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COMPLETE AND SINGLE-POINT BLOW-UP OF THE SOLUTION FOR A DEGENERATE SEMILINEAR PARABOLIC PROBLEM WITH MIXED BOUNDARY CONDITIONS

C. Y. CHAN AND N. E. DYAKEVICH

Department of Mathematics, University of Louisiana at Lafayette,
Lafayette, LA 70504-1010 USA
E-mail: chan@louisiana.edu
Department of Mathematics, California State University San Bernardino, 5500
University Parkway, San Bernardino, CA 92407-2397 USA
E-mail: dyakevic@csusb.edu

ABSTRACT. Let a, σ and q be constants with $a > 0$, $0 < \sigma \leq \infty$, and $q \geq 0$. This article studies the following degenerate semilinear parabolic initial-boundary value problem,

$$\begin{aligned}&\xi^q u_\tau - u_{\xi\xi} = f(u) \text{ for } 0 < \xi < a,\, 0 < \tau < \sigma,\\ &u(\xi, 0) = u_0(\xi) \geq 0 \text{ for } 0 \leq \xi \leq a,\\ &u(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } \tau > 0.\end{aligned}$$

We assume that $f \in C^2([0, \infty))$, $f(0) \geq 0$, $f'(u) > 0$ for $u > 0$, $f''(u) \geq 0$, $(s/f(s))' \leq 0$, and $\int_{k_1}^{\infty} f^{-1}(s)\, ds < \infty$, where k_1 is any positive constant. The function $u_0(\xi) \in C^{2+\alpha}([0, a])$ for some constant $\alpha \in (0, 1)$ is positive for $\xi > 0$ such that $u_0(0) = 0$ and $u_0'(a) = 0$. Existence of a unique classical solution u is shown. A criterion for u to blow up in a finite time τ_b, and an upper bound for τ_b are given. Using a lower solution and an upper solution, we investigate conditions on $u_0(\xi)$, q and $f(u)$ such that either u blows up completely or the blow-up occurs only at the point $x = a$.

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1. INTRODUCTION

Let a, σ and q be constants with $a > 0$, $0 < \sigma \leq \infty$, and $q \geq 0$. We consider the following degenerate semilinear parabolic initial-boundary value problem,

$$\begin{aligned}&\xi^q u_\tau - u_{\xi\xi} = f(u) \text{ for } 0 < \xi < a,\, 0 < \tau < \sigma,\\ &u(\xi, 0) = u_0(\xi) \geq 0 \text{ for } 0 \leq \xi \leq a,\\ &u(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } \tau > 0.\end{aligned}$$

Let $\xi = ax$, $\tau = a^{q+2}t$, $D = (0, 1)$, $\Omega = D \times (0, T)$, $\bar{D}$ and $\bar{\Omega}$ be the closures of D and Ω respectively, and $Lu = x^q u_t - u_{xx}$. In the sequel, let k_i $(i = 1, 2, 3, \ldots, 9)$ denote

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positive constants. The above problem is transformed into

$$(1.1)\qquad \begin{cases} Lu = a^2 f(u) \text{ in } \Omega, \\ u(x,0) = u_0(x) \text{ on } \bar{D}, \\ u(0,t) = 0 = u_x(1,t),\ 0 < t < T, \end{cases}$$

where $T = \sigma/a^{q+2} \le \infty$. We assume that $f \in C^2([0,\infty))$, $f(0) \ge 0$, $f'(u) > 0$ for $u > 0$, $f''(u) \ge 0$, $\int_{k_1}^{\infty} f^{-1}(s)\,ds < \infty$, $(s/f(s))' \le 0$, $u_0(x) \in C^{2+\alpha}(\bar{D})$ for some constant $\alpha \in (0,1)$, $u_0(0) = 0$, $u_0'(1) = 0$, and $u_0(x) > 0$ for $x > 0$.

A solution u of the problem (1.1) is said to blow-up at the point $(\bar{x}, t_b)$ if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (\bar{x}, t_b)$. The blow-up of u is complete at t_b if at t_b, u blows up at every point $x \in \bar{D}$. If at t_b, u blows up at only one point $x \in \bar{D}$, then the blow-up is a single-point blow-up.

The blow-up of the solution for the degenerate semilinear parabolic equation $Lu = u^p$ subject to homogeneous first boundary conditions was studied by Floater [5] for the case $1 < p \le q+1$, and by Chan and Liu [2] for the case $p > q+1$. Chan and Yang [3] investigated the complete blow-up of u for the problem (1.1) with $u_x(1,t)$ and $f(u)$ replaced by $u(1,t)$ and $f(u(x_0,t))$ for some fixed $x_0 \in D$ respectively.

In Section 2, we show existence of a unique classical solution u of the problem (1.1). We investigate the conditions on $u_0(x)$ for u to blow up in a finite time t_b, and give an upper bound for t_b. In Section 3, we establish a criterion for the complete blow-up to occur when $1 < p \le q+1$. For the case $p - 1 > q \ge 0$, we give, in Section 4, a criterion for the single-point blow-up at $x = 1$.

2. EXISTENCE AND UNIQUENESS

Let $\rho(x)$ in $C^1[0,1]$ be an increasing function such that $\rho(x)$ is 0 for $x \le 0$ and 1 for $x \ge 1$. Also, let δ and t_0 be positive constants with $\delta < 1/2$, $D_\delta = (\delta, 1)$, $\omega_\delta = D_\delta \times (0, t_0)$, $\bar{D}_\delta$ and $\bar{\omega}_\delta$ be, respectively, the closures of D_δ and ω_δ,

$$\rho_\delta = \begin{cases} 0, & x \le \delta \\ \rho(\frac{x}{\delta} - 1), & \delta < x < 2\delta \\ 1, & x \ge 2\delta, \end{cases}$$

$$u_{0_\delta}(x) = \rho_\delta(x) u_0(x).$$

We note that

$$\frac{\partial u_{0_\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \le \delta \\ -\frac{x}{\delta^2}\rho'(\frac{x}{\delta} - 1) u_0(x), & \delta < x < 2\delta \\ 0, & x \ge 2\delta. \end{cases}$$

Since ρ is increasing, we have $\partial u_{0_\delta}(x)/\partial\delta \le 0$. It follows from $0 \le \rho_\delta \le 1$ that $u_{0_\delta}(x) \le u_0(x)$.

A proof analogous to that of Lemma 1 by Chan and Yuen [4] gives the following comparison result.

Lemma 2.1. *For any fixed* $\bar{t} \in (0, T)$*, and any bounded and nontrivial function* $b(x,t)$ *on* $\bar{D} \times [0, \bar{t}]$*, if*

$$(L-b)\,u \geq (L-b)\,v \text{ in } D \times (0, \bar{t}],$$
$$u_0(x) \geq v_0(x),\ x \in \bar{D},$$
$$u(0,t) \geq v(0,t),\ u_x(1,t) \geq v_x(1,t),\ t \in (0, \bar{t}],$$

then $u \geq v$ *on* $\bar{D} \times [0, \bar{t}]$.

Let $\omega = D \times (0, t_0)$ for some positive number t_0, and $\bar{\omega}$ be its closure. We modify the proof of Lemma 2 of Chan and Liu [2] to prove the following existence result.

Lemma 2.2. *There exists some positive constant* t_0 $(< T)$ *such that the problem (1.1) has a unique solution* $u \in C(\bar{\omega}) \cap C^{2,1}((0,1] \times [0, t_0])$.

Proof. We consider the problem,

$$\begin{cases} Lu_\delta = a^2 f(u_\delta) \text{ in } D_\delta \times (0, t_0]\,, \\ u_\delta(x,0) = u_{0_\delta}(x) \text{ on } \bar{D}_\delta, \\ u_\delta(\delta, t) = 0 = u_{\delta_x}(1,t) \text{ for } 0 < t \leq t_0. \end{cases} \tag{2.1}$$

Let us construct an upper solution $\mu(x,t)$ for all u_δ with $\mu(x,t) \in C^{2,1}(\bar{\omega})$ as follows: let

$$\theta_1(x) = \frac{x\,(2k_2 + 1 - x)}{2},$$

where k_2 is chosen such that $k_2 > 1/2$, and $u_0(x) \leq a^2\,(1 + f\,(0))\,\theta_1(x)$. We note that $\theta_1'(x) = k_2 + 1/2 - x \geq k_2 - 1/2 > 0$. Let ϵ be some positive constant in $(0, 1/2)$ such that $f(\theta_1(\epsilon)\,[a^2\,(1 + f\,(0))]) < 1 + f\,(0)$. Since f is continuous, there exists some t_1 such that the initial-value problem,

$$\tau'\,(t) = \frac{a^2 f\,(k_2 \tau)}{\epsilon^q \theta_1\,(\epsilon)}, \qquad \tau(0) = a^2\,(1 + f\,(0))\,, \tag{2.2}$$

has a unique solution for $0 \leq t \leq t_1$. Let us choose some constant t_0 in $(0, t_1]$ such that $f\,(\theta_1\,(\epsilon)\,\tau_1\,(t_0)) \leq 1 + f\,(0)$. Let $\mu(x,t) = \theta_1(x)\tau_1(t)$. Since $x^q \theta_1 \tau_1' \geq 0$, and $\theta_1''(x) = -1$, we obtain for any $x \in [0, \epsilon]$ and $t \in (0, t_0]$,

$$L\mu - a^2 f(\mu) \geq \tau_1 - a^2 f(\theta_1 \tau_1) \geq a^2\,(1 + f\,(0) - f(\theta_1\,(\epsilon)\,\tau_1\,(t_0))) \geq 0.$$

For $0 \leq x \leq 1$, $\theta_1\,(x) \leq k_2$. Since θ_1 and f are increasing functions, we have for $x \in (\epsilon, 1]$,

$$L\mu - a^2 f(\mu) \geq \epsilon^q \theta_1 \tau_1'\,(t) - a^2 f(\theta_1 \tau_1) \geq \epsilon^q \theta_1 \left(\tau_1'\,(t) - \frac{a^2 f\,(k_2 \tau_1)}{\epsilon^q \theta_1\,(\epsilon)} \right) = 0$$

by (2.2). From construction, $\mu(x,0) = a^2\,(1 + f\,(0))\,\theta_1(x) \geq u_0(x)$, $\mu(0,t) = 0$, and $\mu_x(1,t) > 0$. By Lemma 2.1, $\mu(x,t) \in C^{2,1}(\bar{\omega})$ is an upper solution.

We note that $x^{-q} \in C^{\alpha,\alpha/2}(\bar{\omega}_\delta)$, $|a^2 x^{-q} f(u_\delta)| \le a^2 \delta^{-q} f(u_\delta)$ for $(x,t,u_\delta) \in \bar{\omega}_\delta \times R$, and $u_{0_\delta}(x) \in C^{2+\alpha}\left(\bar{D}_\delta\right)$. Our boundary conditions are homogeneous, and 0 and μ are lower and upper solutions. By Lemma 2.1, $0 \le u_\delta \le \mu$. Thus, a proof analogous to that of Theorem 4.2.2 of Ladde, Lakshmikantham and Vatsala [6, p. 143] shows that the problem (2.1) has a solution $u_\delta \in C^{2+\alpha,1+\alpha/2}(\bar{\omega}_\delta)$. Since $\partial u_{0_\delta}(x)/\partial\delta \le 0$, we have $u_{\delta_1} \ge u_{\delta_2}$ in $\bar{\Omega}_{\delta_2}$ if $\delta_1 \le \delta_2$. Therefore, $\lim_{\delta\to 0} u_\delta$ exists for all $(x,t) \in \bar{\omega}$. Let $u(x,t) = \lim_{\delta\to 0} u_\delta(x,t)$. Using the singular index 3 (cf. Ladyženskaja, Solonnikov and Ural'ceva [7, p. 351]), a proof similar to that in the proof of Lemma 2 of Chan and Liu [2] shows that $u(x,t) \in C(\bar{\omega}) \cap C^{2,1}((0,1]\times[0,t_0])$ is a solution of the problem (1.1).

By Lemma 2.1, $u(x,t)$ is unique. □

Let T be the supremum over t_0 for which the problem (1.1) has a unique solution $u(x,t) \in C(\bar{\omega}) \cap C^{2,1}((0,1]\times[0,t_0])$. Then, the problem (1.1) has a unique solution $u(x,t) \in C(\bar{D}\times[0,T)) \cap C^{2,1}((0,1]\times[0,T))$. The proof of the following result is a modification of that of Theorem 2.5 by Floater [5].

Theorem 2.3. *If $T < \infty$, then u is unbounded in Ω.*

Proof. Let us suppose that u is bounded above by some positive constant M in Ω. We would like to show that u can be continued into a time interval $[0, T+\tilde{t}_0]$ for some positive $\tilde{t}_0$. Let

$$K = \max\left\{a^2 f(M),\, 1+f(0),\, a^2(1+f(0))\right\},$$

$$\tilde{\theta}_1(x) = \frac{K}{2} x\,(2k_2+1-x)\,.$$

Then in Ω,

$$L\left(\tilde{\theta}_1 - u\right) = K - a^2 f(u) \ge 0.$$

Also, $\tilde{\theta}_1(x) \ge u_0(x)$, $\tilde{\theta}_1(0) = u(0,t)$, $\tilde{\theta}_1' > 0$ for $x \in \bar{D}$, and $\tilde{\theta}_1'(1) = K(k_2 - 1/2) > 0 = u_x(1,t)$ for $t > 0$. By Lemma 2.1, $\tilde{\theta}_1(x) \ge u(x,t)$ for any $t \le T$. With $\tilde{\theta}_1(x)$ as the initial function at T, we are to construct an upper solution $\tilde{\mu}(x,t)$ of $u(x,t)$ on $\bar{D}\times[T, T+\tilde{t}_0]$ for some positive $\tilde{t}_0$. Let $\hat{\epsilon} \in (0,1/2)$ be some fixed positive constant such that $f(a^2\tilde{\theta}_1(\hat{\epsilon})) < 1 + f(0) \le K$. There exists some t_2 such that the initial-value problem,

$$\tau' = \frac{a^2 f(Kk_2\tau(t-T))}{\hat{\epsilon}^q \tilde{\theta}_1(\hat{\epsilon})},\ \tau(T-T) = a^2,$$

has a unique positive solution $\tilde{\tau}_1(t-T)$ for $T \le t \le T+t_2$. Let $\tilde{\mu}(x,t) = \tilde{\theta}_1(x)\tilde{\tau}_1(t-T)$, and $\tilde{t}_0$ be chosen such that $0 < \tilde{t}_0 \le t_2$ and

$$f\left(\tilde{\theta}_1(\hat{\epsilon})\,\tilde{\tau}_1\left(\tilde{t}_0\right)\right) \le 1 + f(0) \le K.$$

Since $x^q\tilde{\theta}_1\tilde{\tau}_1'(t) \ge 0$, and $\tilde{\theta}_1''(x) = -K$, we obtain for any $x \in (0,\hat{\epsilon}]$ and $t \in \left[T, T+\tilde{t}_0\right]$,

$$L\tilde{\mu} - a^2 f(\tilde{\mu}) \ge K\tilde{\tau}_1 - a^2 f(\tilde{\theta}_1\tilde{\tau}_1) \ge a^2\left(K - f(\tilde{\theta}_1(\hat{\epsilon})\,\tilde{\tau}_1(\tilde{t}_0))\right) \ge 0.$$

It follows from $\tilde{\theta}_1''(x) = -K$, $\tilde{\tau}_1(t-T) \geq a^2$ for $t \in [T, T+\tilde{t}_0]$, and $\tilde{\theta}_1(x) \leq Kk_2$ that for $x \in (\hat{\epsilon}, 1]$ and $t \in [T, T+\tilde{t}_0]$,

$$\begin{aligned} L\tilde{\mu} - a^2 f(\tilde{\mu}) &\geq \hat{\epsilon}^q \tilde{\theta}_1 \tilde{\tau}_1'(t-T) - a^2 f(\tilde{\theta}_1 \tilde{\tau}_1(t-T)) \\ &\geq \hat{\epsilon}^q \tilde{\theta}_1 \left(\tilde{\tau}_1'(t-T) - \frac{a^2 f(Kk_2 \tilde{\tau}_1(t-T))}{\hat{\epsilon}^q \tilde{\theta}_1(\hat{\epsilon})} \right) \\ &= 0. \end{aligned}$$

By Lemma 2.1, $\tilde{\mu}(x,t)$ is an upper solution of u on $\bar{D} \times [T, T+\tilde{t}_0]$. As in Lemma 2.2, we can show that the problem (1.1) has a unique solution $u(x,t) \in C(\bar{D} \times [0, T+\tilde{t}_0]) \cap C^{2,1}((0,1] \times [0, T+\tilde{t}_0])$. This contradicts the definition of T, and hence the theorem is proved. □

From Chan and Liu [2], the general solution of the Sturm-Liouville problem,

$$\varphi'' + \lambda x^q \varphi = 0,\ \varphi(0) = 0,\ \varphi'(1) = 0, \tag{2.3}$$

is given by

$$\varphi(x) = A\sqrt{x} J_{\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda}}{q+2} x^{\frac{q+2}{2}} \right) + B\sqrt{x} J_{-\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda}}{q+2} x^{\frac{q+2}{2}} \right),$$

where A and B are arbitrary constants, and $J_{1/(q+2)}$ and $J_{-1/(q+2)}$ denote Bessel functions of the first kind of order $1/(q+2)$ and $-1/(q+2)$ respectively. From McLachlan [8, p. 197],

$$J_\nu(z) = \sum_{r=0}^{\infty} (-1)^r \frac{z^{\nu+2r}}{2^{\nu+2r} r! \Gamma(\nu+r+1)},$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$. From $\varphi(0) = 0$, we obtain $B = 0$, which gives

$$\varphi(x) = A\sqrt{x} J_{\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda}}{q+2} x^{\frac{q+2}{2}} \right).$$

We have

$$\varphi'(x) = A\sqrt{\lambda} x^{\frac{q+1}{2}} J_{-1+\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda}}{q+2} x^{\frac{q+2}{2}} \right).$$

From Watson [9, p. 479], the zeros $2\sqrt{\lambda_i}/(q+2)$ $(i = 1, 2, 3, \cdots)$ of

$$J_{-1+\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda}}{q+2} \right)$$

are positive. Let

$$\varphi_i(x) = \frac{(q+2)^{1/2} x^{1/2} J_{\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda_i}}{q+2} x^{\frac{q+2}{2}} \right)}{\left| J_{1+\frac{1}{q+2}} \left(\frac{2\sqrt{\lambda_i}}{q+2} \right) \right|}.$$

Then, $\{\varphi_i(x)\}$ forms an orthonormal set with the weight function x^q (cf. Chan and Chan [1]).

Let

$$E(t) = \int_D x^q \varphi_1(x) u(x,t)\, dx,$$

where $\varphi_1(x)$ denotes the eigenfunction corresponding to the fundamental eigenvalue λ_1.

Theorem 2.4. *If*

$$\lambda_1 E(0) < a^2 f(E(0)), \tag{2.4}$$

then there exists some $t_b < \infty$ *such that*

$$\lim_{t \to t_b^-} \max_{x \in \bar{D}} u(x,t) = \infty.$$

Furthermore,

$$t_b \le \frac{f(E(0))}{a^2 f(E(0)) - \lambda_1 E(0)} \int_{E(0)}^{\infty} \frac{d\eta}{f(\eta)} < \infty. \tag{2.5}$$

Proof. Multiplying the differential equation $Lu = a^2 f(u)$ by φ_1 and integrating over x from 0 to 1, we obtain

$$E'(t) = \int_D u_{xx} \varphi_1 dx + \int_D a^2 f(u) \varphi_1 dx.$$

Using (2.3), integration by parts, and Jensen's inequality for convex functions, we have

$$\begin{aligned} E'(t) &= -\lambda_1 E(t) + a^2 \int_D f(u) \varphi_1 dx \\ &\ge -\lambda_1 E(t) + a^2 \int_D x^q f(u) \varphi_1 dx \\ &\ge f(E(t)) \left(a^2 - \frac{\lambda_1 E(t)}{f(E(t))} \right). \end{aligned}$$

From $(s/f(s))' \le 0$ and (2.4), we have

$$E' \ge f(E) \left(a^2 - \frac{\lambda_1 E(0)}{f(E(0))} \right) > 0. \tag{2.6}$$

It follows from $E(0) > 0$ that the function $E(t)$ cannot be bounded for all t. Therefore, there exists some $t_b\,(<\infty)$ such that $E(t) \to \infty$ as $t \to t_b^-$. Using the Schwarz inequality, we have

$$E(t) \le \left(\max_{x \in \bar{D}} u(x,t) \right) \left(\int_D x^q \varphi_1^2(x) dx \right)^{1/2} \left(\int_D x^q dx \right)^{1/2} \le \left(\max_{x \in \bar{D}} u(x,t) \right).$$

Hence, u blows up.

Integrating (2.6), we obtain (2.5). □

3. COMPLETE BLOW-UP

It follows from Theorem 2.4 that if the initial data are sufficiently large, then the solution u of the problem (1.1) blows up in a finite time t_b. In the sequel, we assume that the blow-up time t_b is a fixed given number corresponding to a given initial function $u_0(x)$. We would like to construct a lower solution $\psi(x,t) \in C^{2,1}(\bar{D} \times [0,t_b))$ in the form $\theta_2(x)\tau_2(t)\eta(t)$ that blows up completely over $(0,1]$ at t_b.

Theorem 3.1. *Let $1 < p \le q+1$, and*

$$f(u) = u^p \tilde{f}(u), \tag{3.1}$$

where $a^2 \min\limits_{0 \le u < \infty} \tilde{f}(u) = k_3 > 0$. If u blows up, then u blows up completely, provided

$$u_0(x) \ge e^{(3q+5)(q+2)t_b + \frac{1}{p-1}} \left[\frac{2}{k_3 t_b (p-1)}\right]^{\frac{1}{p-1}} x e^{x^{q+1}(1-x)}. \tag{3.2}$$

Proof. Let $\gamma(x,t) = \theta_2(x)\tau_2(t)$, where $\theta_2(x) = xe^{x^{q+1}(1-x)}$, and

$$\tau_2'(t) = -[(3q+5)(q+2)+k_4]\tau_2,\ \tau_2(0) = e^{[(3q+5)(q+2)+k_4]t_b} > 0 \tag{3.3}$$

with k_4 to be chosen later appropriately. We note that

$$\tau_2(t) = e^{[(3q+5)(q+2)+k_4](t_b-t)}$$

is a positive decreasing function for any $t \ge 0$. For any $x \in \bar{D}$ and $t \in [0,t_b]$,

$$\begin{aligned}
&L\gamma + k_4 x^q \gamma \\
&= x^{q+1} e^{x^{q+1}(1-x)} \tau_2' - e^{x^{q+1}(1-x)} \{(q+1)x^q + (q+1)^2 x^{2q+1} \\
&- 2(q+1)(q+2)x^{2q+2} - (q+2)x^{q+1} + (q+2)^2 x^{2q+3} \\
&+ (q+1)^2 x^q - (q+2)^2 x^{q+1}\}\tau_2 + k_4 x^{q+1} e^{x^{q+1}(1-x)} \tau_2 \\
&\le x^{q+1} e^{x^{q+1}(1-x)} \{\tau_2' + [(q+2)(3q+5)+k_4]\tau_2\} \\
&= 0.
\end{aligned}$$

We note that $\gamma(0,t) = \theta_2(0)\tau_2(t) = 0$, and $\gamma_x(1,t) = \theta_2'(1)\tau_2(t) = 0$. Let $\gamma(x,0)$ be denoted by $\gamma_0(x)$. From

$$\gamma(x,t) = xe^{x^{q+1}(1-x)} e^{[(3q+5)(q+2)+k_4](t_b-t)} > 0 \text{ for any } x \in (0,1],$$

we have $\gamma_0(0) = 0$, and $\gamma_0'(1) = 0$.

Let us construct a positive increasing function $\eta(t) \in C^1([0,t_b))$:

$$\eta(t) = \begin{cases} \eta_1(t) & \text{for } t \in \left[0, \frac{t_b}{2}\right], \\ \eta_2(t) & \text{for } t \in \left[\frac{t_b}{2}, t_b\right), \end{cases}$$

where

$$\eta_1' = k_4 \eta_1 \text{ for } 0 < t \le \frac{t_b}{2},\ \eta_1(0) = \eta_{1_0}, \tag{3.4}$$

$$\eta_2' = k_3 \eta_2^p \text{ for } \frac{t_b}{2} < t < t_b,\ \eta_2\left(\frac{t_b}{2}\right) = \eta_{2_0}. \tag{3.5}$$

Here, the constants k_4 and η_{1_0} are to be chosen such that $\eta(t)$ is continuously differentiable at $t = t_b/2$ while the constant η_{2_0} is to be chosen such that η_2 blows up at $t = t_b$.

For $t \in (0, t_b/2]$,

$$x^q \eta' - k_3 \gamma^{p-1} \eta^p - k_4 x^q \eta \le x^q (\eta' - k_4 \eta) = 0. \tag{3.6}$$

Since $q \ge p - 1 > 0$, it follows that for $t \in (t_b/2, t_b)$,

$$x^q \eta' - k_3 \gamma^{p-1} \eta^p - k_4 x^q \eta \le x^q \eta' - k_3 x^{p-1} \eta^p \le x^q (\eta' - k_3 \eta^p) = 0. \tag{3.7}$$

From (3.4) and (3.5),

$$\eta_1(t) = \eta_{1_0} e^{k_4 t} \text{ for } 0 \le t \le \frac{t_b}{2},$$

$$\eta_2(t) = \left[\frac{1}{\eta_{2_0}^{1-p} - k_3 (p-1)\left(t - \frac{t_b}{2}\right)}\right]^{\frac{1}{p-1}} \text{ for } \frac{t_b}{2} \le t < t_b.$$

To ensure that $\eta_2(t)$ blows up at $t = t_b$, we choose $\eta_{2_0}^{1-p} = k_3 (p-1) t_b/2$. Therefore,

$$\eta_2(t) = \left[\frac{1}{k_3 (p-1)(t_b - t)}\right]^{\frac{1}{p-1}} \text{ for } \frac{t_b}{2} \le t < t_b.$$

To ensure that $\eta(t) \in C^1[0, t_b)$, we need to choose η_{1_0} and k_4 such that

$$\eta_{1_0} e^{\frac{k_4 t_b}{2}} = \left[\frac{2}{k_3 (p-1) t_b}\right]^{\frac{1}{p-1}},$$

$$k_4 \eta_{1_0} e^{\frac{k_4 t_b}{2}} = k_3 \left[\frac{2}{k_3 (p-1) t_b}\right]^{\frac{p}{p-1}}.$$

Dividing the second equation by the first, we obtain

$$k_4 = \frac{2}{(p-1) t_b}.$$

Thus,

$$\eta_{1_0} = e^{-\frac{k_4 t_b}{2}} \left[\frac{2}{k_3 (p-1) t_b}\right]^{\frac{1}{p-1}} = e^{-\frac{1}{p-1}} \left[\frac{2}{k_3 (p-1) t_b}\right]^{\frac{1}{p-1}}.$$

Hence, $\eta(t) \in C^1[0, t_b)$.

Let $\psi(x,t) = \gamma(x,t)\eta(t)$. Using (3.6), (3.7) and (3.1), we have

$$\begin{aligned}
&L\psi - a^2 f(\psi) \\
&= x^q \gamma \eta' + x^q \gamma_t \eta - \gamma_{xx} \eta - a^2 \gamma^p \eta^p \tilde{f}(\gamma\eta) \\
&\le x^q \gamma \eta' + x^q \gamma_t \eta - \gamma_{xx} \eta - k_3 \gamma^p \eta^p \\
&\le \gamma \left(x^q \eta' - k_3 \gamma^{p-1} \eta^p - k_4 x^q \eta\right) + \eta \left(x^q \gamma_t - \gamma_{xx} + k_4 x^q \gamma\right) \\
&\le 0.
\end{aligned}$$

Also, $\psi(0,t)=\gamma(0,t)\eta(t)=0$, and $\psi_x(1,t)=\gamma_x(1,t)\eta(t)=0$. Finally, we would like to show that

$$\psi(x,0)\le u_0(x) \text{ for all } x\in\bar{D}.$$

From (3.2) and (3.3),

$$u_0(x)\ge e^{(3q+5)(q+2)t_b+\frac{1}{p-1}}\left[\frac{2}{k_3t_b(p-1)}\right]^{\frac{1}{p-1}}xe^{x^{q+1}(1-x)}=\gamma(x,0)\,\eta_{1_0}=\psi(x,0).$$

By Lemma 2.1, $\psi(x,t)$ is a lower solution of the problem (1.1). Since $\psi(x,t)$ blows up at t_b at all points of $(0,1]$, it follows that $u(x,t)$ blows up at $t=t_b$ at all points of $(0,1]$. For $x=0$, we can find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n)\to(0,t_b)$ and $\lim_{n\to\infty}u(x_n,t_n)\to\infty$. Thus, the blow-up set is $\bar{D}$. □

4. SINGLE-POINT BLOW-UP

Let k_5 be the smallest positive constant such that

$$u_0(x)\le k_5xe^{x^{q+2}(1-x)}. \tag{4.1}$$

Lemma 4.1. *Let $p-1>q\ge0$. If*

$$f(u)=u^p\bar{f}(u),$$

where

$$a^2\max_{0\le u<\infty}\bar{f}(u)=k_6\le\frac{2}{e^{p+4(q+3)^2t_b(p-1)}(p-1)\,t_bk_5^{p-1}}, \tag{4.2}$$

then there exists some positive constant k_7 such that

$$u(x,t)<\frac{k_7x}{(t_b-t)^{\frac{1}{p-1}}} \text{ for } x\in\bar{D} \text{ and } t\in\left[\frac{t_b}{2},t_b\right). \tag{4.3}$$

Proof. We would like to construct an upper solution $\Psi(x,t)$ of u in the form $\theta_3(x)\tau_3(t)\bar{\eta}(t)$. Let $\bar{\gamma}(x,t)=\theta_3(x)\tau_3(t)$, where $\theta_3(x)=xe^{x^{q+2}(1-x)}$, and

$$\tau_3'=4(q+3)^2\tau_3,\ \tau_3(0)=e^{-4(q+3)^2t_b}.$$

We have

$$\tau_3(t)=e^{-4(q+3)^2(t_b-t)}\le1.$$

For any $x\in\bar{D}$ and any $t\in[0,t_b)$,

$$\begin{aligned}L\bar{\gamma}&=x^{q+1}e^{x^{q+2}(1-x)}\tau_3'-e^{x^{q+2}(1-x)}\{(q+2)\,x^{q+1}\\&\quad+(q+2)^2x^{2q+3}-2(q+2)(q+3)\,x^{2q+4}-(q+3)\,x^{q+2}\\&\quad+(q+3)^2x^{2q+5}+(q+2)^2x^{q+1}-(q+3)^2x^{q+2}\}\tau_3\\&\ge e^{x^{q+2}(1-x)}\{x^{q+1}\tau_3'-[(q+2)\,x^{q+1}+(q+2)^2x^{2q+3}\\&\quad+(q+3)^2x^{2q+5}+(q+2)^2x^{q+1}]\tau_3\}\\&\ge x^{q+1}e^{x^{q+2}(1-x)}\{\tau_3'-[(q+2)+(q+2)^2+(q+3)^2\end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&+ (q+2)^2]\tau_3\} \\
&\geq x^{q+1} e^{x^{q+2}(1-x)} \left[\tau_3' - 4\left(q+3\right)^2 \tau_3\right] \\
&= 0.
\end{aligned}
$$

We note that $\bar{\gamma}(x,t)$ is positive for any fixed $x \in (0,1]$, $\bar{\gamma}(0,t) = \theta_3(0)\tau_3(t) = 0$, and $\bar{\gamma}_x(1,t) = \theta_3'(1)\tau_3(t) = 0$. Since $\bar{\gamma}(x,0) = e^{-4(q+3)^2 t_b} x e^{x^{q+2}(1-x)}$, it is 0 at $x=0$, and its derivative with respect to x at $x=1$ is 0.

Let $\bar{\eta}(t) \in C^1([0,t_b))$ be a positive increasing function given by

$$
\bar{\eta}(t) = \begin{cases} \bar{\eta}_1(t) & \text{for } t \in \left[0, \frac{t_b}{2}\right], \\ \bar{\eta}_2(t) & \text{for } t \in \left[\frac{t_b}{2}, t_b\right), \end{cases}
$$

where

$$
\bar{\eta}_1' = k_8 \bar{\eta}_1 \text{ for } 0 < t \leq \frac{t_b}{2},\ \bar{\eta}_1(0) = \bar{\eta}_{1_0}, \tag{4.5}
$$

$$
\bar{\eta}_2'(t) = k_6 e^{p-1} \bar{\eta}_2^p \text{ for } \frac{t_b}{2} < t < t_b,\ \bar{\eta}_2\left(\frac{t_b}{2}\right) = \bar{\eta}_{2_0}. \tag{4.6}
$$

The constants k_8 and $\bar{\eta}_{1_0}$ are to be chosen such that $\bar{\eta}(t)$ is continuously differentiable at $t = t_b/2$ while the constant $\bar{\eta}_{2_0}$ is to be chosen in such a way that $\bar{\eta}_2(t)$ blows up at $t = t_b$. From (4.5) and (4.6),

$$
\bar{\eta}_1(t) = \bar{\eta}_{1_0} e^{k_8 t} \text{ for } t \in \left[0, \frac{t_b}{2}\right],
$$

$$
\bar{\eta}_2(t) = \left[\frac{1}{\bar{\eta}_{2_0}^{1-p} - k_6 e^{p-1}(p-1)\left(t - \frac{t_b}{2}\right)}\right]^{\frac{1}{p-1}} \text{ for } t \in \left[\frac{t_b}{2}, t_b\right).
$$

To ensure that $\bar{\eta}_2(t)$ blows up at $t = t_b$, we choose

$$
\bar{\eta}_{2_0}^{1-p} = \frac{k_6 e^{p-1}(p-1) t_b}{2}.
$$

Therefore,

$$
\bar{\eta}_2(t) = \left[\frac{1}{k_6 e^{p-1}(p-1)(t_b - t)}\right]^{\frac{1}{p-1}} \text{ for } t \in \left[\frac{t_b}{2}, t_b\right).
$$

In order to ensure that $\bar{\eta}(t) \in C^1[0,t_b)$, we set $\bar{\eta}_1(t_b/2) = \bar{\eta}_2(t_b/2)$, and $\bar{\eta}_1'(t_b/2) = \bar{\eta}_2'(t_b/2)$. We have

$$
\bar{\eta}_{1_0} e^{\frac{k_8 t_b}{2}} = \left[\frac{2}{k_6 e^{p-1}(p-1) t_b}\right]^{\frac{1}{p-1}}, \tag{4.7}
$$

$$
k_8 \bar{\eta}_{1_0} e^{\frac{k_8 t_b}{2}} = k_6 e^{p-1} \left[\frac{2}{k_6 e^{p-1}(p-1) t_b}\right]^{\frac{p}{p-1}}.
$$

Dividing the second equation by the first, we obtain

$$
k_8 = k_6 e^{p-1} \left[\frac{2}{k_6 e^{p-1}(p-1) t_b}\right] = \frac{2}{(p-1) t_b}.
$$

This and (4.7) give

$$\bar{\eta}_{1_0} = e^{-\frac{1}{p-1}} \left[\frac{2}{k_6 e^{p-1} (p-1) t_b} \right]^{\frac{1}{p-1}}.$$

Thus, $\bar{\eta}(t) \in C^1 [0, t_b)$.

Since $q < p - 1$ and

$$\max_{t \in [0, t_b/2]} \bar{\eta}(t) = \left[\frac{2}{k_6 e^{p-1} (p-1) t_b} \right]^{\frac{1}{p-1}},$$

it follows that for any $t \in (0, t_b/2]$,

$$\begin{aligned} & x^q \bar{\eta}' - k_6 \bar{\gamma}^{p-1} \bar{\eta}^p \\ & \geq x^q \bar{\eta}' - k_6 x^{p-1} e^{x^{q+2}(1-x)(p-1)} e^{-4(p-1)(q+3)^2 (t_b - t)} \bar{\eta}^p \\ & \geq x^q \bar{\eta}' - k_6 e^{p-1} x^{p-1} \bar{\eta}^p \\ & \geq x^q \bar{\eta}' - k_6 e^{p-1} x^q \bar{\eta}^{p-1} \bar{\eta} \\ & \geq x^q \bar{\eta}' - \frac{2}{(p-1) t_b} x^q \bar{\eta} \\ & = x^q (\bar{\eta}' - k_8 \bar{\eta}) \\ & = 0, \end{aligned} \tag{4.8}$$

and for any $t \in [t_b/2, t_b)$,

$$\begin{aligned} & x^q \bar{\eta}' - k_6 \bar{\gamma}^{p-1} \bar{\eta}^p \\ & \geq x^q \bar{\eta}' - k_6 x^{p-1} e^{x^{q+2}(1-x)(p-1)} e^{-(p-1)4(q+3)^2 (t_b - t)} \bar{\eta}^p \\ & \geq x^q \bar{\eta}' - k_6 e^{p-1} x^{p-1} \bar{\eta}^p \\ & \geq x^q \left(\bar{\eta}' - k_6 e^{p-1} \bar{\eta}^p \right) \\ & = 0. \end{aligned} \tag{4.9}$$

For $\Psi(x,t) = \bar{\gamma}(x,t) \bar{\eta}(t)$, we have

$$\begin{aligned} & L\Psi - a^2 f(\Psi) \\ & = x^q \bar{\gamma} \bar{\eta}' + x^q \bar{\gamma}_t \bar{\eta} - \bar{\gamma}_{xx} \bar{\eta} - a^2 f(\bar{\gamma} \bar{\eta}) \\ & \geq x^q \bar{\gamma} \bar{\eta}' + x^q \bar{\gamma}_t \bar{\eta} - \bar{\gamma}_{xx} \bar{\eta} - k_6 \bar{\gamma}^p \bar{\eta}^p \\ & = \bar{\gamma} \left(x^q \bar{\eta}' - k_6 \bar{\gamma}^{p-1} \bar{\eta}^p \right) + (L\bar{\gamma}) \bar{\eta}. \end{aligned}$$

From (4.4), (4.8) and (4.9),

$$L\Psi - a^2 f(\Psi) \geq 0.$$

We note that $\Psi(0,t) = \bar{\gamma}(0,t)\bar{\eta}(t) = 0$, and $\Psi_x(1,t) = \bar{\gamma}_x(1,t)\bar{\eta}(t) = 0$. It follows from (4.1) and (4.2) that for $x \in \bar{D}$,

$$\begin{aligned}\Psi(x,0) &= xe^{x^{q+2}(1-x)}e^{-4(q+3)^2 t_b}e^{-\frac{1}{p-1}}\left[\frac{2}{k_6 e^{p-1}(p-1)\,t_b}\right]^{\frac{1}{p-1}}\\ &\geq k_5 x e^{x^{q+2}(1-x)}\\ &\geq u_0(x).\end{aligned}$$

Therefore, $\Psi(x,t)$ is an upper solution of the problem (1.1). Thus for $t \in [t_b/2, t_b)$,

$$\begin{aligned}u(x,t) &\leq xe^{x^{q+2}(1-x)}e^{-4(q+3)^2(t_b-t)}\left[\frac{1}{k_6 e^{p-1}(p-1)(t_b-t)}\right]^{\frac{1}{p-1}}\\ &\leq xe\left(\frac{1}{k_6 e^{p-1}(p-1)(t_b-t)}\right)^{\frac{1}{p-1}}\\ &= x\left[\frac{1}{k_6(p-1)(t_b-t)}\right]^{\frac{1}{p-1}}\\ &< \frac{k_7 x}{(t_b-t)^{\frac{1}{p-1}}}\end{aligned} \tag{4.10}$$

for some positive constant $k_7 > 1/[k_6(p-1)]^{1/(p-1)}$. Hence, (4.3) holds. □

Let k_9 be an arbitrary constant such that $0 < k_9 < 1$. Also, let us choose the constant β sufficiently large to satisfy the following conditions:

$$\begin{cases}\beta > \max\left\{\frac{q+2}{2}, p-q-1\right\},\\[2ex] 1 - k_9^{p-q-1} - \frac{4p^2 k_9^{2\beta-q-2}}{p-1} - \frac{2k_9^{\beta+p-q-1}}{\beta t_b} \geq 0.\end{cases} \tag{4.11}$$

Let us choose

$$k_7 = \left[\frac{1}{k_6(p-1)} + \frac{2k_9^\beta}{\beta k_6(p-1)\,t_b}\right]^{\frac{1}{p-1}}.$$

Lemma 4.2. *Under the hypotheses of Lemma 4.1,*

$$u(x,t_b) \leq \frac{k_7 x}{\left[\frac{1}{\beta}\left(k_9^\beta - x^\beta\right)\right]^{\frac{2}{p-1}}} < \infty \quad \textit{for any } x \in [0,k_9).$$

Proof. Let

$$\Phi(x,t) = \frac{k_7 x}{D^{1/(p-1)}},$$

where

$$D(x,t) = \left[\frac{1}{\beta}\left(k_9^\beta - x^\beta\right)\right]^2 + t_b - t.$$

Using (4.11), we obtain for any $x \in (0, k_9)$ and $t_b/2 < t \le t_b$,

$$
\begin{aligned}
& L\Phi - a^2\Phi^p \bar{f}(\Phi) \\
& \ge \frac{k_7}{(p-1)\,D^{\frac{p}{p-1}}}\left[x^{q+1} - \frac{2}{\beta}\left(k_9^\beta - x^\beta\right)x^{\beta-1}\right. \\
& \left. - \frac{4p}{\beta^2 (p-1)\,D}\left(k_9^\beta - x^\beta\right)^2 x^{2\beta-1} - 2\left(k_9^\beta - x^\beta\right)x^{\beta-1} - k_6 k_7^{p-1}(p-1)\,x^p\right] \\
& \ge \frac{k_7 x^{q+1}}{(p-1)\,D^{\frac{p}{p-1}}} \\
& \times \left[1 - 2k_9^\beta x^{\beta-q-2} - \frac{4p x^{2\beta-q-2}}{p-1} - 2k_9^\beta x^{\beta-q-2} - k_7^{p-1} k_6 (p-1)\, x^{p-q-1}\right] \\
& \ge \frac{k_7 x^{q+1}}{(p-1)\,D^{\frac{p}{p-1}}} \\
& \times \left[1 - 2pk_9^{2\beta-q-2} - \frac{4pk_9^{2\beta-q-2}}{p-1} - 2pk_9^{2\beta-q-2} - k_7^{p-1} k_6 (p-1)\, k_9^{p-q-1}\right] \\
& \ge \frac{k_7 x^{q+1}}{(p-1)\,D^{\frac{p}{p-1}}} \\
& \times \left\{1 - \frac{4p^2 k_9^{2\beta-q-2}}{p-1} - \left[\frac{1}{k_6 (p-1)} + \frac{2k_9^\beta}{\beta k_6 (p-1)\, t_b}\right] k_6 (p-1)\, k_9^{p-q-1}\right\} \\
& = \frac{k_7 x^{q+1}}{(p-1)\,D^{\frac{p}{p-1}}}\left(1 - k_9^{p-q-1} - \frac{4p^2 k_9^{2\beta-q-2}}{p-1} - \frac{2k_9^{\beta+p-q-1}}{\beta t_b}\right) \\
& \ge 0.
\end{aligned}
$$

It follows from (4.10), $\beta \ge 1$ and $0 < k_9 < 1$ that

$$
\begin{aligned}
\Phi\left(x, \frac{t_b}{2}\right) &= \frac{k_7 x}{\left\{\frac{t_b}{2} + \left[\frac{1}{\beta}\left(k_9^\beta - x^\beta\right)\right]^2\right\}^{\frac{1}{p-1}}} \ge \frac{k_7 x}{\left(\frac{t_b}{2} + \frac{k_9^{2\beta}}{\beta^2}\right)^{\frac{1}{p-1}}} \ge \frac{k_7 x}{\left(\frac{t_b}{2} + \frac{k_9^{\beta}}{\beta}\right)^{\frac{1}{p-1}}} \\
&= \frac{\left[\frac{1}{k_6(p-1)} + \frac{2k_9^\beta}{\beta k_6 (p-1) t_b}\right]^{\frac{1}{p-1}} x}{\left(\frac{t_b}{2} + \frac{k_9^\beta}{\beta}\right)^{\frac{1}{p-1}}} = \frac{\left[\frac{\beta t_b + 2k_9^\beta}{\beta k_6 (p-1) t_b}\right]^{\frac{1}{p-1}} x}{\left(\frac{\beta t_b + 2k_9^\beta}{2\beta}\right)^{\frac{1}{p-1}}} = \left[\frac{2}{k_6 (p-1)\, t_b}\right]^{\frac{1}{p-1}} x \\
&\ge u\left(x, \frac{t_b}{2}\right) \text{ on } [0, k_9].
\end{aligned}
$$

Since

$$
\Phi(0,t) = 0,\ \Phi(k_9, t) = \frac{k_7 k_9}{(t_b - t)^{\frac{1}{p-1}}},
$$

it follows from Lemma 1 of Chan and Yuen [4] that $\Phi(x,t)$ is an upper solution of the problem (1.1) for $0 \le x \le k_9$. The lemma is then proved. □

Since $k_9 \in (0,1)$, we have the following result.

Theorem 4.3. *Under the hypotheses of Lemma 4.1, if u blows up, then the blow-up set is $x = 1$.*

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