}$, ϵ, K be positive numbers and let n be a natural number. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ and numbers $l > T_f$, $K_* > K$, $\delta > 0$ such that the following assertion holds:*

For each $g \in \mathcal{U}$, each $T \geq 2l$ and each $w \in W^{2,1}([0,T])$ which satisfies

$$|(w(0), w'(0))|, \ |(w(T), w'(T))| \leq K,$$

$$I^g(0, T, w) \leq U_T^g((w(0), w'(0)), (w(T), w'(T))) + \delta$$

the inequialty $|(w(t), w'(t))| \leq K_*$ holds for all $t \in [0,T]$ and there exist $\tau_1 \in [0, l]$, $\tau_2 \in [T-l, T]$ such that $\tau_2 - \tau_1 \geq nT_f$ and that for each $\tau \in [\tau_1, \tau_2 - nT_f]$ the inequality (2.6) holds with some $h \geq 0$. Moreover, if $d((w(0), w'(0)), \Omega(w_f)) \leq \delta$, then $\tau_1 = 0$ and if $d((w(T), w'(T)), \Omega(w_f)) \leq \delta$, then $\tau_2 = T$.

We set for simplicity

$$\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, a), \ \bar{\mathfrak{M}} = \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a).$$

3. PERIODIC GOOD FUNCTIONS

In the sequel we use the following result established in (Zaslavski, 1996).

Proposition 3.1. *Assume that $f \in \bar{\mathfrak{M}}$, $\{f_k\}_{k=1}^\infty \subset \bar{\mathfrak{M}}$, $\{T_k\}_{k=1}^\infty \subset (0, \infty)$, $T \geq 0$, $\{w_k\}_{k=1}^\infty \subset W_{loc}^{2,1}([0,\infty))$, $f_k \to f$ in $\bar{\mathfrak{M}}$, $T_k \to T$ as $k \to \infty$, $w_k(t + T_k) = w_k(t)$ for all $t \in [0, \infty)$, $k = 1, 2, \dots$ and that*

$$I^{f_k}(0, T_k, w_k) = \mu(f_k) T_k, \ k = 1, 2, \dots$$

Then there exist $\tau > 0$ and $w \in W_{loc}^{2,1}([0,\infty))$ such that:

$$w(t + \tau) = w(t) \text{ for all } t \in [0, \infty), \ I^f(0, \tau, w) = \tau \mu(f);$$

if $T > 0$, then $\tau = T$ and if $T = 0$, then $w(t) = w(0)$ for all $t \in [0, \infty)$.

Proposition 3.2. *Assume that $f \in \bar{\mathfrak{M}}$ satisfies*

$$\mu(f) < \inf\{f(z, 0, 0) : \ z \in R\}. \tag{3.1}$$

Then there exist $T_f > 0$ and $v \in \mathcal{P}(f)$ such that T_f is a period of v and if $w \in \mathcal{P}(f)$ and $T > 0$ is a period of w, then $T \geq T_f$.

Proof. By Theorem 1.1, $\mathcal{P}(f) \neq \emptyset$. Set

$$T_f = \inf\{T \in (0, \infty) : \ \text{there is } w \in \mathcal{P}(f) \text{ such that } T \text{ is a period of } w\}. \tag{3.2}$$

Clearly T_f is well defined. By the definition of T_f there exist sequences

$$\{T_k\}_{k=1}^\infty \subset (0, \infty) \text{ and } \{w_k\}_{k=1}^\infty \subset \mathcal{P}(f)$$

such that $\lim_{k\to\infty} T_k = T$ and the number T_k is a period of w_k for each integer $k \ge 1$. In view of Proposition 3.1 there exist $v \in \mathcal{P}(f)$ and $\tau > 0$ such that τ is a period of v and

$$\text{if } T_f > 0, \quad \text{then } \tau = T_f \text{ and if } T_f = 0, \quad \text{then } v(t) = v(0) \text{ for all } t \ge 0.$$

The inequality (3.1) implies that $T_f > 0$ and $\tau = T_f$. This completes the proof of Proposition 3.2. □

Corollary 3.1. *Assume that $f \in \bar{\mathfrak{M}}$ satisfies (3.1) and let T_f and v be as guaranteed by Proposition 3.2. Then for each t_1, t_2 satisfying $0 \le t_1 < t_2 < T_f$ the relation $X_v(t_1) \ne X_v(t_2)$ holds.*

Proof. Let us assume the converse. Then there exist t_1, t_2 such that

$$0 \le t_1 < t_2 < T_f, \; X_v(t_1) = X_v(t_2). \tag{3.3}$$

Since $v \in \mathcal{P}(f)$ it follows from the representation formula (2.4) and (2.5) that

$$I^f(s_1, s_2, v) - (s_2 - s_1)\mu(f) - \pi^f(X_v(s_1)) + \pi^f(X_v(s_2)) = 0$$

for each $s_1 \ge 0$, $s_2 > s_1$. In particular

$$I^f(t_1, t_2, v) = (t_2 - t_1)\mu(f) + \pi^f(X_v(t_1)) - \pi^f(X_v(t_2)). \tag{3.4}$$

In view of (3.3) there exists $w \in W^{2,1}_{loc}([0,\infty))$ such that

$$w(t + (t_2 - t_1)) = w(t) \text{ for all } t \ge 0 \text{ and } w(t) = v(t), \; t \in [t_1, t_2]. \tag{3.5}$$

By (3.4) and (3.5), $w \in \mathcal{P}(f)$ and $t_2 - t_2$ is a period of w. The inequality $0 < t_2 - t_1 < T_f$ contradicts the defnition of T_f. The contradiction we have reached proves Corollary 3.1. □

Proposition 3.3. *Assume that $f \in \bar{\mathfrak{M}}$ satisfies (3.1), let T_f and v be as guaranteed by Proposition 3.2 and let $\epsilon \in (0, T_f)$. Then there exists a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that if $g \in \mathcal{U}$ and $w \in \mathcal{P}(g)$ with a period $T > 0$, then $T \ge T_f - \epsilon$.*

Proof. Let us assume the converse. Then there exist a sequence $\{f_k\}_{k=1}^{\infty} \subset \bar{\mathfrak{M}}$ satisfying $f_k \to f$ as $k \to \infty$ in $\bar{\mathfrak{M}}$, a sequence $\{w_k\}_{k=1}^{\infty} \subset W^{2,1}_{loc}([0,\infty))$ such that $w_k \in \mathcal{P}(f_k)$ for all integers $k \ge 1$ and a sequence $\{T_k\}_{k=1}^{\infty} \subset (0, T_f - \epsilon)$ such that T_k is a period of w_k for all integers k. Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exists $T = \lim_{k\to\infty} T_k$. Clearly, $0 \le T \le T_f - \epsilon$. Applying Proposition 3.1 we obtain that there exists $w \in \mathcal{P}(f)$ with a period $T \le T_f - \epsilon$. This contradicts the definition of T_f. The contradiction we have reached proves Proposition 3.3. □

In the sequel we will also use the following result established in (Zaslavski, 1996).

Proposition 3.4. *The function $f \to \mu(f)$, $f \in \bar{\mathfrak{M}}$ is continuous.*

4. TURNPIKE PROPERTIES

Let $f \in \bar{\mathfrak{M}}$. We say that f has the asymptotic turnpike property if there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for every (f)-good function w.

The following result was established in (Marcus & Zaslavski, 1999b). It shows that if an integrand $g \in \mathfrak{M}$ has the asymptotic turnpike property, then it also possess the turnpike property.

Theorem 4.1. *Assume that $g \in \mathfrak{M}$ has the asymptotic turnpike property. Let w be a periodic (g)-good function and let $T_w > 0$ be a period of w. The for each $\epsilon, M > 0$ there exist a neighborhood $\mathcal{U}$ of g in $\bar{\mathfrak{M}}$ and positive numbers δ, l such that the following assertion holds:*

If $f \in \mathcal{U}$, $T \geq T_w + 2l$ and if $v \in W^{2,1}([0,T])$ satisfies

$$|X_v(0)| \leq M, \ |X_v(T)| \leq M, \ I^f(0,T,v) \leq U_T^f(X_v(0), X_v(T)) + \delta,$$

then there exist $\tau_1 \in [0,l]$ and $\tau_2 \in [T-l,T]$ such that for every $s \in [\tau_1, \tau_2 - T_w]$ there exists $\xi \in [0,T_w]$ such that

$$|X_v(s+t) - X_w(\xi+t)| \leq \epsilon \text{ for all } t \in [0,T_w].$$

Furthermore, if $d(X_v(0), \Omega(w)) \leq \delta$, (respectively, $d(X_v(T), \Omega(w)) \leq \delta$), then $\tau_1 = 0$ (respectively, $\tau_2 = T$).

The next result was also established in (Marcus & Zaslavski, 1999b).

Theorem 4.2. *Let $f \in \mathfrak{M}$. Then there exists a nonnegative function $\phi \in C^\infty(R^1)$ such that $\phi(t) > 0$ for all large $|t|$, $\phi^{(m)}$ is bounded for any $m > 0$ and for each $r \in (0,1)$ the function*

$$f_r(x_1,x_2,x_3) = f(x_1,x_2,x_3) + r\phi(x_1), \ (x_1,x_2,x_3) \in R^3$$

belongs to $\mathfrak{M}$ and posesses the asymptotic turnpike property.

5. PROOFS OF THE MAIN RESULTS

Denote by E the set of all $f \in \mathfrak{M}$ which have the asymptotic turnpike property. By Theorem 4.2 E is an everywhere dense subset of $\bar{\mathfrak{M}}$.

The next result was established in (Marcus & Zaslavski, 1999b).

Proposition 5.1. *Assume that $g \in E$ and w is a periodic (g)-good function with a period $T_w > 0$. Then for every $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of g in $\bar{\mathfrak{M}}$ such that for each $f \in \mathcal{U}$, each (f)-good function v and each large enough positive number s there is $\xi \in [0,T_w)$ such that*

$$|X_v(s+t) - X_w(\xi+t)| \leq \epsilon, \ t \in [0,T_w].$$

Another useful ingredient in our proof is the following result for which we refer the reader to the proof of Proposition 4.4 of (Leizarowitz & Mizel, 1989) and (Zaslavski, 1995b).

Proposition 5.2. *Let $g \in \bar{\mathfrak{M}}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of g in $\bar{\mathfrak{M}}$ and $M_3 > 0$ such that for each $T \geq 0$, each $f \in \mathcal{U}$ and each $v \in W^{2,1}([0,T])$ which satisfies*

$$|X_v(0)| \leq M_1,\ |X_v(T)| \leq M_1,$$

$$I^f(0, T_f, v) \leq U_T^f(X_v(0), X_v(T_2)) + M_2$$

the following inequality holds:

$$|X_v(t)| \leq M_3,\ t \in [0,T].$$

Let $f \in E$. By Proposition 5.1 there exists a unique (up to translations) periodic (f)-good function which will be denoted by w_f.

If

$$\mu(f) = \inf\{f(t,0,0) :\ t \in R\},$$

then set $T_f = 1$. If

$$\mu(f) < \inf\{f(t,0,0) :\ t \in R\},$$

then we may assume without loss of generality that

$$w_f(0) = \inf\{w(t) :\ t \in R^1\}.$$

Then by Proposition 1.1 there exist $T_f > T_{f,0} > 0$ such that T_f is a period of w_f and w_f is strictly increasing in $[0, T_{f,0}]$ and strictly decreasing in $[T_{f,0}, T_f]$.

By Proposition 5.1, Proposition 5.2, Theorem 4.1 and Proposition 3.3 for each $f \in E$ and each integer $p \geq 1$ there exist an open neighborhood $\mathcal{U}(f,p)$ of f in $\bar{\mathfrak{M}}$ and numbers $M(f,p) > p$, $l(f,p) > 0$, $\delta(f,p) \in (0, p^{-1})$ such that the following properties hold:

(Pi) For each $g \in \mathcal{U}(f,p)$ and each (g)-good function v if a number s is large enough, then there is $\xi \in [0, T_f)$ such that

$$|X_v(s+t) - X_{w_f}(\xi + t)| \leq 4^{-1}\delta(f,p),\ t \in [0, 2pT_f].$$

(Pii) For each $g \in \mathcal{U}(f,p)$, each $T \geq pT_f + 2l(f,p)$ and each $v \in W^{2,1}[0,T])$ which satisfies

$$|X_v(0)| \leq p,\ |X_v(T)| \leq p,\ I^g(0,T,v) \leq U_T^g(X_v(0), X_v(T)) + \delta(f,p)$$

the inequality $|X_v(t)| \leq M(f,p)$ holds for all $t \in [0,T]$ and there exist $\tau_1 \in [0, l(f,p)]$, $\tau_2 \in [T - l(f,p), T]$ such that:

if $s \in [\tau_1, \tau_2 - pT_f]$, then there is $\xi \in [0, T_f]$ such that

$$|X_v(s+t) - X_{w_f}(\xi + t)| \le p^{-1} \text{ for all } t \in [0, pT_f];$$

if $d(X_v(0), \Omega(w_f)) \le \delta(f,p)$, then $\tau_1 = 0$ and if $d(X_v(T), \Omega(w_f)) \le \delta(f,p)$, then $\tau_2 = T$.

(Piii) If $\mu(f) < \inf\{f(t,0,0): \ t \in R\}$, then for each $g \in \mathcal{U}(f,p)$ and each $v \in \mathcal{P}(g)$ with a period $T > 0$ the inequality $T \ge T_f - p^{-1}\min\{1, T_f\}$ holds.

(Piv)

$$\mathcal{U}(f,p) \subset \{g \in \bar{\mathfrak{M}}: \ \rho(f,g) < p^{-1}\}.$$

Define

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(f,p): \ f \in E\}. \tag{5.1}$$

It is clear that $\mathcal{F}$ is a countable intersection of open everywhere dense subests of $\bar{\mathfrak{M}}$.

Let $f \in \mathcal{F}$. First we show that f has the asymptotic turnpike property. Let v_1, v_2 be (f)-good functions and let $p \ge 1$ be an integer. By (5.1) there exists $h \in E$ such that $f \in \mathcal{U}(h,p)$. By this inclusion and property (Pi) for any large enough $s > 0$ there are $\xi_1, \xi_2 \in [0, T_h)$ such that

$$|X_{v_i}(s+t) - X_{w_h}(\xi_i + t)| \le 4^{-1}\delta(h,p) \le 4^{-1}p^{-1}, \ t \in [0, T_h].$$

This inequality implies that

$$\text{dist}(\Omega(v_i), \Omega(w_h)) \le (4p)^{-1}, \ i = 1,2$$

and

$$\text{dist}(\Omega(v_1), \Omega(v_2)) \le (2p)^{-1}.$$

Since p is any natural number we conclulde that $\Omega(v_1) = \Omega(v_2)$ and f has the asymptotic turnpike property. By Theorem 1.1 f has a periodic (f)-good function. We show that this function is unique up to translations.

If $\mu(f) = \inf\{f(t,0,0): \ t \in R\}$, then the uniqueness of a periodic (f)-good function follows from the asymptotic turnpike property.

Assume now that

$$\mu(f) < \inf\{f(t,0,0): \ t \in R\}. \tag{5.2}$$

In view of (5.2) and Propositions 3.3 and 3.4 there exist $r_0 \in (0,1)$, $\Delta > 0$ such that the following property holds:

(Pv) if $g \in \bar{\mathfrak{M}}$, $\rho(f,g) \le r_0$ and $v \in \mathcal{P}(g)$ with a period $T > 0$, then $T \ge \Delta$ and $\mu(g) < \inf\{g(t,0,0): \ t \in R\}$.

Assume that

$$v_1, v_2 \in \mathcal{P}(f), \ T_i \text{ is a period of } v_i, \ i = 1,2. \tag{5.3}$$

The property (Pv) and (5.3) imply that

$$T_1, T_2 \geq \Delta. \tag{5.4}$$

Assume that an integer

$$p > 4r_0^{-1}. \tag{5.5}$$

The relation (5.1) implies that there exists $h \in E$ such that

$$f \in \mathcal{U}(h, p). \tag{5.6}$$

It follows from the property (Piv), (5.6) and (5.5) that

$$\rho(f, h) < p^{-1} < r_0/4. \tag{5.7}$$

Combined with the property (Pv) this inequality implies that

$$\mu(h) < \inf\{h(t, 0, 0) : t \in R\}, \tag{5.8}$$

$$T_h \geq \Delta. \tag{5.9}$$

By (5.6), (5.8), (5.3), (5.5) and the property (Piii)

$$T_1, T_2 \geq T_h/2. \tag{5.10}$$

In view of (5.6), the property (Pi) and (5.3) the following property holds:

For $i \in \{1, 2\}$ and each $s \geq 0$ there is $\xi \in [0, T_h)$ such that

$$|X_{v_i}(s + t) - X_{w_h}(\xi + t)| \leq 4^{-1}\delta(f, p) < 4^{-1}p^{-1},\ t \in [0, 2pT_h].$$

This property implies that there are $\xi_1, \xi_2 \in [0, T_h)$ such that for i=1,2

$$|X_{v_i}(t) - X_{w_h}(\xi_i + t)| \leq 4^{-1}p^{-1},\ t \in [0, 2pT_h]. \tag{5.11}$$

The inequality (5.11) implies that for each $t \in [0, (p - 1)T_h]$

$$|X_{v_1}(t + T_h - \xi_1) - X_{v_2}(t + T_h - \xi_2)|$$

$$\leq |X_{v_1}(t + T_h - \xi_1) - X_{w_h}(t + T_h)| + |X_{w_h}(t + T_h) - X_{v_2}(t + T_h - \xi_2)|$$

$$\leq (4p)^{-1} + (4p)^{-1} = (2p)^{-1}. \tag{5.12}$$

Set $\eta_1 = T_h - \xi_1,\ \eta_2 = T_h - \xi_2$. Since $\xi_1, \xi_2 \in [0, T_h)$ it follows from (5.10) that

$$0 \leq \eta_1, \eta_2 \leq T_h \leq 2\min\{T_1, T_2\}.$$

By (5.12), the definition of η_1, η_2 and (5.9)

$$|X_{v_1}(t + \eta_1) - X_{v_2}(t + \eta_2)| \leq (2p)^{-1} \text{ for all } t \in [0, (p - 1)\Delta].$$

Thus we have shown that for each integer p satisfying (5.5) there exist

$$\eta_1^{(p)},\ \eta_2^{(p)} \in [0, 2\min\{T_1, T_2\}]$$

such that

$$|X_{v_1}(t+\eta_1^{(p)}) - X_{v_2}(t+\eta_2^{(p)})| \le 2^{-1}p^{-1} \text{ for all } t \in [0,(p-1)\Delta]. \tag{5.13}$$

Extracting a subsequence and re-indexing, if necessary, we may assume that there exist

$$\bar{\eta}_i = \lim_{p\to\infty} \eta_i^{(p)},\ i=1,2. \tag{5.14}$$

Combined with (5.13) this relation implies that for all $t \in [0,\infty)$

$$|X_{v_1}(t+\bar{\eta}_1) - X_{v_2}(t+\bar{\eta}_2)| = \lim_{p\to\infty} |X_{v_1}(t+\eta_1^{(p)}) - X_{v_2}(t+\eta_2^{(p)})|$$

$$\le \lim_{p\to\infty} (2p)^{-1} = 0$$

and $v_1(t+\bar{\eta}_1) = v_2(t+\bar{\eta}_2)$ for all $t \ge 0$. This equality implies that v_2 is a translation of v_1. Thus there exists a unique (up to translations) periodic (f)-good function.

Let w_f be a periodic (f)-good function such that

$$w_f(0) = \inf\{w(t):\ t \in R\}. \tag{5.15}$$

If $\mu(f) = \inf\{f(t,0,0):\ t \in R\}$, then set $T_f = 1$. If $\mu(f) < \inf\{f(t,0,0):\ t \in R\}$, then by Theorem 1.1 there are numbers $T_f > T_{f,0} > 0$ such that

$$w_f \text{ is strictly increasing in } [0,T_{f,0}] \text{ and strictly decreasing in } [T_{f,0},T_f] \tag{5.16}$$

and

$$w_f(t+T_f) = w_f(t) \text{ for all } t \in [0,T_f]. \tag{5.17}$$

Now we can complete the proof of Theorem 2.3. Assertions 1 and 3 have already been proved. Let us prove Assertion 2. Assume that

$$\mu(f) < \inf\{f(t,0,0):\ t \in R\}.$$

In order to prove Assertion 2 we need to show that $X_{w_f}(t_1) \ne X_{w_f}(t_2)$ for each t_1, t_2 satisfying $0 \le t_1 < t_2 < T_f$.

Let us assume the converse. Then there exist numbers t_1, t_2 such that

$$0 \le t_1 < t_2 < T_f,\ X_{w_f}(t_1) = X_{w_f}(t_2).$$

Then it is not difficult to see that there exists $u \in \mathcal{P}(f)$ such that $u(t) = w_f(t)$ for all $t \in [t_1,t_2]$ and $t_2 - t_1$ is a period of u. Clearly u is not a translation of w_f. This contradicts Assertion 3. The contradiction we have reached proves Assertion 2.

Let us prove Assertion 4. There are two cases:

$$\mu(f) = \inf\{f(t,0,0):\ t \in R\} \tag{5.18}$$

and

$$\mu(f) < \inf\{f(t,0,0):\ t \in R\}. \tag{5.19}$$

Assume that (5.18) holds. Then w_f is a constant and $T_f = 1$. Choose a natural number

$$p > 8\epsilon^{-1}. \tag{5.20}$$

Since $f \in \mathcal{F}$ it follows from (5.1) that there exists $h \in E$ such that

$$f \in \mathcal{U}(h,p). \tag{5.21}$$

In view of (5.21) and the property (Pi)

$$|X_{w_f}(0) - X_{w_h}(t)| \le 4^{-1}\delta(h,p) \le (4p)^{-1} \text{ for all } t \in [0,T_h]. \tag{5.22}$$

Assume that

$$g \in \mathcal{U}(h,p), \tag{5.23}$$

v is a (g)-good function and that a positive number s is large enough. By (5.23) and the property (Pi) there is $\xi \in [0,T_h)$ such that

$$|X_v(s) - X_{w_h}(\xi)| \le 4^{-1}\delta(h,p) \le (4p)^{-1}.$$

Together with (5.22) and (5.20) this inequality implies that

$$|X_v(s) - X_{w_f}(0)| \le |X_v(s) - X_{w_h}(\xi)| + |X_{w_h}(\xi) - X_{w_f}(0)| \le (2p)^{-1} < \epsilon.$$

Since this inequality holds for all sufficiently large s Assertion 4 holds when (5.18) is true.

Assume that (5.19) holds. In view of (5.19), Propositions 3.3 and 3.4, the definition of T_f (see (5.16), (5.17)) and Assertion 3 there exists $r_0 \in (0,1)$ such that for each $g \in \bar{\mathfrak{M}}$ satisfying $\rho(f,g) \le r_0$

$$\mu(g) < \inf\{g(t,0,0) : \ t \in R\}, \tag{5.24}$$

$$\text{if } v \in \mathcal{P}(g) \text{ and if } T > 0 \text{ is a period of } v, \text{ then } T \ge T_f/2. \tag{5.25}$$

Choose a natutal number

$$p > 4r_0^{-1} + 4/\epsilon + 4n. \tag{5.26}$$

Since $f \in \mathcal{F}$ it follows from (5.1) that there exists $h \in E$ such that

$$f \in \mathcal{U}(h,p). \tag{5.27}$$

Inclusion (5.27), the property (Piv) and (5.26) imply that

$$\rho(f,h) < 1/p < r_0/4. \tag{5.28}$$

Together with (5.24), (5.25) and the definition of w_h, T_h this inequality implies that

$$\mu(h) < \inf\{h(t,0,0) : \ t \in R\}, \tag{5.29}$$

$$T_h \ge T_f/2. \tag{5.30}$$

By (5.27), (5.26), the property (Piii), (5.29) and the definition of w_f, T_f (see (5.15)-(5.17))

$$T_f \geq 2^{-1} T_h. \tag{5.31}$$

Assume that

$$g \in \mathcal{U}(h, p) \tag{5.32}$$

and v is a (g)-good function. It follows from (Pi) and (5.32) that there is $s_0 > 0$ such that the following property holds:

(Pvi) For each $s \geq s_0$ there is $\xi \in [0, T_h)$ such that

$$|X_v(s+t) - X_{w_h}(\xi + t)| \leq 4^{-1}\delta(h, p), \ t \in [0, 2pT_h]. \tag{5.33}$$

In view of (Pi), (5.27) and the definition of w_f (see (5.15)-(5.17)) there is $\xi_0 \in [0, T_h)$ such that

$$|X_{w_f}(t) - X_{w_h}(\xi_0 + t)| \leq 4^{-1}\delta(h, p), \ t \in [0, 2pT_h]. \tag{5.34}$$

Let $s \geq s_0$. By the property (Pvi) there exists $\xi \in [0, T_h)$ such that (5.33) is true. Since T_h is a period of w_h it follows from (5.26), (5.33) and (5.34) that for each $t \in [0, (p-2)T_h]$

$$|X_v(s+t) - X_{w_f}(t + \xi + T_h - \xi_0)| \leq |X_v(s+t) - X_{w_h}(\xi + T_h + t)|$$

$$+ |X_{w_h}(\xi + T_h + t) - X_{w_f}(\xi + T_h - \xi_0 + t)| \leq 4^{-1}\delta(h, p) + 4^{-1}\delta(h, p) \leq (2p)^{-1}.$$

Together wih (5.26) and (5.30) this inequality implies that

$$|X_v(s+t) - X_{w_f}(t + \xi - \xi_0 + T_h)| \leq (2p)^{-1} \text{ for all } t \in [0, 4^{-1}pT_f].$$

By this inequality and (5.26)

$$|X_v(s+t) - X_{w_f}(t + \xi - \xi_0 + T_h)| \leq \epsilon \text{ for all } t \in [0, nT_f].$$

This completes the proof of Assertion 4 and the proof of Theorem 2.3.

Now we are ready to complete the proof of Theorem 2.4. There are two cases:

$$\mu(f) = \inf\{f(t, 0, 0) : \ t \in R\} \tag{5.35}$$

and

$$\mu(f) < \inf\{f(t, 0, 0) : \ t \in R\}. \tag{5.36}$$

Assume that (5.35) holds. Then w_f is constant and $T_f = 1$. Choose a natural number

$$p > 8\epsilon^{-1} + K. \tag{5.37}$$

Since $f \in \mathcal{F}$ it follows from (5.1) that there exists $h \in E$ such that

$$f \in \mathcal{U}(h, p). \tag{5.38}$$

In view of (5.38) and the property (Pi)

$$|X_{w_f}(0) - X_{w_h}(t)| \leq 4^{-1}\delta(h,p) \text{ for all } t \in [0, T_h]. \tag{5.39}$$

Choose positive numbers

$$\delta < \delta(h,p)/4,\ l > pT_h + 2l(h,p) + 2n. \tag{5.40}$$

Assume that

$$g \in \mathcal{U}(h,p),\ T \geq 2l \tag{5.41}$$

and $w \in W^{2,1}([0,T])$ satisfies

$$|X_w(0)| \leq K,\ |X_w(T)| \leq K,\ I^g(0,T,v) \leq U_T^g(X_w(0), X_w(T)) + \delta. \tag{5.42}$$

It follows from (5.41), the property (Pii), (5.40), (5.37) and (5.42) that

$$|X_w(t)| \leq M(h,p) \text{ for all } t \in [0,T]. \tag{5.43}$$

By the property (Pii), (5.41), (5.40), (5.42) and (5.37) there exist

$$\tau_1 \in [0, l(h,p)] \subset [0,l],\ \tau_2 \in [T - l(h,p), T] \subset [T-l, T] \tag{5.44}$$

such that the following property holds:

(Pvii) For every $s \in [\tau_1, \tau_2 - pT_h]$ there is $\xi \in [0, T_h)$ such that

$$|X_w(s+t) - X_{w_h}(\xi + t)| \leq p^{-1} \text{ for all } t \in [0, pT_h], \tag{5.45}$$

$$\text{if } d(X_w(0), \Omega(w_h)) \leq \delta(h,p), \text{ then } \tau_1 = 0, \tag{5.46}$$

$$\text{if } d(X_w(T), \Omega(w_h)) \leq \delta(h,p), \text{ then } \tau_2 = T. \tag{5.47}$$

Let $s \in [\tau_1, \tau_2 - pT_h]$. By the property (Pvii) there is $\xi \in [0, T_h)$ such that (5.45) holds. It follows from (5.45) and (5.39) that for all $t \in [0, pT_h]$

$$|X_w(s+t) - X_{w_f}(0)| \leq |X_w(s+t) - X_{w_h}(\xi+t)|$$

$$+|X_{w_h}(\xi+t) - X_{w_f}(0)| \leq 1/p + \delta(h,p)/4 \leq 1/p + 1/(4p) \leq (2p)^{-1}.$$

Hence

$$|X_w(t) - X_{w_f}(0)| \leq (2p)^{-1} \text{ for all } t \in [\tau_1, \tau_2].$$

Together with (5.37) this inequality implies that

$$|X_w(t) - X_{w_f}(0)| \leq \epsilon \text{ for all } t \in [\tau_1, \tau_2]. \tag{5.48}$$

Relations (5.44), (5.40) and (5.41) imply that

$$\tau_2 - \tau_1 \geq T - 2l(h,p) > 2l - 2l(h,p) > 2n = 2nT_f. \tag{5.49}$$

Assume that

$$d(X_w(0), \Omega(w_f)) = |X_w(0) - X_{w_f}(0)| \leq \delta. \tag{5.50}$$

Then it follows from (5.50), (5.39) and (5.40) that

$$d(X_w(0),\Omega(w_h)) \le |X_w(0)-X_{w_f}(0)| + d(X_{w_f}(0),\Omega(w_h)) \le \delta + 4^{-1}\delta(h,p) \le \delta(h,p).$$

Together with (5.46) this inequality implies that $\tau_1 = 0$. Thus we have shown that

$$\text{if } |X_w(0)-X_{w_f}(0)| \le \delta, \quad \text{then } \tau_1 = 0. \tag{5.51}$$

Assume that

$$d(X_w(T),\Omega(w_f)) = |X_w(T)-X_{w_f}(0)| \le \delta. \tag{5.52}$$

Then it follows from (5.52), (5.39) and (5.40) that

$$d(X_w(T),\Omega(w_h)) \le |X_w(T)-X_{w_f}(0)| + d(X_{w_f}(0),\Omega(w_h))$$

$$\le \delta + 4^{-1}\delta(h,p) < \delta(h,p).$$

Together with (5.47) this inequality implies that $\tau_2 = T$. Thus we have shown that

$$\text{if } |X_w(T)-X_{w_f}(0)| \le \delta, \text{ then } \tau_2 = T. \tag{5.53}$$

It follows from (5.53), (5.52), (5.49) and (5.48) that the assertion of Theorem 2.4 holds if (5.35) is valid.

Assume now that (5.36) holds. In view of (5.36), Proposition 3.4, the definition of T_f (see (5.16), (5.17)), Assertion 3 of Theorem 2.3 and Proposition 3.3 there exists $r_0 \in (0,1)$ such that the following properties hold:

$$\mu(g) < \inf\{g(t,0,0) : \ t \in R\} \text{ for each } g \in \bar{\mathfrak{M}} \text{ satisfying } \rho(f,g) \le r_0; \tag{5.54}$$

$$\text{for each } g \in \bar{\mathfrak{M}} \text{ satisfying } \rho(f,g) \le r_0 \text{ and each } v \in \mathcal{P}(g)$$

$$\text{with a period } T > 0 \text{ we have } T \ge T_f/2. \tag{5.55}$$

Choose a natural number

$$p > 8\epsilon^{-1} + K + 8r_0^{-1} + 8n. \tag{5.56}$$

Since $f \in \mathcal{F}$ it follows from (5.1) that there exists $h \in E$ such that

$$f \in \mathcal{U}(h,p). \tag{5.57}$$

Relations (5.57), (5.56) and the property (Piv) imply that

$$\rho(f,h) < p^{-1} < r_0/8. \tag{5.58}$$

Together with (5.54), (5.55) and the definition of w_h, T_h this inequality implies that

$$\mu(h) < \inf\{h(t,0,0) : \ t \in R\}, \tag{5.59}$$

$$T_h \ge T_f/2. \tag{5.60}$$

In view of (Pi), (5.57) and the definition of w_f (see (5.15)-(5.17)) there is $\xi_0 \in [0, T_h)$ such that

(5.61) $$|X_{w_f}(t) - X_{w_h}(\xi_0 + t)| \le 4^{-1}\delta(h,p),\ t \in [0, 2pT_h].$$

By (5.57), (5.56), (Piii), (5.59) and the definition of w_f, T_f (see (5.15)-(5.17))

(5.62) $$T_f \ge 2^{-1}T_h.$$

Choose positive numbers

(5.63) $$\delta < \delta(h,p)/4,$$

(5.64) $$l > pT_h + 2l(h,p) + 2n + 2nT_f.$$

Assume that

(5.65) $$g \in \mathcal{U}(h,p),\ T \ge 2l$$

and $w \in W^{2,1}([0,T])$ satisfies

(5.66) $$|X_w(0)|, |X_w(T)| \le K,\ I^g(0,T,w) \le U_T^g(X_w(0), X_w(T)) + \delta.$$

It follows from (5.65), (5.66), (5.63) and the property (Pii) that

(5.67) $$|X_w(t)| \le M(h,p),\ t \in [0,T].$$

By the property (Pii), (5.65), (5.63), (5.64) and (5.56) there exist

(5.68) $$\tau_1 \in [0, l(h,p)] \subset [0,l],\ \tau_2 \in [T - l(h,p), T] \subset [T-l, T]$$

such that the following property holds:

(Pviii) For every $s \in [\tau_1, \tau_2 - pT_h]$ there is $\xi \in [0, T_h]$ such that

(5.69) $$|X_w(s+t) - X_{w_h}(\xi + t)| \le 1/p \text{ for all } t \in [0, pT_h];$$

(5.70) $$\text{if } d(X_w(0), \Omega(w_h)) \le \delta(h,p),\ \text{then } \tau_1 = 0;$$

(5.71) $$\text{if } d(X_w(T), \Omega(w_h)) \le \delta(h,p),\ \text{then } \tau_2 = T.$$

In view of (5.68), (5.65) and (5.64)

(5.72) $$\tau_2 - \tau_1 \ge T - 2l(h,p) \ge l - 2l(h,p) > nT_f,\ pT_h.$$

Assume that

(5.73) $$s \in [\tau_1, \tau_2 - nT_f].$$

Clearly

(5.74) $$[s, s + nT_f] \subset [\tau_1, \tau_2].$$

It follows from (5.60), (5.56), (5.74) and (5.72) that there exists a number s_0 such that

$$[s, s + nT_f] \subset [s_0, s_0 + pT_h] \subset [\tau_1, \tau_2]. \tag{5.75}$$

By (5.75) and the property (Pviii) there exists $\xi \in [0, T_h]$ such that

$$|X_w(s_0 + t) - X_{w_h}(\xi + t)| \le 1/p \text{ for all } t \in [0, pT_h]. \tag{5.76}$$

Since T_h is a period of w_h it follows from (5.61) and (5.56) that for all $t \in [0, pT_h]$

$$|X_{w_h}(\xi + t) - X_{w_f}(t + \xi - \xi_0 + T_h)| = |X_{w_h}(\xi + T_h + t) - X_{w_f}(t + T_h - \xi_0 + \xi)|$$

$$= |X_{w_h}(\xi_0 + T_h - \xi_0 + \xi + t) - X_{w_f}(T_h - \xi_0 + \xi + t)| \le 4^{-1}\delta(h, p).$$

Hence

$$|X_{w_h}(\xi + t) - X_{w_f}(t + \xi - \xi_0 + T_h)| \le 4^{-1}\delta(h, p), \; t \in [0, T_h p].$$

Combined with (5.76) this inequality implies that for $t \in [0, pT_h]$

$$|X_w(s_0 + t) - X_{w_f}(t + \xi - \xi_0 + T_h)| \le 1/p + 4^{-1}\delta(h, p) \le 2p^{-1}.$$

Together with (5.75) this inequality implies that

$$|X_w(s + t) - X_{w_f}(t + s - s_0 + \xi - \xi_0 + T_h)| \le 2/p < \epsilon, \; t \in [0, nT_f].$$

Thus we have shown that for each number s satisfying (5.73) there is $\xi_1 \in [0, T_f)$ such that

$$|X_w(s + t) - X_{w_f}(t + \xi_1)| \le \epsilon, \; t \in [0, nT_f]. \tag{5.77}$$

Assume that

$$d(X_w(0), \Omega(w_f)) \le \delta. \tag{5.78}$$

It follows from (5.61), (5.62) and (5.60) that

$$\mathrm{dist}(\Omega(w_f), \Omega(w_h)) \le 4^{-1}\delta(h, p).$$

Together with (5.78) and (5.63) this inequality implies that

$$d(X_w(0), \Omega(w_h)) \le \delta + 4^{-1}\delta(h, p) \le \delta(h, p).$$

Combined with (5.70) this inequality implies that $\tau_1 = 0$. Thus we have shown that

$$\text{if } d(X_w(0), \Omega(w_f)) \le \delta, \; \text{then } \tau_1 = 0. \tag{5.79}$$

Analogously we can show that

$$\text{if } d(X_w(T), \Omega(w_f)) \le \delta, \; \text{then } \tau_2 = T. \tag{5.80}$$

Relations (5.67), (5.72), (5.77), (5.79) and (5.80) imply the validity of the assertion of Theorem 2.4 if (5.36) holds. Theorem 2.4 is proved.

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