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OSCILLATION OF SECOND-ORDER NEUTRAL DELAY DYNAMIC EQUATIONS OF EMDEN-FOWLER TYPE

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ABSTRACT. In this paper, we will establish some new oscillation criteria for the second-order superlinear neutral delay dynamic equation of Emden-Fowler type

$$\left[a(t)(y(t)+r(t)y(\tau(t)))^{\Delta}\right]^{\Delta}+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0,$$

on a time scale $\mathbb{T}$; here $\gamma>1$, $a(t)$, $r(t)$, $\tau(t)$, $p(t)$ and $\delta(t)$real-valued positive functions defined on $\mathbb{T}$. Our results in the special case when $\mathbb{T}=\mathbb{R}$, improve the oscillation results for superlinear neutral delay differential equations and are essentially new on the other different types of time scales. We illustrate the main results by some examples. To the best of our knowledge nothing is known regarding the qualitative behavior of these equations on time scales, so this paper initiates the study of these equations.

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1. INTRODUCTION

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [18], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice - once for differential equations and once again for difference equations.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is

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equal to the reals or to the integers represent the classical theories of differential and of difference equations. Since we are interested in asymptotic behavior of solutions, we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus that has interesting applications on physics. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Many other interesting time scales exist, and they give rise to many applications (see [5]). A book on the subject of time scales, by Bohner and Peterson [5], summarizes and organizes much of time scale calculus, see also the book by Bohner and Peterson [6] for advances results of dynamic equations on time scales.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales, we refer the reader to the papers [2-4, 7-16, 19-24]. Recently, some authors have been interested in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of first and second-order linear and nonlinear neutral delay dynamic equations on time scales, we refer to the papers [3] and [19].

In [19], Mathsen et. al. considered the first-order neutral delay dynamic equation

$$[y(t) - r(t)y(\tau(t))]^{\Delta} + p(t)y(\delta(t)) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

and established some new oscillation criteria which as a special case involve some well-known oscillation results for first-order neutral delay differential equations.

In [3], Agarwal et. al. considered the second-order nonlinear neutral delay dynamic equation

$$(r(t)((y(t) + p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + f(t, y(\delta(t))) = 0, \tag{1.2}$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is a quotient of odd positive integers, and the delay functions $\tau(t)$ and $\delta(t)$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $r(t)$ and $p(t)$ are real valued positive functions defined on $\mathbb{T}$, and

(h_1). $r(t) > 0, \int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty$ and $0 \leq p(t) < 1$,

(h_2). $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a nonnegative function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \geq q(t) |u^{\gamma}|$, and established some oscillation criteria for all solutions of Eq.(1.2).

In this paper, we are concerned with oscillation of the second-order neutral delay dynamic equation of Emden-Fowler type

$$\left[a(t)(y(t)+r(t)y(\tau(t)))^{\Delta}\right]^{\Delta}+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0, \tag{1.3}$$

on a time scale $\mathbb{T}$.

Our aim in this paper is motivated by the question posed in [19]: What can be said about higher-order neutral dynamic equations on time scales and the various generalizations?

Equation (1.3) is the prototype of a wide class of nonlinear dynamic equations called Emden-Fowler superlinear dynamic equations. It is interesting to study Eq.(1.3) because the continuous version, i.e., when t is continuous variable, has several physical applications, see, e.g. [25] and when t is a discrete variable it becomes the difference equation of Emden-Fowler type and also is important in application.

Recall a solution $y(t)$ of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.3) which exist on some half line $[t_1,\infty)$ and satisfy $\sup\{|y(t)|: t>t_2\}>0$ for any $t_2\geq t_1$.

Throughout this paper we assume that:

(H_1). $\gamma>1$;

(H_2). *The delay functions* $\tau(t):\mathbb{T}\to\mathbb{T}$, $\delta(t):\mathbb{T}\to\mathbb{T}$, $\tau(t)\leq t$, $\delta(t)\leq t$ *for all* $t\in\mathbb{T}$, *and* $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$.

(H_3). $a(t)$, $r(t)$ *and* $p(t)$ *are rd-positive continuous functions defined on* $\mathbb{T}$ *such that* $a^{\Delta}(t)\geq 0$, $\int_{t_0}^{\infty}\frac{1}{a(t)}\Delta t=\infty$ and $0\leq r(t)<1$.

We note that in the special case when $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=0$, $\mu(t)=0$, $f^{\Delta}(t)=f'(t)$ and (1.3) becomes the second-order superlinear neutral delay differential equation

$$[a(t)(y(t)+r(t)y(\tau(t)))']'+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0,\quad t\in[t_0,\infty), \tag{1.4}$$

that has been studied in [1]. Unfortunately, most of the oscillation criteria in [1] are unsatisfactory since additional assumptions have to be imposed on the unknown solutions. Also, the author proved that if

$$\int_{t_0}^{\infty}\frac{1}{a(t)}dt=\int_{t_0}^{\infty}p(t)dt=\infty,$$

then every solution of (1.4) oscillates for every $r(t)>0$. But one can easily see that this result cannot be applied when $p(t)=t^{-\alpha}$ for $\alpha>0$.

In the case when $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$, $\mu(t)=1$,

$$y^{\Delta}(t)=\Delta y(t)=y(t+1)-y(t),$$

and (1.3) becomes the second-order superlinear neutral delay difference equation

$$\Delta[a(t)\Delta(y(t)+r(t)y(\tau(t)))]+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0,\ t\in[t_0,\infty). \tag{1.5}$$

In the case when $\mathbb{T}=h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h,\ \mu(t)=h$,

$$y^{\Delta}(t)=\Delta_h y(t)=\frac{y(t+h)-y(t)}{h},$$

and (1.3) becomes the second-order superlinear neutral delay difference equation with constant step size

$$\Delta_h[a(t)\Delta_h(y(t)+r(t)y(\tau(t)))]+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0,\ \ t\in[t_0,\infty). \tag{1.6}$$

In the case when $\mathbb{T}=q^{\mathbb{N}}=\{t:t=q^k,\,k\in\mathbb{N},\,q>1\}$, we have $\sigma(t)=q\,t$, $\mu(t)=(q-1)t$,

$$x^{\Delta}(t)=\Delta_q x(t)=\frac{x(q\,t)-x(t)}{(q-1)\,t},$$

and (1.3) becomes the second-order superlinear $q-$neutral delay difference equation

$$\Delta_q[a(t)\Delta_q(y(t)+r(t)y(\tau(t)))]+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0. \tag{1.7}$$

In the case when $\mathbb{T}=\mathbb{N}_0^2=\{t^2:t\in\mathbb{N}_0\}$, we have $\sigma(t)=(\sqrt{t}+1)^2$ and $\mu(t)=1+2\sqrt{t}$,

$$\Delta_N y(t)=\frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}},\ \text{ for }\ t\in[t_0^2,\infty),$$

and (1.3) becomes the second-order superlinear neutral delay difference equation

$$\Delta_N(a(t)\Delta_N(y(t)+r(t)y(\tau(t))))+p(t)\left|y(\delta(t))\right|^{\gamma} signy(\delta(t))=0. \tag{1.8}$$

In the case when $\mathbb{T}=\mathbb{T}_n=\{H_n:n\in\mathbb{N}_0\}$ where H_n are the so-called harmonic numbers defined by

$$H_0=0,\quad H_n=\sum_{k=1}^{n}\frac{1}{k},\quad n\in\mathbb{N}_0,$$

we have $\mu(H_n)=\frac{1}{n+1}$,

$$y^{\Delta}(H_n)=\Delta_n(y(H_n))=(n+1)\Delta y(H_n),\ \text{ for }H_n\in[0,\infty),$$

and (1.3) becomes the second-order superlinear neutral difference equation

$$\Delta_n(a(H_n)\Delta_n(y(H_n)+r(H_n)y(\tau(H_n))))+p(H_n)\left|y(\delta(H_n))\right|^{\gamma} signy(\delta(H_n))=0. \tag{1.9}$$

The paper is organized as follows: In the next Section, we present some basic definitions concerning the calculus on time scales. In Section 3, by developing the Riccati transformation technique, we establish some new sufficient conditions for oscillation of Eq.(1.3). Our results in the special case when $\mathbb{T}=\mathbb{R}$, improve the oscillation results established in [1] for second-order neutral delay differential equation (1.4), and are essentially new for equations (1.5)-(1.9). Also, our results includes as a special some well-known oscillation results for second order dynamic equations without delay. Some examples are considered in Section 4 to illustrate our main results.

2. SOME PRELIMINARIES ON TIME SCALES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T},\ s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left–dense if $\rho(t) = t$, right–dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \Re^+$) if it is rd-continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{2.2}$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \tag{2.3}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f^\sigma = f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{2.4}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma = g(\sigma(t))$) of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f g^\Delta + f^\Delta g^\sigma, \tag{2.5}$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}. \tag{2.6}$$

For a, $b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t) \Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t) g^\Delta(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t) g^\sigma \Delta t, \tag{2.7}$$

and infinite integral is defined as

$$\int_a^\infty f(t) \Delta t = \lim_{b \to \infty} \int_a^b f(t) \Delta t.$$

3. MAIN RESULTS

In this Section, by using the Riccati transformation technique and the formula

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_{0}^{1} [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dhx^{\Delta}(t),$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90], we will establish some new sufficient conditions for oscillation of Eq.(1.3). Throughout this Section, these assumptions will be supposed to hold. Let $T_0 = \min_{t\in\mathbb{T}}\{\tau(t),\, \delta(t) : t \geq t_0\}$ and let $\delta_{-1}(t)$ and $\tau_{-1}(t)$ are the inverse functions of τ and δ. Clearly $\delta_{-1}(t)$ and $\tau_{-1}(t) \geq t$ for $t \geq T_0$, $\tau_{-1}(t)$ is nondecreasing and coincides with the inverses of τ and δ when the inverses exist. We define the functions $Q(t)$ and $Q_1(t)$ by

$$Q(t) := p(t)(1 - r(\delta(t)))^{\gamma}, \text{ and } Q_1(t) := Q(t)\left(\frac{\delta(t)}{t}\right)^{\gamma}.$$

In what follows it will be assumed that

$$\int_{t_0}^{\infty} Q(t)(\delta(t))^{\gamma}\Delta t = \infty, \tag{3.1}$$

is fulfilled.

Theorem 3.1. *Assume that $(H_1)-$ (H_3) hold. Furthermore, assume that there exists a positive $\Delta-$differentiable function $\eta(t)$ such that for all constants $M > 0$*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\eta(s)Q_1(s) - \frac{a(s)\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s = \infty. \tag{3.2}$$

Then every solution of equation (1.3) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that y is an eventually positive solution of (1.3) with $y(t)$, $y(\tau(t))$ and $y(\delta(t)) > 0$ for all $t > t_1$ sufficiently large. Set

$$x(t) = y(t) + r(t)y(\tau(t)). \tag{3.3}$$

In view of (1.3) and (3.3), we have

$$(a(t)\left(x^{\Delta}(t)\right))^{\Delta} = -p(t)y^{\gamma}(\delta(t)) < 0, \quad for\ t \geq t_1. \tag{3.4}$$

Since $p(t)$ is a positive function, then $a(t)x^{\Delta}(t)$ is an eventually decreasing function and it is either eventually positive or eventually negative. Suppose that there exists $t_2 \geq t_1$ such that $a(t_2)x^{\Delta}(t_2) = c < 0$. Then from (3.4), we have $a(t)x^{\Delta}(t) < a(t_2)x^{\Delta}(t_2) = c$ for $t \geq t_2$, and so

$$x^{\Delta}(t) \leq c\frac{1}{a(t)},$$

which implies by (H_3) that

$$x(t) \leq x(t_1) + c\int_{t_2}^{t} \frac{1}{a(s)}\Delta s \to -\infty \quad \text{as} \quad t \to \infty.$$

This contradicts the fact that $x(t) > 0$ for all $t \geq t_1$, and hence $a(t)x^\Delta(t)$ is eventually nonnegative. Therefore, there is some $t_2 > t_1$ such that

$$x(t) > 0, \;\; x^\Delta(t) \geq 0, \;\; (a(t)x^\Delta(t))^\Delta < 0, \;\; t \geq t_2. \tag{3.5}$$

This implies for $t \geq \tau_{-1}(t_2)$, that

$$\begin{aligned} y(t) &= x(t) - r(t)y(\tau(t)) = x(t) - r(t)[x(\tau(t)) - r(\tau(t))y(\tau(\tau(t)))] \\ &\geq x(t) - r(t)x(\tau(t)) \geq (1 - r(t))x(t). \end{aligned}$$

Then for $t \geq t_3 = \delta_{-1}(\tau_{-1}(t_2))$, we have

$$y(\delta(t)) \geq (1 - r(\delta(t))x(\delta(t)).$$

From (3.4) and the last inequality, we obtain

$$(a(t)x^\Delta(t))^\Delta + Q(t)x^\gamma(\delta(t)) \leq 0, \quad t \geq t_3. \tag{3.6}$$

Now, we define the function $w(t)$ by the *Riccati substitution*

$$w(t) := \eta(t)\frac{a(t)x^\Delta(t)}{x^\gamma(t)}, \;\; for \; t \geq t_3. \tag{3.7}$$

Then $w(t) > 0$, and using (2.5) and (2.6) yield that

$$\begin{aligned} w^\Delta(t) = \left(ax^\Delta\right)^\sigma \left[\frac{\eta(t)}{x^\gamma(t)}\right]^\Delta + \frac{\eta(t)}{x^\gamma(t)}\left(a(t)x^\Delta(t)\right)^\Delta = \frac{\eta(t)}{x^\gamma(t)}\left(a(t)x^\Delta(t)\right)^\Delta \\ + \left(ax^\Delta\right)^\sigma \left[\frac{x^\gamma(t)\eta^\Delta(t) - \eta(t)(x^\gamma(t))^\Delta}{x^\gamma(t)(x^\sigma)^\gamma}\right]. \end{aligned} \tag{3.8}$$

In view of (3.6) and (3.8), we have

$$w^\Delta(t) \leq -\eta(t)Q(t)\left(\frac{x(\delta(t))}{x(t)}\right)^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma}w^\sigma - \frac{\eta(t)\left(ax^\Delta\right)^\sigma (x^\gamma(t))^\Delta}{x^\gamma(t)(x^\sigma)^\gamma}. \tag{3.9}$$

Now, since $\gamma > 1$, the chain rule and (3.5) imply that

$$\begin{aligned} (x^\gamma(t))^\Delta &= \gamma\int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1}\, dh x^\Delta(t) \\ &\geq \gamma\int_0^1 [hx(t) + (1-h)x]^{\gamma-1}\, dh x^\Delta(t) = \gamma\int_0^1 [x(t)]^{\gamma-1}\, dh x^\Delta(t) \\ &= \gamma\,[x(t)]^{\gamma-1}\, x^\Delta(t) \geq \gamma\,[x(t_0)]^{\gamma-1}\, x^\Delta(t) = \gamma M^{\gamma-1}x^\Delta(t), \end{aligned} \tag{3.10}$$

where we put $M = x(t_0) > 0$. Also, from (3.5) since $a^\Delta(t) \geq 0$ we can easily verify that $x^{\Delta\Delta}(t) \leq 0$ for $t \geq t_3$.

Now, we claim that $x(t)/t$ is nonincreasing. Define $X(t) := x(t) - tx^{\Delta}(t)$. Since $X^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t \geq t_3$, we have either $x(t) - tx^{\Delta}(t) \geq 0$ or $x(t) - tx^{\Delta}(t) < 0$. To prove that $x(t) \geq tx^{\Delta}(t)$ it suffices to prove that the latter case is impossible. Indeed, otherwise

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} > 0, \quad \text{for } t \geq t_3,$$

whence $x(t) \geq dt$ with some $d > 0$, and then $x(\delta(t)) \geq d(\delta(t))$ for $t \geq t_3$. Integrating (3.6) from t_3 to t, we have

$$a(t)x^{\Delta}(t) - a(t_3)x^{\Delta}(t_3) + \int_{t_3}^{t} Q(s)x^{\gamma}(\delta(s))\Delta s \leq 0.$$

This implies that

$$\begin{aligned} a(t_3)x^{\Delta}(t_3) &\geq a(t)x^{\Delta}(t) + \int_{t_3}^{t} Q(s)x^{\gamma}(\delta(s))\Delta s \\ &\geq \int_{t_3}^{t} Q(s)x^{\gamma}(\delta(s))\Delta s \geq d^{\gamma}\int_{t_3}^{t} Q(s)(\delta(s))^{\gamma}\Delta s, \end{aligned}$$

which is a contradiction with (3.1). Thus

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} < 0.$$

This implies that $x(\delta(t))/(\delta(t)) \geq x(t)/t$, and so we have $x(\delta(t))/x(t) \geq (\delta(t))/t$. This, (3.9) and (3.10) imply that

$$w^{\Delta}(t) \leq -\eta(t)Q_1(t) + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}}w^{\sigma} - \frac{\gamma M^{\gamma-1}\eta(t)\left(ax^{\Delta}\right)^{\sigma}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma})^{\gamma}}.$$

Also, from (3.5) since $x(t)$ is positive and nondecreasing and $a(t)x^{\Delta}(t)$ is nonincreasing, we see that

$$w^{\Delta}(t) \leq -\eta(t)Q_1(t) + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}}w^{\sigma} - \frac{\gamma M^{\gamma-1}\eta(t)\left(\left(ax^{\Delta}\right)^{\sigma}\right)^2}{a(t)(x^{\sigma})^{2\gamma}}. \tag{3.11}$$

From (3.7) and (3.11), we have

$$w^{\Delta}(t) \leq -\eta(t)Q_1(t) + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}}w^{\sigma} - \frac{\gamma M^{\gamma-1}\eta(t)}{a(t)(\eta^{\sigma})^2}\left(w^{\sigma}\right)^2. \tag{3.12}$$

Integrating (3.12) from t_3 to t $(t \geq t_3)$, we obtain

$$\int_{t_3}^{t}\eta(s)Q_1(s)\Delta s \leq -\int_{t_3}^{t} w^{\Delta}(s)\Delta s + \int_{t_3}^{t}\frac{\eta^{\Delta}(s)}{\eta^{\sigma}}w^{\sigma}\Delta s - \int_{t_3}^{t}\frac{\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^{\sigma})^2}\left(w^{\sigma}\right)^2\Delta s. \tag{3.13}$$

Hence

$$\int_{t_3}^{t}\eta(s)Q_1(s)\Delta s \leq w(t_3) + \int_{t_3}^{t}\frac{\eta^{\Delta}(s)}{\eta^{\sigma}}w^{\sigma}(s)\,\Delta s - \int_{t_1}^{t}\frac{\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^{\sigma})^2}(w^{\sigma})^2\Delta s. \tag{3.14}$$

Then, we have

$$\int_{t_3}^{t}\left[\eta(s)Q_1(s)-\frac{a(s)\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s \le w(t_3)$$

$$-\int_{t_3}^{t}\left[\sqrt{\frac{\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^{\sigma})^2}}w^{\sigma}+\frac{\sqrt{a(s)}\eta^{\Delta}(s)}{2\sqrt{\gamma M^{\gamma-1}\eta(s)}}\right]^2\Delta s. \tag{3.15}$$

Hence

$$\int_{t_3}^{t}\left[\eta(s)Q_1(s)-\frac{a(s)\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s < w(t_3),$$

which contradicts the condition (3.2). Thus every solution of (1.3) oscillates. The proof is complete.

Remark 3.1. From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.3) by different choices of $\eta(t)$. For instance, let $\eta(t)=t$, for $t\ge t_0$. From Theorem 3.1, we have the following oscillation result.

Corollary 3.1. *Assume that* $(H_1)-(H_3)$ *hold. Furthermore, assume that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sp(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}-\frac{a(s)}{4\gamma M^{\gamma-1}s}\right]\Delta s=\infty, \tag{3.16}$$

then every solution of (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Let $\eta(t)=1$, $t\ge t_0$. From Theorem 3.1, we have the following result which can be considered as the extension of the Leighton–Wintner Theorem.

Corollary 3.2. (*Leighton-Wintner extension Theorem*). *Assume that* $(H_1)-(H_3)$ *hold, and*

$$\int_{t_0}^{\infty}p(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}\Delta s=\infty, \tag{3.17}$$

then every solution of (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Note that when $\gamma=1$, $r(t)=0$ and $\tau(t)=\delta(t)=t$, Eq.(1.3) reduces to the second-order dynamic equation

$$(a(t)y^{\Delta}(t))^{\Delta}+p(t)y(t)=0, \tag{3.18}$$

and Corollary 3.2 becomes the following well-known *Leighton-Wintner Theorem.*

Corollary 3.3 [5] (*Leighton-Wintner Theorem*). *Assume that*

$$\int_{t_0}^{\infty}\frac{1}{a(s)}\Delta s=\infty,\ \text{and}\ \int_{t_0}^{\infty}p(s)\Delta s=\infty,$$

then every solution of (3.18) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

This show that our results include as a special case some of the well-known oscillation results in the literature. In the following theorem, we present new oscillation criteria for Eq.(1.3) of Kamenev-type.

Theorem 3.2. *Assume that $(H_1)-(H_3)$ hold. Let $\eta(t)$ be as defined in Theorem 3.1. If*

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}\left[(t-s)^m\eta(s)Q_1(s)-\frac{(\eta^\sigma)^2B^2(t,s)}{4\gamma M^{\gamma-1}\eta(s)(t-s)^m}\right]\Delta s=\infty, \tag{3.19}$$

where $m>1$ is a positive integer, and

$$B(t,s)=(t-s)^m\frac{\eta^\Delta(s)}{\eta^\sigma}-m(t-\sigma(s))^{m-1},\ t\geq\sigma(s)\geq t_0.$$

Then every solution of (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. We proceed as in the proof of Theorem 3.1 to get (3.12) for all $t\geq t_3$ sufficiently large. Multiplying (3.12) by $(t-s)^m$ and integrating from t_3 to t, we have

$$\begin{aligned}\int_{t_3}^{t}(t-s)^m\eta(s)Q_1(s)\Delta s &\leq -\int_{t_3}^{t}(t-s)^m w^\Delta(s)\Delta s+\int_{t_3}^{t}(t-s)^m\frac{\eta^\Delta(s)}{\eta^\sigma}w^\sigma\Delta s\\ &\quad -\int_{t_3}^{t}\frac{(t-s)^m\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^\sigma)^2}\left(w^\sigma\right)^2\Delta s.\end{aligned} \tag{3.20}$$

Using formula (2.7), we have

$$-\int_{t_3}^{t}(t-s)^m w^\Delta(s)\Delta s=-\left.(t-s)^m w(s)\right|_{t_3}^{t}+\int_{t_3}^{t}\left((t-s)^m\right)^{\Delta_s}w^\sigma\Delta s. \tag{3.21}$$

Now, we prove that

$$\left((t-s)^m\right)^{\Delta_s}\leq -m(t-\sigma(s))^{m-1}. \tag{3.22}$$

We consider the following two cases: (i) $\mu(t)=0$, (ii) $\mu(t)\neq 0$. If (i) holds, then

$$\left((t-s)^m\right)^{\Delta_s}=-m(t-s)^{m-1}. \tag{3.23}$$

If (ii) holds, then we have

$$\begin{aligned}\left((t-s)^m\right)^{\Delta} &= \frac{1}{\mu(s)}\left[((t-\sigma(s))^m)-((t-s)^m)\right]\\ &= -\frac{1}{\sigma(s)-s}\left[((t-s)^m)-((t-\sigma(s))^m)\right].\end{aligned}$$

Using the inequality (cf. [17])

$$x^m-y^m\geq\gamma y^{m-1}(x-y)\ \ for\ all\ x\geq y>0\ and\ m\geq 1,$$

we have

$$\left[(t-s)^m-(t-\sigma(s))^m\right]\geq m((t-\sigma(s))^{m-1}(\sigma(s)-s).$$

Hence

$$\left((t-s)^m\right)^{\Delta_s}\leq -m(t-\sigma(s))^{m-1}. \tag{3.24}$$

Then, from (3.23) and (3.24), since in the general case $\sigma(s) \geq s$, we see that (3.22) holds. From (3.20)-(3.24), we obtain

$$\begin{aligned} &\int_{t_3}^{t} (t-s)^m \eta(s) Q_1(s) \Delta s \\ &\leq w(t_3)\,(t-t_3)^m + \int_{t_3}^{t} \left[(t-s)^m \frac{\eta^{\Delta}(s)}{\eta^{\sigma}} - m(t-\sigma(s))^{m-1} \right] w^{\sigma}(s)\,\Delta s \\ &- \int_{t_3}^{t} \frac{(t-s)^m \gamma M^{\gamma-1} \eta(s)}{a(s)(\eta^{\sigma})^2} (w^{\sigma})^2 \Delta s. \end{aligned} \tag{3.25}$$

Hence

$$\int_{t_3}^{t} \left[(t-s)^m \eta(s) Q_1(s) - \frac{a(s)(\eta^{\sigma})^2 B^2(t,s)}{4\gamma M^{\gamma-1} \eta(s)(t-s)^m} \right] \Delta s \leq w(t_3)\,(t-t_3)^m\,,$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_3}^{t} \left[(t-s)^m \eta(s) Q_1(s) - \frac{a(s)(\eta^{\sigma})^2 B^2(t,s)}{4\gamma M^{\gamma-1} \eta(s)(t-s)^m} \right] \leq w(t_3).$$

This contradicts the condition (3.19). Thus every solution of (1.3) oscillates. The proof is complete.

In the following theorem, we present new oscillation criteria for Eq.(1.3), which can be considered as the generalization of the Kamenev-type oscillation criteria.

First, we define $\Re$ by $H \in \Re$ provided $H : [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ satisfies

$$H(t,t) \geq 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad t > s \geq t_0,$$

$H^{\Delta_s}(t,s) \leq 0$, for $t \geq s \geq a$, and for each fixed t, $H(t,s)$ is right-dense continuous with respect to s. As a simple and important example, note that if $\mathbb{T} = \mathbb{R}$, then $H(t,s) := (t-s)^n$ is in $\Re$.

Theorem 3.3. *Assume that* $(H_1)-$ (H_3) *hold and let* $h, H : \mathbf{D} \to \mathbb{R}$ *be* $rd-$*continuous functions such that* H *belongs to the class* $\Re$ *and*

$$h(t,s) = -\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}. \tag{3.26}$$

If there exists a positive $\Delta-$*differentiable function* $\eta(t)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left[\eta(s) Q_1(s) - \frac{a(s)\,(\eta^{\sigma})^2}{4\gamma M^{\gamma-1} \eta(s)} R^2(t,s) \right] \Delta s = \infty, \tag{3.27}$$

where

$$R(t,s) = \left[h(t,s)/\sqrt{H(t,s)} - \frac{\eta^{\Delta}(s)}{\eta^{\sigma}} \right].$$

Then every solution of equation (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. We proceed as in the proof of Theorem 3.1 to prove that (3.12) holds for $t \geq t_3$. Multiplying (3.12) by $H(t,s)$ and integrating from t_3 to t, we have

$$\int_{t_3}^{t} H(t,s)\eta(s)Q_1(s)\Delta s \leq -\int_{t_3}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_3}^{t} H(t,s)\frac{\eta^{\Delta}(s)}{\eta^{\sigma}} w^{\sigma}\Delta s$$
$$-\int_{t_3}^{t} H(t,s)\frac{\gamma M^{\gamma-1}\eta(s)}{a(s)\left(\eta^{\sigma}\right)^2}(w^{\sigma})^2\Delta s. \tag{3.28}$$

Using formula (2.7), we have

$$\int_{t_3}^{t} H(t,s)w^{\Delta}(s)\Delta s = \left|H(t,s)w(s)\right|_{t_3}^{t} - \int_{t_3}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s$$
$$= -H(t,t_3)w(t_3) - \int_{t_3}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s. \tag{3.29}$$

where $H(t,t) = 0$. Substituting from (3.29) in (3.28) and use (3.26), we get

$$\int_{t_3}^{t} H(t,s)\eta(s)Q_1(s)\Delta s \leq H(t,t_3)w(t_3) - \int_{t_3}^{t} h(t,s)\sqrt{H(t,s)}w^{\sigma}\Delta s$$
$$+\int_{t_3}^{t} H(t,s)\frac{\eta^{\Delta}(s)}{\eta^{\sigma}} w^{\sigma}\Delta s - \int_{t_3}^{t} H(t,s)\frac{\gamma M^{\gamma-1}\eta(s)}{a(s)\left(\eta^{\sigma}\right)^2}(w^{\sigma})^2\Delta s. \tag{3.30}$$

Hence,

$$\int_{t_3}^{t} H(t,s)\eta(s)Q_1(s)\Delta s \leq H(t,t_3)w(t_3) - \int_{t_3}^{t} \left[h(t,s)\sqrt{H(t,s)} - \frac{H(t,s)\eta^{\Delta}(s)}{\eta^{\sigma}}\right] w^{\sigma}\Delta s$$
$$-\int_{t_3}^{t} \frac{\gamma H(t,s)M^{\gamma-1}\eta(s)(w^{\sigma})^2}{a(s)\left(\eta^{\sigma}\right)^2}\Delta s. \tag{3.31}$$

Therefore

$$\int_{t_3}^{t} H(t,s)\eta(s)Q_1(s)\Delta s \leq H(t,t_3)w(t_3)$$
$$-\int_{t_3}^{t} \left[\frac{\sqrt{\gamma H(t,s)M^{\gamma-1}\eta(s)}}{\sqrt{a(s)}\eta^{\sigma}}w^{\sigma} + \frac{\eta^{\sigma}\sqrt{a(s)}\left[h(t,s)\sqrt{H(t,s)} - \frac{H(t,s)\eta^{\Delta}(s)}{\eta^{\sigma}}\right]}{2\sqrt{\gamma H(t,s)M^{\gamma-1}\eta(s)}}\right]^2 \Delta s$$
$$+\int_{t_3}^{t} \frac{a(s)\left(\eta^{\sigma}\right)^2 H(t,s)}{4\gamma M^{\gamma-1}\eta(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\eta^{\Delta}(s)}{\eta^{\sigma}}\right]^2 \Delta s. \tag{3.32}$$

Then

$$\int_{t_3}^{t} H(t,s)\left[\eta(s)Q_1(s) - \frac{a(s)\,(\eta^\sigma)^2}{4\gamma M^{\gamma-1}\eta(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\eta^\Delta(s)}{\eta^\sigma}\right]^2\right]\Delta s$$

$$< H(t,t_3)w(t_3), \tag{3.33}$$

and this implies that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t} H(t,s)\left[\eta(s)Q_1(s) - \frac{a(s)\,(\eta^\sigma)^2 R^2(t,s)}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s < w(t_3), \tag{3.34}$$

for all large t, which contradicts (3.27). Thus every solution of (1.3) oscillates. The proof is complete.

As an immediate consequence of Theorem 3.3 we have the following oscillation result.

Corollary 3.4. *Let the assumption (3.27) in Theorem 3.1 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\eta(s)Q_1(s)\Delta s = \infty, \tag{3.35}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{a(s)H(t,s)\,(\eta^\sigma)^2}{\eta(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\eta^\Delta(s)}{\eta^\sigma}\right]^2\Delta s < \infty,$$

then every solution of equation (1.3) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Remark 3.2. With an appropriate choice of the functions H and h we can derive a number of oscillation criteria for Eq.(1.3) on different types of time scales. Consider, for example the function $H(t,s) = (t-s)^\lambda$, $(t,s) \in \mathbf{D}$ with $\lambda \geq 1$ is an odd integer. Evidently H belongs to the class $\Re$ and then (3.27) reduces to the oscillation criterion of Kamenev-type. Also, one can use the factorial function $H(t,s) = (t-s)^{(k)}$ where $t^{(k)} = t(t-1)...(t-k+1)$, $t^{(0)} = 1$. In this case

$$\begin{aligned} H^{\Delta_s}(t-s)^{(\lambda)} &= \frac{(t-\sigma(s))^{(k)} - (t-s)^{(k)}}{\mu(s)} = -\frac{(t-s)^{(k)} - (t-\sigma(s))^{(k)}}{\mu(s)} \\ &\geq -(k)(t-s)^{(k-1)}. \end{aligned}$$

4. EXAMPLES

In this Section, we give some examples which illustrate our main results.

Example 4.1. Consider the following second-order superlinear neutral delay dynamic equation

$$\left[y(t) + \frac{1}{t}y(\tau(t))\right]^{\Delta\Delta} + \frac{\lambda}{t(\delta(t)-1)^2}\left|y(\delta(t))\right|^2 signy(\delta(t)) = 0, \qquad t \in [t_0,\infty)_{\mathbb{T}}, \tag{4.1}$$

where $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$, $(t_0 > 0)$, $\gamma = 2$ and λ is a nonnegative constant and $\tau(t) \leq t$ and $\delta(t) < t$ are defined on $\mathbb{T}$ such that $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$ and

$\int_{t_0}^{\infty}\left[\frac{(\delta(s))^2}{s^3}\right]\Delta s=\infty$. In Eq.(4.1) $a(t)=1$, $p(t)=\frac{\lambda}{t(\delta(t)-1)^2}$, $r(t)=\frac{1}{t}$. It is easy to see that the assumptions $(H_1)-(H_3)$ hold. To apply Corollary 3.2 it remains to satisfy the conditions (3.1) and (3.17). First, we prove that (3.1) holds. From the definitions of $p(t)$ and $r(t)$ we have

$$\begin{aligned}\int_{t_0}^{\infty}(1-r(\delta(t)))^{\gamma}(\delta(t))^{\gamma}p(t)\Delta t &= \int_{t_0}^{\infty}\left(1-\frac{1}{(\delta(t))}\right)^2(\delta(t))^2\frac{1}{t(\delta(t)-1)^2}\Delta t\\ &= \int_{t_0}^{\infty}\frac{1}{t}\Delta t=\infty.\end{aligned}$$

Hence (3.1) is fulfilled. Also, we note that

$$\begin{aligned}&\lim_{t\to\infty}\sup\int_{t_0}^{t}(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}p(s)\Delta s\\ =\;&\lim_{t\to\infty}\sup\int_{1}^{t}\left(1-\frac{1}{(\delta(s))}\right)^2(\delta(s))^2\frac{\lambda}{s(\delta(s)-1)^2}\left(\frac{\delta(s)}{s}\right)^2\Delta s\\ =\;&\lambda\lim_{t\to\infty}\sup\int_{t_0}^{t}\left[\frac{(\delta(s))^2}{s^3}\right]\Delta s=\infty\quad if\quad \lambda>0.\end{aligned}$$

Hence, (3.17) holds and by Corollary 3.2 every solution of Eq.(4.1) oscillates if $\lambda>0$.

As a second example we consider the case when $\mathbb{T}=\mathbb{R}$.

Example 4.2. Consider the following second-order superlinear neutral delay differential equation

$$\left(y(t)+\frac{1}{t^2}\,y(t-\frac{\pi}{2})\right)''+\frac{(t-1)^3}{t\,(t-2)^3}\left|y(t-1)\right|^3 signy(t-1)=0,\qquad t\geq 2. \tag{4.2}$$

Then, the functions $a(t)=1$, $r(t)=\frac{1}{t^2}$, $p(t)=\frac{(t-1)^3}{t\,(t-2)^3}$ and $Q_1(t)=\frac{1}{t}$ satisfy conditions $(H_1)-(H_3)$ and (3.1). Now, we apply Corollary 3.4, when $\mathbb{T}=\mathbb{R}$. By taking $\eta(t)=\frac{1}{t}$ and $H(t,s)=\ln^2\frac{t}{s}$ for $t\geq s\geq 2$, we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,2)}\int_2^t H(t,s)\eta(s)Q_1(s)\,ds=\limsup_{t\to\infty}\frac{1}{\ln^2\frac{t}{2}}\int_2^t\ln^2\frac{t}{s}\cdot\frac{ds}{s^2}=\infty,$$

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,2)}\int_2^t H(t,s)\frac{a(s)\,(\eta^{\sigma})^2}{4\gamma M^{\gamma-1}\eta(s)}\left[h(t,s)/\sqrt{H(t,s)}-\frac{\eta'(s)}{\eta^{\sigma}}\right]^2 ds\\ =\;&\liminf_{t\to\infty}\frac{1}{\ln^2\frac{t}{2}}\int_2^t\frac{1}{s^3}\left(2+\ln\frac{t}{s}\right)^2<\infty\,.\end{aligned}$$

Conditions of Corollary 3.4 are satisfied and hence every solution of (4.2) is oscillatory.

Remark 4.1. Note, in the special case when $\mathbb{T}=\mathbb{R}$, the results in [1] cannot be applied for Eq.(4.1) and (4.2), so our results improve the results in [1] for differential equations, and essentially new for difference equation (1.4) and also are essentially new for the equations 1.5-1.9.

It remains an open problem to extend the above results in the sublinear case, i. e., when $0 < \gamma < 1$ and this will be of our interest in future work.

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