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THE STRONG NONLINEAR LIMIT-POINT/LIMIT-CIRCLE PROPERTIES FOR SUB-HALF-LINEAR EQUATIONS

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ABSTRACT. The authors consider the nonlinear second order differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda}\operatorname{sgn} y = 0, \tag{E}$$

where $p > 0$, $\lambda > 0$, $a(t) > 0$, $r(t) > 0$, and $\lambda \le p$ (the sub-half-linear case). They give necessary and sufficient conditions for equation (E) to be of the strong nonlinear limit-circle type and for (E) to be of the strong nonlinear limit-point type. Examples illustrating the results are also included.

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1. INTRODUCTION

The study of the limit–point/limit–circle problem has its origins in the work of Hermann Weyl [11] who considered the second order linear eigenvalue problem

$$y'' + r(t)y = \theta y, \quad \theta \in \mathbb{C}, \tag{C}$$

and classified this equation to be of the *limit–circle* type if every solution belongs to L^2, i.e.,

$$\int_0^\infty y^2(\sigma)\,d\sigma < \infty,$$

and to be of the *limit–point* type if at least one solution $y(t)$ does not belong to L^2, i.e.,

$$\int_0^\infty y^2(\sigma)\,d\sigma = \infty.$$

The limit–point/limit–circle problem then becomes that of determining conditions on the coefficient function r that allows us to distinguish between these two cases. Weyl showed that the linear equation (C) always has at least one square integrable solution provided Im $\theta \neq 0$. Thus, the problem reduces to whether the equation (C) has one

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(limit–point case) or two (limit–circle case) square integrable solutions; this is known as the *Weyl Alternative*.

Weyl was also able to prove that if (C) is limit–circle for some $\theta_0 \in \mathbb{C}$, then it is limit–circle for all $\theta \in \mathbb{C}$. In particular, this is true for $\theta = 0$, so that if we can show that the equation

$$y'' + r(t)y = 0 \tag{L}$$

is limit–circle, then equation (C) is limit–circle for all values of θ, and if (L) is not limit–circle, then equation (C) is not limit–circle for any value of θ. However, for this equation (L) we are not guaranteed that there is at least one square integrable solution. Over the ensuing years there has been considerable interest in this problem due to its relationship to the solution of certain boundary value problems. The analogous problem for nonlinear equations is relatively new by comparison and is not as extensively studied as the linear case.

In this paper, we consider the second order nonlinear differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda}\operatorname{sgn} y = 0, \tag{1.1}$$

where $p > 0$, $\lambda > 0$, $a \in C^1(R_+)$, $a^{1/p}r \in AC'_{\mathrm{loc}}(R_+)$, $a(t) > 0$, and $r(t) > 0$. Observe that if $\lambda = p$, then equation (1.1) is the well-known *half-linear* equation. Where convenient, we will refer to equation (1.1) as being of the *super-half-linear* type if $\lambda > p$ and of the *sub-half-linear* type if $\lambda < p$. Throughout this paper we will assume that $\lambda \le p$.

We should immediately point out that the functions a and r as given here are smooth enough to ensure that all nontrivial solutions are defined on R_+ and are different from zero in any neighborhood of ∞ (see, for example, Theorem 1 in [1] or Lemma 1 in [4]). We will let

$$y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t),$$

and define the function $R : R_+ \to R$ by

$$R(t) = a^{1/p}(t)r(t).$$

For the nonlinear equation (1.1), the limit–point and limit–circle properties take the following form (see [3, 4, 5, 6, 7]).

Definition 1.1. A solution y of equation (1.1) defined on R_+ is said to be of the *nonlinear limit–circle* type if

$$\int_0^{\infty} |y(\sigma)|^{\lambda+1} d\sigma < \infty, \tag{NLC}$$

and it is said to be of the *nonlinear limit–point* type otherwise, i.e., if

$$\int_0^{\infty} |y(\sigma)|^{\lambda+1} d\sigma = \infty. \tag{NLP}$$

Equation (1.1) will be said to be of the *nonlinear limit–circle* type if every solution y of (1.1) defined on R_+ satisfies (NLC) and to be of the *nonlinear limit–point* type if there is at least one solution y defined on R_+ for which (NLP) holds.

A survey of known results on the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero, can be found in the recent monograph by Bartušek, Došlá, and Graef [3]. Additional results can be found in the papers of Bartušek and Graef [4, 5, 6, 7].

Our focus in this paper is on what we call *strong nonlinear limit-point* and *strong nonlinear limit-circle* solutions of (1.1). The notion of a strong nonlinear limit-point solution was first introduced in [5]. We let δ denote the constant

$$\delta = \frac{p+1}{p}.$$

Definition 1.2. A solution y of (1.1) is said to be of the strong nonlinear limit-point type if

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1}\,d\sigma = \infty$$

and

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty\,.$$

Equation (1.1) is said to be of the strong nonlinear limit-point type if every nontrivial solution is of the strong nonlinear limit-point type.

Analogously, we have the following definition.

Definition 1.3. A solution y of (1.1) is said to be of the strong nonlinear limit-circle type if

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1}\,d\sigma < \infty$$

and

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma < \infty\,.$$

Equation (1.1) is said to be of the strong nonlinear limit-circle type if every solution is of the strong nonlinear limit-circle type.

It will be convenient to define the following constants:

$$\alpha = \frac{p+1}{(\lambda+2)p+1}\,, \quad \beta = \frac{(\lambda+1)p}{(\lambda+2)p+1}\,, \quad \gamma = \frac{p+1}{p(\lambda+1)}\,,$$

$$\beta_1 = \frac{(\lambda+2)p+1}{(\lambda+1)(p+1)}\,, \quad \omega = \frac{(\lambda+1)(p+1)}{\lambda-p} \quad \text{for} \quad \lambda \neq p, \quad \text{and}$$

$$\omega_1 = \frac{\lambda-p}{(\lambda+1)(p+1)}\,.$$

Notice that $\alpha = 1-\beta$. We define the function $g : R_+ \to R$ by

$$g(t) = -\frac{a^{1/p}(t)R'(t)}{R^{\alpha+1}(t)}\,.$$

For any continuous function $h : R_+ \to R$, we let $h_+(t) = \max\{h(t), 0\}$ and $h_-(t) = \max\{-h(t), 0\}$ so that $h(t) = h_+(t) - h_-(t)$. For any solution $y : R_+ \to R$ of (1.1), we let

$$F(t) = R^\beta(t)\left[\frac{a(t)}{r(t)}|y'(t)|^{p+1} + \gamma|y(t)|^{\lambda+1}\right]$$

$$= R^\beta(t)\left(\frac{|y^{[1]}(t)|^\delta}{R(t)} + \gamma|y(t)|^{\lambda+1}\right). \tag{1.2}$$

Note that $F > 0$ on R_+ for every nontrivial solution of (1.1) (see Theorem 1 in [1]). Throughout the remainder of this paper, we will assume that

$$\lim_{t\to\infty} g(t) = 0, \quad \int_0^\infty |g'(\sigma)|\,d\sigma < \infty, \tag{1.3}$$

and

$$\int_0^\infty \left(a^{-\frac{1}{p}}(\sigma) + r(\sigma)\right) d\sigma = \infty\,. \tag{1.4}$$

A solution y of (1.1) is said to be *oscillatory* if there exists a sequence of its zeros tending to ∞; otherwise, it is *nonoscillatory.*

The next section contains some lemmas that are used to prove the main results in this paper. The main results, their proofs, and some examples to illustrate our results appear in Section 3.

2. PRELIMINARY LEMMAS

In this section, we will present some lemmas that will give some insight into the behavior of solutions of equation (1.1) as well as facilitate the proofs of our main results. Our first lemma is for both the half-linear and sub-half-linear cases.

Lemma 2.1. (i) *If $\lambda = p$, then for every nontrivial solution y of* (1.1), *we have*

$$\lim_{t\to\infty} F(t) = C \in (0,\infty)\,. \tag{2.1}$$

(ii) *If $\lambda < p$, then for every nontrivial solution y of* (1.1), *either* (2.1) *holds, or*

$$F(t) \geq M\Big(\int_t^{\infty} |g'(\sigma)|\, d\sigma\Big)^{\omega}, \tag{2.2}$$

where M is a positive constant depending on λ and p.

Proof. Parts (i) and (ii) follow from Theorems 17.6 and 17.5 in Mirzov [10], respectively. Notice that apart from the values of λ, the behaviors of F in (2.1) and (2.2) are mutually exclusive. $\square$

The following lemma is a consequence of Lemma 2.5 of Bartušek and Graef [6].

Lemma 2.2. *If $\lambda < p$ and*

$$\liminf_{t\to\infty} R^{\beta}(t)\Big(\int_t^{\infty} |g'(\sigma)|\, ds\Big)^{-\omega} \exp\Big\{\int_0^t \big(R^{-1}(\sigma)\big)'_+ R(\sigma)\, d\sigma\Big\} = 0, \tag{2.3}$$

then (2.1) *holds for every nontrivial solution y of* (1.1).

In addition to yielding useful expressions for y and $y^{[1]}$, the following lemma gives a characterization of the oscillatory solutions of (1.1).

Lemma 2.3. *For every nontrivial solution y of equation* (1.1), *there exists a positive function $\varphi \in C^1(R_+)$ such that*

$$\begin{aligned} y(t) &= R^{-\frac{\beta}{\lambda+1}}(t)F^{\frac{1}{\lambda+1}}(t)w\big(\varphi(t)\big)\,, \\ y^{[1]}(t) &= R^{\frac{\beta}{\lambda+1}}(t)F^{\frac{p}{p+1}}(t)w^{[1]}\big(\varphi(t)\big), \end{aligned} \tag{2.4}$$

where $w^{[1]}(s) = |w'(s)|^{p-1}w'(s)$ and w is a periodic solution of the problem

$$\big(|w'|^{p-1}w'\big)' + |w|^{\lambda+1}w = 0\,, \quad w(0) = \gamma^{\frac{-1}{\lambda+1}}\,, \quad w'(0) = 0\,.$$

Moreover,

$$\varphi'(t) = a^{-\frac{1}{p}}(t)R^{\alpha}(t)\left[F^{\omega_1}(t) - \frac{1}{\lambda+1}g(t)w\big(\varphi(t)\big)w^{[1]}\big(\varphi(t)\big)\right]\,, \tag{2.5}$$

and y is oscillatory if and only if $\lim_{t\to\infty}\varphi(t) = \infty$.

This result follows from Lemma 2 and Theorem 1 in [2]. Moreover, it follows from (14) in [2] that

$$\max_{t\in R_+} |w(t)| = \gamma^{-\frac{1}{\lambda+1}} \tag{2.6}$$

and

$$|w^{[1]}(t)| = 1 \quad \text{at all relative extrema of } \ w^{[1]}. \tag{2.7}$$

Our next lemma shows that equation (1.1) always has an oscillatory solution and provides additional information about the behavior of the function F for nonoscillatory solutions.

Lemma 2.4. (i) *Equation* (1.1) *has an oscillatory solution.*

(ii) *For every nonoscillatory solution* y *of* (1.1),

$$\lim_{t\to\infty} F(t) = \infty.$$

Proof. Part (i) follows from Theorem 2 in [2]. To prove (ii), notice that Lemma 2.1 (ii) implies $\lim\limits_{t\to\infty} F(t) \in (0,\infty]$. Let y be a nonoscillatory solution and suppose, to the contrary, that

$$\lim_{t\to\infty} F(t) = C \in (0,\infty)\,.$$

In view of the first part of (1.3), we then see that $\varphi'(t)$ is eventually positive, and since y is nonoscillatory, by Lemma 2.3 we have that $\varphi(t)$ is bounded. Since (1.4) holds, Lemma 6 in [2] implies

$$\int_0^\infty a^{-\frac{1}{p}}(\sigma) R^\alpha(\sigma)\, d\sigma = \infty\,.$$

Thus, by L'Hôpital's Rule, we have

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{\varphi(t)}{\int_0^t a^{-1/p}(\sigma) R^\alpha(\sigma)\, d\sigma} = \lim_{t\to\infty} \frac{\varphi'(t)}{a^{-1/p}(t) R^\alpha(t)} \\ &= \lim_{t\to\infty} \left[F^{\omega_1}(t) - \frac{1}{\lambda+1} g(t) w\big(\varphi(t)\big) w^{[1]}\big(\varphi(t)\big) \right] \\ &= C^{\omega_1} \neq 0\,, \end{aligned}$$

and this contradiction completes the proof of the lemma. □

Lemma 2.5. *Assume that*

$$\lim_{t\to\infty} \frac{a'(t)}{a^{1-\frac{\beta}{p}}(t) r^\alpha(t)} = 0\,, \tag{2.8}$$

$$\int_0^\infty R^{-\beta}(\sigma)\, d\sigma = \infty, \tag{2.9}$$

and let y *be a nontrivial solution of* (1.1) *such that* $\lim\limits_{t\to\infty} F(t) = C \in (0,\infty)$. *Then*

$$\int_0^\infty |y(\sigma)|^{\lambda+1}\, d\sigma = \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\,. \tag{2.10}$$

Proof. Let y be a nontrivial solution of (1.1) such that $\lim\limits_{t\to\infty} F(t) = C \in (0,\infty)$. By Lemma 2.4(ii), y is oscillatory. Moreover, there exists $t_0 \in R_+$ such that

$$\frac{C}{2} \le F(t) \le 2C \quad \text{for} \quad t \ge t_0\,. \tag{2.11}$$

From this and (1.2), we have

$$\begin{aligned}\int_{t_0}^t \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma + \gamma\int_{t_0}^t |y(\sigma)|^{\lambda+1}\,d\sigma &= \int_{t_0}^t F(\sigma)R^{-\beta}(\sigma)\,d\sigma \\ &\ge \frac{C}{2}\int_{t_0}^t R^{-\beta}(\sigma)\,d\sigma\end{aligned} \tag{2.12}$$

for $t \ge t_0$. Now, (2.9) and (2.12) imply that either

$$\int_0^\infty |y(\sigma)|^{\lambda+1}\,d\sigma = \infty \quad \text{or} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty. \tag{2.13}$$

Furthermore,

$$\begin{aligned}\int_{t_0}^t |y(\sigma)|^{\lambda+1}\,d\sigma = -\int_{t_0}^t \frac{y(\sigma)(y^{[1]}(\sigma))'}{r(\sigma)}\,d\sigma &= -\frac{y(t)y^{[1]}(t)}{r(t)} + \frac{y(t_0)y^{[1]}(t_0)}{r(t_0)} \\ &+ \int_{t_0}^t \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma + \int_{t_0}^t \Big(\frac{1}{r(\sigma)}\Big)' y(\sigma)y^{[1]}(\sigma)\,d\sigma.\end{aligned} \tag{2.14}$$

In view of (2.4), (2.6), and (2.7), we have

$$\big|y(t)y^{[1]}(t)\big| \le \gamma^{-\frac{1}{\lambda+1}}F^{\beta_1}(t) \le \gamma^{-\frac{1}{\lambda+1}}(2C)^{\beta_1} := C_1\,, \quad t \ge t_0\,. \tag{2.15}$$

Moreover,

$$\begin{aligned}\int_{t_0}^t \Big|\Big(\frac{1}{r(\sigma)}\Big)'\Big|\,d\sigma &= \int_{t_0}^t \Big|\Big(\frac{a^{\frac{1}{p}}(\sigma)}{R(\sigma)}\Big)'\Big|\,d\sigma \\ &\le \int_{t_0}^t \frac{|a'(\sigma)|}{p\,a(\sigma)r(\sigma)}\,d\sigma + \int_{t_0}^t |g(\sigma)|R^{-\beta}(\sigma)\,d\sigma,\end{aligned} \tag{2.16}$$

and using L'Hôpital Rule, (2.8), and (2.9), we obtain

$$\begin{aligned}\lim_{t\to\infty}\int_{t_0}^t \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\,d\sigma\Big(\int_{t_0}^t R^{-\beta}(\sigma)\,d\sigma\Big)^{-1} &= \lim_{t\to\infty}\frac{|a'(t)|}{a(t)r(t)}R^\beta(t) \\ = \lim_{t\to\infty}\frac{|a'(t)|}{a^{1-\frac{\beta}{p}}(t)r^\alpha(t)} = 0\,.\end{aligned} \tag{2.17}$$

We will now prove that the first integral in (2.10) is divergent, so suppose

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1}\,d\sigma < \infty\,. \tag{2.18}$$

Then (2.13) implies

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty\,. \tag{2.19}$$

From the fact that $\lim_{t\to\infty} g(t) = 0$ and that (2.12), (2.17), (2.18), and (2.19) hold, we can choose $T \ge t_0$ such that

$$\big|g(t)\big| \le \frac{C}{4C_1}, \quad \int_{t_0}^{t} \frac{|y^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\,d\sigma \ge \frac{C}{3}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma,$$

$$\text{and} \quad \frac{C_1}{p}\int_{t_0}^{t} \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\,d\sigma \le \frac{C}{24}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma$$

for $t \ge T$. Hence, (2.14), (2.15) and (2.16) yield

$$\begin{aligned}\int_{t_0}^{t} \big|y(\sigma)\big|^{\lambda+1}\,d\sigma &\ge C_2 - \frac{y(t)y^{[1]}(t)}{r(t)} + \int_{t_0}^{t} \frac{|y^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\,d\sigma \\ &\quad - \frac{C_1}{p}\int_{t_0}^{t} \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\,d\sigma - \frac{C}{4}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma \\ &\ge C_2 - \frac{y(t)y^{[1]}(t)}{r(t)} + \frac{C}{24}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma \end{aligned} \tag{2.20}$$

for $t \ge T$ and some constant C_2. Since y is oscillatory, if we take an increasing sequence $\{t_n\}_1^{\infty}$ of zeros of y with $t_1 \ge T$, then (2.20) contradicts (2.9). Thus,

$$\int_0^{\infty} \big|y(\sigma)\big|^{\lambda+1}\,dt = \infty\,.$$

We next prove the second integral in (2.10) is divergent. To the contrary, suppose that

$$\int_0^{\infty} \frac{|y^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\,d\sigma < \infty\,; \tag{2.21}$$

then, the first integral in (2.10) diverges. Similar to what we did above, in view of (2.10), (2.11), (2.17), (2.21), and the equality in (2.12), we can choose $T_1 \ge t_0$ so that

$$\begin{aligned} &\big|g(t)\big| \le \frac{C}{12C_1}\,, \quad \int_{t_0}^{t} \big|y(\sigma)\big|^{\lambda+1}\,d\sigma \ge \frac{C}{3}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma, \quad \text{and} \\ &\frac{C_1}{p}\int_{t_0}^{t} \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\,d\sigma \le \frac{C}{12}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma \end{aligned} \tag{2.22}$$

for $t \ge T_1$. Then (2.14), (2.15), (2.16), and (2.22) yield

$$\begin{aligned} \frac{C}{3}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma &\le \int_{t_0}^{t} |y(\sigma)|^{\lambda+1}\,d\sigma \\ &\le C_3 - \frac{y(t)y^{[1]}(t)}{r(t)} + \frac{C_1}{p}\int_{t_0}^{t} \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\,d\sigma + \frac{C}{12}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma \\ &\le C_3 - \frac{y(t)y^{[1]}(t)}{r(t)} + \frac{C}{6}\int_{t_0}^{t} R^{-\beta}(\sigma)\,d\sigma \end{aligned} \tag{2.23}$$

for $t \ge T_1$ and some constant C_3. Again taking an increasing sequence of zeros of y tending to ∞, (2.23) contradicts (2.9). □

Remark 2.6. We wish to point out that, except for part (ii) of Lemma 2.4, Lemmas 2.3–2.5 hold without requiring $\lambda \le p$. A version of part (ii) of Lemma 2.4 can be proved for $\lambda > p$, but in that case, $\lim_{t\to\infty} F(t) = 0$.

Remark 2.7. From the proof of Lemma 2.5, we see that we could replace condition (2.8) by asking instead that

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{|a'(\sigma)|}{a(\sigma)r(\sigma)}\, d\sigma \Big(\int_{t_0}^{t} R^{-\beta}(\sigma)\, d\sigma \Big)^{-1} = 0. \tag{2.24}$$

An easy modification of the proof of Lemma 2.5 also shows that (2.8) can be replaced by the the condition

$$\int_0^\infty \left| \left(\frac{1}{r(\sigma)} \right)' \right| d\sigma < \infty. \tag{2.25}$$

The following lemma gives some properties of the nonoscillatory solutions of equation (1.1).

Lemma 2.8. (i) *If*

$$\int_0^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma = \infty, \tag{2.26}$$

then any nonoscillatory solution y of (1.1) *satisfies $y(t)y'(t) > 0$ for all large t.*

(ii) *If*

$$\int_0^\infty a^{-\frac{1}{p}}(\sigma)\, d\sigma < \infty, \tag{2.27}$$

then there exists $R_0 > 0$ such that $R_0 \le R(t)$ for $t \in R_+$.

(iii) *If $\lambda < p$,* (2.2) *and* (2.27) *hold, and*

$$\lim_{t\to\infty} R^{\beta}(t) \left(\int_t^\infty |g'(\sigma)|\, d\sigma \right)^{-\omega} = 0, \tag{2.28}$$

then no nonoscillatory solution y of (1.1) *satisfies $\lim\limits_{t\to\infty} y(t) = C \in (-\infty, \infty)$.*

Proof. Part (i) is straight forward. To prove (ii), first note that $\lim\limits_{t\to\infty} g(t) = 0$ implies there exists $t_0 \in R_+$ such that

$$-1 \le -g(t) = \frac{a^{1/p}(t)R'(t)}{R^{1+\alpha}(t)}\,,$$

for $t \ge t_0$. Thus,

$$-a^{-1/p}(t) \le \frac{R'(t)}{R^{1+\alpha}(t)}$$

for $t \ge t_0$, and so

$$-\infty < -\int_0^\infty a^{-1/p}(\sigma)\, d\sigma \le \frac{1}{\alpha}\big(R^{-\alpha}(t_0) - R^{-\alpha}(t) \big) \tag{2.29}$$

for $t \ge t_0$. If $\liminf\limits_{t\to\infty} R(t) = 0$, then (2.29) gives us a contradiction. Hence, there exists $R_0 > 0$ such that $R_0 \le R(t)$ for $t \in R_+$.

Finally, to prove (iii), suppose that y satisfies

$$\lim_{t\to\infty} \big|y(t)\big| = C \in [0,\infty)\,. \tag{2.30}$$

By Lemma 2.3 and inequality (2.2),

$$\big|y(t)\big|^{\lambda+1} = R^{-\beta}(t)F(t)\big|w^{\lambda+1}\left(\varphi(t)\right)\big| \ge MR^{-\beta}(t)\Big(\int_t^\infty \big|g'(\sigma)\big|\,d\sigma\Big)^{\omega}\big|w^{\lambda+1}\big(\varphi(t)\big)\big|\,.$$

Hence, (2.30) and the assumptions in this part of the lemma imply $\lim\limits_{t\to\infty} w^{\lambda+1}\big(\varphi(t)\big) = 0$.

From Lemma 2.4, we have that

$$\lim_{t\to\infty} F(t) = \infty.$$

In view of (1.3), (2.6), and (2.7), we see from (2.5) that $\varphi(t)$ is increasing. Since y is nonoscillatory, $\varphi(t)$ is bounded. It then follows from the differential equation for w that $w\big(\varphi(t)\big) \to 0$ implies $w^{[1]}\big(\varphi(t)\big)$ approaches a relative extrema, so (2.7) implies

$$\lim_{t\to\infty} \big|w'\big(\varphi(t)\big)\big| = 1. \tag{2.31}$$

By part (ii) of this lemma, $R(t) \ge R_0 > 0$ for $t \in R_+$, so (2.31), the second equality in (2.4), and (2.2) imply

$$\begin{aligned}\big|y'(t)\big|^{p+1} &= a^{-\frac{p+1}{p}}(t)R^{\alpha}(t)F(t)\mid w'\big(\varphi(t)\big)\big|^{p+1}\\ &\ge a^{-\frac{p+1}{p}}(t)R^{-\beta}(t)\Big(\int_t^\infty \big|g'(\sigma)\big|\,d\sigma\Big)^{\omega}\big|w'\big(\varphi(t)\big)\big|^{p+1}R_0M.\end{aligned}$$

Let $t_0 \in R_+$ be such that $y \ne 0$ and $\big|w'\big(\varphi(t)\big)\big| \ge \frac{1}{2}$ on $[t_0,\infty)$. Then,

$$\begin{aligned}\big|y(t) - y(t_0)\big| \ge &\left(\frac{R_0M}{2}\right)^{\frac{1}{p+1}}\\ &\times \min_{t_0\le t\le\infty}\Big[\Big(R^{-\beta}(t)\Big(\int_t^\infty |g'(\sigma)|\,d\sigma\Big)^{\omega}\Big)^{\frac{1}{p+1}}\Big]\int_{t_0}^t a^{-1/p}(\sigma)\,d\sigma \to \infty\end{aligned}$$

as $t \to \infty$. This contradiction to (2.30) completes the proof of the lemma. ☐

Example 2.9. Consider equation (1.1) with $a \equiv 1$, $p = 1$, $\lambda < 1$, $r(t) = t^{\delta}$, $\delta \in R$, and $t \ge 1$, that is, the equation

$$y'' + t^b|y|^{\lambda}\operatorname{sgn} y = 0\,, \quad t \ge 1\,. \tag{2.32}$$

Then (1.3) holds if $b > -(\lambda+3)/2$ and (2.3) holds if $b > -(\lambda+1)$. (Since $\lambda < 1$, these reduce to $b > -(\lambda+1)$.) Condition (2.9) holds if $b \le (\lambda+3)/(\lambda+1)$ while (1.4) and (2.8) are automatic. Part (i) of Lemma 2.8 also applies to equation (2.32).

Our next two lemmas require that R be small in some sense.

Lemma 2.10. *Assume that* $\lambda < p$ *and that* (2.2) *and* (2.28) *hold. Then any oscillatory solution* y *of equation* (1.1) *is unbounded. Moreover, if*

$$a^{-1/p}(t)R^{\frac{1}{p+1}}(t) \le B_1 < \infty \tag{2.33}$$

for $t \in R_+$, *then*

$$\int_0^\infty \left|y(\sigma)\right|^{\lambda+1} d\sigma = \infty\,.$$

Proof. Let y be an oscillatory solution of (1.1). Then from (2.2) and Lemma 2.3, a function φ exists such that, if $\{t_k\}_1^\infty$ is an increasing sequence of relative extrema of y, then

$$\begin{aligned} \left|y^{\lambda+1}(t_k)\right| &= R^{-\beta}(t_k)F(t_k)\left|w^{\lambda+1}\big(\varphi(t_k)\big)\right| \\ &\ge MR^{-\beta}(t_k)\Big(\int_{t_k}^\infty \left|g'(\sigma)\right| d\sigma\Big)^\omega \left|w^{\lambda+1}\big(\varphi(t_k)\big)\right|. \end{aligned} \tag{2.34}$$

At the same time, (2.4) implies that y has a local extrema if and only if w has a local extrema, and by (2.6), we have $w^{\lambda+1}\big(\varphi(t_k)\big) = \gamma^{-1}$. From this and (2.28), we obtain that $\lim_{k\to\infty} \left|y(t_k)\right| = \infty$, and so y is unbounded.

Now let $\{\tau_k\}_1^\infty$ be a sequence such that $\tau_k < t_k < \tau_{k+1}$, $y(\tau_k) = 0$, and $y(t) > 0$ on (τ_k, t_k); note that $y'(t_k) = 0$, $k = 1, 2, \ldots$. From (2.2), we have

$$F^{\omega_1}(t) \le M^{\omega_1} \int_t^\infty \left|g'(\sigma)\right| d\sigma,$$

so from (2.5), we obtain

$$\varphi'(t) \le a^{-\frac{1}{p}}(t)R^\alpha(t)\Big[C_1 \int_t^\infty \left|g'(\sigma)\right| d\sigma - \frac{1}{\lambda+1} g(t)w\big(\varphi(t)\big)w^{[1]}\big(\varphi(t)\big)\Big],$$

where $C_1 = M^{\omega_1}$. Now (1.3) implies that g is of bounded variation on R_+, and since $\lim_{t\to\infty} g(t) = 0$, we see that $\left|g(t)\right| \le \int_t^\infty \left|g'(\sigma)\right| d\sigma$ for $t \in R_+$. From this, (2.6), and (2.7), we have

$$\varphi'(t) \le C_2 a^{-1/p}(t)R^\alpha(t) \int_t^\infty \left|g'(\sigma)\right| d\sigma\,, \quad C_2 = C_1 + \gamma^{-\frac{1}{\lambda+1}}/(\lambda+1)\,. \tag{2.35}$$

Then, from (2.28), (2.33), (2.34) with t_k replaced by t, and (2.35), we have

$$\begin{aligned} &\int_{\tau_k}^{t_k} \left|y(\sigma)\right|^{\lambda+1} d\sigma \\ &\ge M \max_{\tau_k \le t \le t_k} \Big[R^{-\beta}(t)\Big(\int_t^\infty \left|g'(\sigma)\right| d\sigma\Big)^\omega a^{\frac{1}{p}}(t)R^{-\alpha}(t)\Big(\int_t^\infty \left|g'(\sigma)\, d\sigma\right.\Big)^{-1}\Big] \\ &\quad\times \int_{\tau_k}^{t_k} a^{-\frac{1}{p}}(s)R^\alpha(s) \int_s^\infty \left|g'(\sigma)\right| d\sigma w^{\lambda+1}\big(\varphi(s)\big)\, ds \\ &\ge \frac{M}{C_2} \max_{\tau_k \le t \le t_k} \Big\{\Big[R^{-\beta}(t)\Big(\int_t^\infty \left|g'(\sigma)\right| d\sigma\Big)^\omega\Big]^{1-\frac{1}{\omega}} a^{\frac{1}{p}}(t)R^{-\frac{1}{p+1}}\Big\} \\ &\quad\times \int_{\tau_k}^{t_k} w^{\lambda+1}\big(\varphi(\sigma)\big)\varphi'(\sigma)\, d\sigma \to \infty \end{aligned}$$

as $k \to \infty$ since $\int\limits_{\varphi(\tau_k)}^{\varphi(t_k)} w^{\lambda+1}(z)\,dz = \text{const.} > 0$ for any $k \in \{1, 2, \dots\}$ due to the periodicity of w. This completes the proof of the lemma. $\square$

Lemma 2.11. *Let y be an unbounded solution of* (1.1) *and assume that there is a positive constant B_2 such that*

$$R'_-(t)/R(t) < B_2 \tag{2.36}$$

for $t \in R_+$. Then

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = \infty.$$

Proof. Let $Z(t) = F(t)R^{-\beta}(t)$. Then (1.2) and the fact that y is unbounded imply Z is unbounded as well. On the other hand, suppose that

$$\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\,d\sigma = C < \infty.$$

Then, it is easy to see that $Z'(t) = -\frac{R'(t)}{R^2(t)}\left|y^{[1]}(t)\right|^\delta$ and

$$\begin{aligned} Z(t) &= Z(0) + \int_0^\infty Z'(s)\,ds = Z(0) - \int_0^t \frac{R'(\sigma)}{R^2(\sigma)}\left|y^{[1]}(\sigma)\right|^\delta d\sigma \\ &\le Z(0) + \int_0^t \frac{R'_-(\sigma)}{R^2(\sigma)}\left|y^{[1]}(\sigma)\right|^\delta d\sigma \le Z(0) + B_2 C < \infty\,. \end{aligned}$$

This contradiction proves the lemma. $\square$

Our next and final lemma gives a useful representation for the function F.

Lemma 2.12. *Let y be a nontrivial solution of* (1.1). *Then*

$$F(t) = R^{-\alpha}(t)\Big[K_1 + K_2\int_0^t R'(\sigma)\big|y(\sigma)\big|^{\lambda+1}\,d\sigma\Big]$$

with $K_1 = R^\alpha(0)F(0) > 0$ and $K_2 = \alpha(1+\gamma) > 0$.

Proof. Let $C_1 = 1 + \gamma$; then (1.2) yields

$$\begin{aligned} F'(t) &= \beta R^{-\alpha}(t)R'(t)\Big(\frac{|y^{[1]}(t)|^\delta}{R(t)} + \gamma|y(t)|^{\lambda+1}\Big) + R^\beta(t)\big(R^{-1}(t)\big)'\big|y^{[1]}(t)\big|^\delta \\ &= -\alpha\frac{R'(t)}{R^\alpha(t)}\Big(\frac{|y^{[1]}(t)|^\delta}{R(t)} - \big|y(t)\big|^{\lambda+1}\Big) \\ &= -\alpha\frac{R'(t)}{R^\alpha(t)}\Big(\frac{F(t)}{R^\beta(t)} - C_1\big|y(t)\big|^{\lambda+1}\Big) \\ &= -\alpha\frac{R'(t)}{R(t)}F(t) + \alpha C_1\frac{R'(t)}{R^\alpha(t)}\left|y(t)\right|^{\lambda+1}. \end{aligned}$$

Integrating, we obtain

$$F(t) = \exp\left\{-\alpha \int_0^t \frac{R'(\sigma)}{R(\sigma)}\, d\sigma\right\} \left[\int_0^t \alpha C_1 \frac{R'(\sigma)}{R^\alpha(\sigma)} \big|y(\sigma)\big|^{\lambda+1} \right.$$
$$\left. \times \exp\left\{\alpha \int_0^\sigma \frac{R'(s)}{R(s)}\, ds\right\} d\sigma + F(0)\right],$$

or

$$F(t) = R^{-\alpha}(t)\left[K_1 + K_2 \int_0^t R'(\sigma)\big|y(t)\big|^{\lambda+1}\, d\sigma\right]$$

for $t \in R_+$. □

3. MAIN RESULTS

Theorem 3.1. *Suppose that* (2.8) *holds and that either* $\lambda = p$*, or* $\lambda < p$ *and* (2.3) *holds.*

(a) The following statements are equivalent:

(i) *Equation* (1.1) *is of the nonlinear limit-circle type;*

(ii)

$$\int_0^\infty R^{-\beta}(\sigma)\, d\sigma < \infty\,; \tag{3.1}$$

(iii) $\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma < \infty$ *for all solutions of* (1.1).

That is, equation (1.1) *is of the strong nonlinear limit-circle type if and only if* (3.1) *holds.*

(b) The following statements are equivalent:

(iv) *Every nontrivial solution of* (1.1) *is of the nonlinear limit-point type;*

(v)

$$\int_0^\infty R^{-\beta}(\sigma)\, d\sigma = \infty; \tag{3.2}$$

(vi) $\int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty$ *for every nontrivial solutions of* (1.1).

That is, equation (1.1) *is of the strong nonlinear limit-point type if and only if* (3.2) *holds.*

Proof. (a) Lemmas 2.1 and 2.2 imply (2.1) holds for every nontrivial solution of (1.1). Therefore, F is bounded, say

$$C_1 < F(t) < C_2 \tag{3.3}$$

for some constants C_1, $C_2 > 0$ and all $t \in R_+$. From (1.2) and (3.3), we have

$$\infty > C_2 \int_0^\infty R^{-\beta}(\sigma)\, d\sigma \geq \int_0^\infty F(\sigma) R^{-\beta}(\sigma)\, d\sigma$$
$$= \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma + \gamma \int_0^\infty \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma \geq C_1 \int_0^\infty R^{-\beta}(\sigma)\, d\sigma.$$

From this and Lemma 2.5 we see that the three conclusions in part (a) are equivalent.

(b) If (3.2) holds, then Lemma 2.5 implies that every nontrivial solution y of (1.1) satisfies

$$\int_0^\infty \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma = \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\,.$$

Hence, (v) implies both (iv) and (vi).

Suppose that either (iv) or (vi) holds. Then there exists a nontrivial solution y of (1.1) such that either

$$\int_0^\infty \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma = \infty \quad \text{or} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\,.$$

But from part (a), this implies (3.1) fails. This completes the proof of the theorem. □

Remark 3.2. Theorem 3.1 includes as special cases the linear and sublinear parts of Theorems 3.1 and 3.2 in [5]. If we use condition (2.25) in place of (2.8) in Lemma 2.5, then Theorem 3.1 (b) includes a part of Theorem 3.4 (i) in [6] as a special case.

Theorem 3.3. *Assume that* (2.8), (2.26), (2.33), *and* (2.36) *hold and there is a positive constant* R_1 *such that*

$$R(t) \leq R_1 \tag{3.4}$$

for $t \in R_+$. *Then every nontrivial solution of* (1.1) *satisfies*

$$\int_0^\infty \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma = \infty \quad \textit{and} \quad \int_0^\infty \frac{|y^{[1]}(\sigma)|^\delta}{R(\sigma)}\, d\sigma = \infty\,, \tag{3.5}$$

that is, equation (1.1) *is of the strong nonlinear limit-point type.*

Proof. First note that the hypotheses of the theorem imply that conditions (2.9) and (2.28) hold. Let y be a nontrivial solution of (1.1). Then, by Lemma 2.1, either (2.1) or (2.2) holds. If (2.1) holds, the conclusion follows from Lemma 2.5.

Suppose (2.2) holds. In view of Lemma 2.1, we have $\lambda < p$. If y is an oscillatory solution, then the conclusion follows from Lemmas 2.10 and 2.11. If y is nonoscillatory, then Lemma 2.8(i) implies $|y|$ is increasing for large t. If $\lim_{t\to\infty} \bigl|y(t)\bigr| = \infty$, then clearly the first integral in (3.5) diverges, and the rest of the statement follows from Lemma

2.11. If $\lim_{t\to\infty} \big|y(t)\big| = C \in (0,\infty)$, then again the first integral in (3.5) diverges, and (1.2) and (2.2) imply

$$\gamma C^{\lambda+1} + \frac{|y^{[1]}(t)|^{\delta}}{R(t)} \geqq F(t)R^{-\beta}(t) \geq R_1^{-\beta} M\Big(\int_t^{\infty} \big|g'(\sigma)\big|\, d\sigma\Big)^{\omega} \to \infty$$

as $t \to \infty$. Hence, $\lim_{t\to\infty} |y^{[1]}(t)|^{\delta}/R(t) = \infty$ and so the second integral in (3.5) diverges. This completes the proof of the theorem. □

Remark 3.4. It should be clear from the proof of Theorem 3.3 that condition (3.4) can be removed if we instead require that conditions (2.9) and (2.28) hold.

Remark 3.5. Theorem 3.3 includes Theorem 3.6 (i) in [5] as a special case.

Our final theorem is also a strong nonlinear limit–point result.

Theorem 3.6. *Assume that* $\lambda < p$, (2.8), (2.27), *and* (2.36) *hold, and there is a positive constant* B_3 *such that*

$$R'(t) \leq B_3 \tag{3.6}$$

for $t \in R_+$. *If* (2.28) *holds, then any nontrivial solution* y *of* (1.1) *satisfies*

$$\int_0^{\infty} \big|y(\sigma)\big|^{\lambda+1}\, d\sigma = \infty \quad \textit{and} \quad \int_0^{\infty} \frac{|y^{[1]}(\sigma)|^{\delta}}{R(\sigma)}\, d\sigma = \infty\,, \tag{3.7}$$

that is, equation (1.1) *is of the strong nonlinear limit-point type.*

Proof. First observe that (3.6) implies that (2.9) holds. Let y be a nontrivial solution of (1.1); then, by Lemma 2.1, either (2.1) or (2.2) holds.

If (2.1) holds, then the statement follows from Lemma 2.5, so assume (2.2) holds. If y is oscillatory, then Lemmas 2.10 and 2.11 imply the second integral in (3.7) diverges. To prove that the first integral in (3.7) diverges, first observe that Lemma 2.8 (ii) implies $R(t) \geq R_0$ for $t \in R_+$ and some $R_0 > 0$. Then, from Lemma 2.12 and inequality (2.2), we have

$$\begin{aligned} R^{-\beta}(t)\Big(\int_t^{\infty} \big|g'(\sigma)\big|\, d\sigma\Big)^{\omega} &\leq R^{-\beta}(t)F(t)/M \\ &= R^{-1}(t)\Big[K_1 + K_2\int_0^t R'(\sigma)\big|y(\sigma)\big|^{\lambda+1}\, d\sigma\Big]\Big/M \\ &\leq R_0^{-1}\Big[K_1 + K_2 B_3 \int_0^t \big|y(\sigma)\big|^{\lambda+1}\, d\sigma\Big]\Big/M, \end{aligned}$$

and the desired conclusion follows from (2.28). Finally, if y is nonoscillatory, Lemma 2.8 (iii) implies $\lim_{t\to\infty} \big|y(t)\big| = \infty$, and the divergence of the first integral in (3.7) is immediate. The divergence of the second integral follows from Lemma 2.11. This completes the proof of the theorem. □

Remark 3.7. Alternate versions of Theorems 1, 2, and 3 hold if we replace condition (2.8) with either (2.24) or (2.25).

We conclude this paper with some examples.

Example 3.8. Consider the equation

$$\left(|y'|^3 y'\right)' + t^b y^3 = 0, \quad t \ge 1, \tag{3.8}$$

that is, in equation (1.1) we have $a(t) \equiv 1$, $r(t) = t^b$, $p = 4$, and $\lambda = 3$. Condition (1.3) holds if $b\alpha > -1$, that is, if $b > -21/5$, and condition (2.3) holds if $b > -4$. By Theorem 3.1, we then have that equation (3.8) is of the strong nonlinear limit–circle (limit–point) type if and only $b > 21/16$ $(-4 < b \le 21/16)$.

Example 3.9. Consider the equation

$$\left(e^{at}|y'|^{p-1}y'\right)' + e^{bt}|y|^\lambda \operatorname{sgn} y = 0, \quad t \ge 0, \tag{3.9}$$

with $\lambda \le p$. Assume that

$$\frac{a}{p} + b > 0, \;\; b\left(\frac{p+1}{\lambda+1}\right) > a,$$

and either $a \le 0$ or $b \ge 0$. If either (i) $\lambda = p$, or (ii) $\lambda < p$ and $a < b$, then Theorem 3.1 (a) implies that equation (3.9) is of the strong nonlinear limit–circle type.

Example 3.10. Consider the equation

$$\left(|y'|^{p-1}y'\right)' + t^b|y|^\lambda \operatorname{sgn} y = 0, \quad t \ge 1. \tag{3.10}$$

If $p \ge \lambda > 0$ and $0 \ge b > -[(\lambda+2)p+1]/(p+1)$, then all the hypotheses of Theorem 3.3 are satisfied, so equation (3.10) is of the strong nonlinear limit–point type.

Example 3.11. Consider the equation

$$\left(|y'|^{\lambda-1}y'\right)' + t^b|y|^\lambda \operatorname{sgn} y = 0, \quad t \ge 1. \tag{3.11}$$

By Theorem 3.1, (3.11) is of the strong nonlinear limit-circle type if and only if $b > (\lambda+1)/\lambda$, and it is of the strong nonlinear limit-point type if and only if $(\lambda+1)/\lambda \ge b > -(\lambda+1)$.

A concluding note about the covering hypothesis (1.3) seems appropriate. It should not be a surprise that an integral condition like this one must hold when discussing the limit-point/limit-circle behavior of solutions. In fact, if we let $p = 1$, $\lambda = 1$, and $a(t) \equiv 1$, then

$$\int_0^\infty |g'(\sigma)|\, d\sigma = \int_0^\infty \left| \frac{r''(\sigma)}{r^{3/2}(\sigma)} - \frac{3}{2}\frac{(r'(\sigma))^2}{r^{5/2}(\sigma)} \right| d\sigma < \infty,$$

which is essentially the well-known condition of Dunford and Schwartz [9, p. 1414]

$$\int_0^\infty \left| \frac{r''(\sigma)}{r^{3/2}(\sigma)} - \frac{5}{4}\frac{(r'(\sigma))^2}{r^{5/2}(\sigma)} \right| d\sigma < \infty$$

for second order linear equations. For a discussion of the relationship between the linear and nonlinear limit-point/limit-circle properties, we refer the reader to the monograph of Bartušek, Došlá, and Graef [3].

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