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OSCILLATIONS OF FIRST ORDER DELAY DYNAMIC EQUATIONS

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ABSTRACT. Consider the first order linear delay dynamic equation of the form

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0. \tag{E}$$

New oscillation criteria are established which contain well-known criteria for delay differential and difference equations as special cases. Illustrative examples are given to show that the results obtained essentially improve known oscillation results for Eq. (E).

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1. INTRODUCTION AND PRELIMINARIES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced in 1988 by Hilger [7] in his Ph.D. Thesis in order to unify continuous and discrete analysis. Several authors have expounded on various aspect of this new theory, see [1, 2, 8] and the references cited therein.

First we give a short review on the time scales calculus extracted from [1]. For any $t \in \mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \qquad \text{and} \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively, while the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. Also, t is said to be right-scattered if $\sigma(t) > t$, left-scattered if $t > \rho(t)$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, if there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with $|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|$,

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for all $s \in U$, then f is Δ-differentiable at t, and we call α the derivative of f at t and denote it by $f^{\Delta}(t)$.

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if t is right-scattered. When t is a right-dense point, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$, then f is continuous at t. Furthermore, we assume that $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. The following formulae are useful:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t), \qquad (fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t).$$

A function F with $F^{\Delta} = f$ is called an antiderivative of f, and then we define

$$\int_a^b f(t) \Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$. It is well known that rd-continuous functions possess antiderivatives.

Note that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt, \tag{1.1}$$

and if $\mathbb{T} = \mathbb{N}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$ and

$$\int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t). \tag{1.2}$$

If f is rd-continuous, then

$$\int_t^{\sigma(t)} f(s) \Delta s = \mu(t) f(t). \tag{1.3}$$

Intermediate Value Theorem [8]. The continuous mapping $f : [r, s] \to \mathbb{R}$, is assumed to fulfill the condition $f(r) < 0 < f(s)$, $r, s \in \mathbb{T}$. Then there is a $\delta \in [r, s)$ with $f(\delta) f(\sigma(\delta)) \leq 0$.

In recent years, there has been an increasing interest in the oscillation of solutions of some dynamic equations. See [1, 2, 14] and the references cited therein. However, few papers only ([3, 13, 18, 19]) deal with delay dynamic equations even in the case of first order linear equations. In this paper, we are concerned with the oscillatory behavior of the first order linear delay dynamic equation

$$x^{\Delta}(t) + p(t) x(\tau(t)) = 0, \tag{E}$$

where $t \in \mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$ is nondecreasing, $\tau(t) < t$, $\lim_{t \to \infty} \tau(t) = \infty$ and $p : \mathbb{T} \to \mathbb{R}$ is a nonnegative rd-continuous function.

If $x : \mathbb{T} \to \mathbb{R}$ is defined and Δ-differentiable for $t \in \mathbb{T}$ and satisfies Eq. (E) for $t \in \mathbb{T}$, then x is called a solution of Eq. (E). A solution x has a generalized zero at t

in case $x(t) = 0$. We say x has a generalized zero on $[a, b]$ in case $x(t)x(\sigma(t)) < 0$ or $x(t) = 0$ for some $t \in [a, b)$, where a, $b \in \mathbb{T}$ and $a \leq b$ (x has a generalized zero at b, in case $x(\rho(b))x(b) < 0$ or $x(b) = 0$). A nontrivial solution of Eq. (E) is said to be oscillatory on $[t_x, \infty)$ if it has infinitely many generalized zeros when $t \geq t_x$. Finally, Eq. (E) is called oscillatory if all its solutions are oscillatory.

We list the following well-known oscillation criteria for the equation (E) in special cases of $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then Eq. (E) reduces to the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0. \tag{E_R}$$

In 1972, Ladas et. al. [11] proved that Eq. (E_R) is oscillatory if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{S_R}$$

while, in 1979, Ladas [10] and in 1982 Koplatadze and Chanturia [9], proved that the same conclusion holds if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{I_R}$$

Similarly, in case that $\mathbb{T} = \mathbb{N}$, Eq. (E) reduces to the first order delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \qquad k \in \mathbb{N}, \quad n > k \geq 1. \tag{E_N}$$

In 1989, Erbe and Zhang [5] proved that Eq. (E_N) is oscillatory provided that

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1. \tag{S_N}$$

In the same year, Ladas et. al. [12] presented the condition

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1} \tag{I_N}$$

for Eq. (E_N) to be oscillatory.

In 2002, Zhang and Deng [18], using cylinder transforms, and in 2005, Bohner [3], using exponential functions notation, proved the following result for any time scale $\mathbb{T}$.

Theorem 1.1. *Define*

$$\alpha := \limsup_{t\to\infty} \sup_{\lambda\in E} \{\lambda \exp_{-\lambda p}(\tau(t), t)\} \tag{1.4}$$

where

$$\exp_{-\lambda p}(\tau(t), t) = \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s,$$

$E = \{\ \lambda \mid \lambda > 0,\ 1 - \lambda p(t)\mu(t) > 0,\ t \in \mathbb{T}\}$, *and*

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h} & \textit{if} \quad h \neq 0 \\ \\ z & \textit{if} \quad h = 0. \end{cases}$$

If Eq. (E) has an eventually positive solution, then $\alpha \geq 1$.

The following corollary was also given in [18].

Corollary 1.2. *If* $\alpha < 1$, *then all solutions of Eq. (E) are oscillatory.*

In 2004, Zhang and Lian [19] studied the distribution of generalized zeros of solutions of the delay dynamic Eq. (E). Note that in [18] and [3], the conditions (I_R) and (I_N) are derived as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, respectively.

It is obvious that there is a gap between the conditions (S_R) and (I_R) (or similarly between the conditions (S_N) and (I_N)) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \qquad \Big(\text{or} \quad \lim_{n\to\infty} \sum_{i=n-k}^{n-1} p_i\Big)$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors. See [4, 15, 16, 17] and the references cited therein.

The purpose of this paper is to establish new sufficient conditions for the oscillation of all solutions to the dynamic equation (E). Moreover, the above mentioned problem is handled for some special cases of time scales $\mathbb{T}$, and some results previously obtained are compared with the results presented in this paper. Several illustrative examples are given.

2. MAIN RESULTS

Since we deal with the oscillatory behavior of the dynamic equation (E) on time scales, we assume throughout the paper that the time scale $\mathbb{T}$ under consideration satisfies $\sup \mathbb{T} = \infty$. The following lemmas are needed in our proofs.

Lemma 2.1. *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is nonincreasing and* $\tau : \mathbb{T} \to \mathbb{T}$ *is nondecreasing. If* $v < u$, *then*

$$\int_{v}^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_{v}^{\sigma(u)} f(s)\Delta s. \tag{2.1}$$

Proof. Since $v < u$, we can divide the integral into two parts

$$\int_{v}^{\sigma(u)} f(s)g(\tau(s))\Delta s = \int_{v}^{u} f(s)g(\tau(s))\Delta s + \int_{u}^{\sigma(u)} f(s)g(\tau(s))\Delta s.$$

Using the fact that τ is nondecreasing and g is nonincreasing, the first part gives

$$\int_v^u f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_v^u f(s)\Delta s. \tag{2.2}$$

Since $f(g \circ \tau)$ is rd-continuous, in view of (1.3), the second part yields

(2.3)

$$\int_u^{\sigma(u)} f(s)g(\tau(s))\Delta s = \mu(u)f(u)g(\tau(u))g(\tau(u))\Big(\mu(u)f(u)\Big)g(\tau(u)) \int_u^{\sigma(u)} f(s)\Delta s.$$

Combining (2.2) and (2.3), we obtain

$$\int_v^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u))\left(\int_v^u f(s)\Delta s + \int_u^{\sigma(u)} f(s)\Delta s\right) = g(\tau(u)) \int_v^{\sigma(u)} f(s)\Delta s.$$

The proof is complete. □

In case $v = u$, the monotonocity property of τ is not needed. In view of (2.3), the following corollary is immediate.

Corollary 2.2. *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are rd-continuous and $\tau : \mathbb{T} \to \mathbb{T}$ is rd-continuous. Then*

$$\int_u^{\sigma(u)} f(s)g(\tau(s))\Delta s = g(\tau(u)) \int_u^{\sigma(u)} f(s)\Delta s. \tag{2.4}$$

For the following lemma see [3] and [19].

Lemma 2.3. *Assume that x is an eventually positive solution of Eq. (E) and that for some positive constant M*

$$\liminf_{t\to\infty} \int_{\tau(t)}^t p(s)\Delta s > M. \tag{2.5}$$

Then

$$\frac{x(\tau(t))}{x(t)} \leq \frac{4}{M^2}, \quad \textit{for all large } t. \tag{2.6}$$

Proof. For some sufficiently large t_0, $x(t) > 0$ when $t \geq t_0$. From Eq. (E), we have $x^\Delta(t) \leq 0$ for $t \geq \tau^{-1}(t_0) = t_1$. By (2.5), it is possible to find a sufficiently large number $t_2 \geq \tau^{-1}(t_1)$ such that

$$\int_{\tau(t)}^t p(s)\Delta s \geq M, \qquad \forall t \geq t_2. \tag{2.7}$$

Since $\tau^{-1}(t) > t \geq t_2$, by (2.7) we get

$$\int_t^{\tau^{-1}(t)} p(s)\Delta s \geq M, \quad \forall t \geq t_2. \tag{2.8}$$

Define

$$G(r) := \int_t^r p(s)\Delta s - \frac{M}{2},$$

for $r \in [t, \tau^{-1}(t)]$. It is clear that $G : [t, \tau^{-1}(t)] \to \mathbb{R}$ is continuous and nondecreasing. We also have

$$G(t) = -\frac{M}{2} < 0 \qquad \text{and} \qquad G(\tau^{-1}(t)) \geq M - \frac{M}{2} = \frac{M}{2} > 0.$$

By Intermediate Value Theorem for time scales, there exists a $t_* \in [t, \tau^{-1}(t))$ such that $G(t_*)G(\sigma(t_*)) \leq 0$. Since G is nondecreasing, we conclude that $G(t_*) \leq 0 < G(\sigma(t_*))$. Hence there exists a $t_* \in [t, \tau^{-1}(t))$ such that

$$\int_t^{t_*} p(s)\Delta s \leq \frac{M}{2} \qquad \text{and} \qquad \int_t^{\sigma(t_*)} p(s)\Delta s > \frac{M}{2}, \quad \text{for} \quad t \geq t_2. \tag{2.9}$$

By (2.5) and the first part of (2.9), we also have

$$\int_{\tau(t_*)}^{\sigma(t)} p(s)\Delta s \geq \int_{\tau(t_*)}^{t_*} p(s)\Delta s - \int_t^{t_*} p(s)\Delta s \geq \frac{M}{2}, \quad \text{for} \quad t \geq t_2. \tag{2.10}$$

Using (2.1) and the decreasing character of x, we obtain

$$\int_{\tau(t_*)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \geq x(\tau(t))\frac{M}{2}, \tag{2.11}$$

and

$$\int_t^{\sigma(t_*)} p(s)x(\tau(s))\Delta s \geq x(\tau(t_*))\frac{M}{2}. \tag{2.12}$$

Integrating Eq. (E) from t to $\sigma(t_*)$ and using (2.11) and (2.12), we obtain for $t \geq t_2$,

$$\begin{aligned} x(t) &\geq x(t) - x(\sigma(t_*)) = \int_t^{\sigma(t_*)} p(s)x(\tau(s))\Delta s \\ &\geq \frac{M}{2}x(\tau(t_*)) \geq \frac{M}{2}[x(\tau(t_*)) - x(\sigma(t))]\frac{M}{2}\int_{\tau(t_*)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \\ &\geq \frac{M^2}{4}x(\tau(t)). \end{aligned}$$

The proof is complete. □

Note that Lemma 2.3 is a generalization of the results in [9] and [6].

Theorem 2.4. *If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{2.13}$$

then Eq. (E) is oscillatory.

Proof. Assume, for the sake of contradiction, that Eq. (E) has a nonoscillatory solution x. We may assume that x is eventually positive by replacing x by $-x$, otherwise. Since $\tau(t) \to \infty$ as $t \to \infty$, there is a positive number $t_1 \geq t_0$, such that $x(\tau(t)) > 0$ for $t \geq t_1$. In view of Eq. (E),

$$x^{\Delta}(t) = -p(t)x(\tau(t)) < 0, \qquad t \geq t_1. \tag{2.14}$$

Integrating Eq. (E) from $\tau(t)$ to $\sigma(t)$, we have

$$x(\sigma(t)) - x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} p(s)x(\tau(s))\Delta s = 0. \tag{2.15}$$

Since x is Δ-differentiable, it is rd-continuous, Lemma 2.1 is applicable for the integral term in the previous equation. In view of (2.1), it is easy to see that

$$\int_{\tau(t)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \geq x(\tau(t)) \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s. \tag{2.16}$$

Using (2.16) in (2.15), we obtain

$$x(\sigma(t)) + x(\tau(t))\Big(\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s - 1\Big) \leq 0, \qquad \text{for} \qquad t \geq t_1, \tag{2.17}$$

which, by (2.13), leads to a contradiction. The proof is complete. □

Observe that Theorem 2.4 unifies previous results related with the oscillation of first order delay equations in the continuous and discrete case. In particular, if $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{N}$, condition (2.13) of Theorem 2.4 takes the form (S_R) or (S_N), respectively.

Theorem 2.5. *Assume that there exists a positive constant M such that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > M \tag{2.18}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{M^2}{4}. \tag{2.19}$$

Then Eq. (E) is oscillatory.

Proof. Assume that x is an eventually positive solution of Eq. (E) such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. As in the proof of Theorem 2.4, integrating Eq. (E) from $\tau(t)$ to t, we have

$$\begin{aligned} 0 &= x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)x(\tau(s))\Delta s \\ &\geq x(t) + x(\tau(t))\Big(\int_{\tau(t)}^{t} p(s)\Delta s - 1\Big) \\ &= x(\tau(t))\Big(\frac{x(t)}{x(\tau(t))} + \int_{\tau(t)}^{t} p(s)\Delta s - 1\Big), \qquad \text{for} \qquad t \geq t_1. \end{aligned}$$

Now using Lemma 2.3, we obtain

$$x(\tau(t))\Big(\frac{M^2}{4} + \int_{\tau(t)}^{t} p(s)\Delta s - 1\Big) \leq 0, \qquad \text{for all} \qquad t \geq t_1,$$

which, in view of condition (2.19), leads to a contradiction. The proof is complete. □

Note that, in view of Theorem 2.4, it makes sense to consider in Theorem 2.5 the case when $0 < M < 1$. Also observe that [4, Theorem 2.2] for Eq. (E_R) can be derived from Theorem 2.5 when the time scale $\mathbb{T}$ is chosen as $\mathbb{R}$.

Consider a time scale of the form

$$\mathbb{T} = \{t_n : n \in \mathbb{Z}\}, \tag{2.20}$$

where $\{t_n\}$ is a strictly increasing sequence of real numbers such that $\mathbb{T}$ is closed. For such time scales, Bohner [3] presented the following result.

Theorem 2.6. [3, Theorem 2] *Consider a time scale as described in (2.20). Let $k \in \mathbb{N}$ and $\tau(t_n) = t_{n-k}$ for all $n \in \mathbb{N}$. If Eq. (E) has an eventually positive solution, then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \le \left(\frac{k}{k+1}\right)^{k+1}. \tag{2.21}$$

An immediate consequence is the following result.

Corollary 2.7. *Consider a time scale as described in (2.20). Let $k \in \mathbb{N}$ and $\tau(t_n) = t_{n-k}$ for all $n \in \mathbb{N}$. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > \left(\frac{k}{k+1}\right)^{k+1}, \tag{2.22}$$

then Eq. (E) is oscillatory.

Observe that when condition (2.22) is not satisfied, then from Corollary 2.7, we cannot conclude anything about the oscillatory behavior of Eq. (E). However, from Theorem 2.5, we have the following conclusion.

Corollary 2.8. *Assume that there exist a positive real number M such that*

$$M < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \le \left(\frac{k}{k+1}\right)^{k+1} \tag{2.23}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{M^2}{4}.$$

Then Eq. (E) is oscillatory on the time scales described in (2.20) with $k \in \mathbb{N}$ and $\tau(t_n) = t_{n-k}$ for all $n \in \mathbb{N}$.

Note that for Eq. (E_N), [17, Theorem 2.6] can be derived from Corollary 2.8 when the time scale $\mathbb{T}$ is chosen as $\mathbb{N}$.

3. EXAMPLES

Example 3.1. Let $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$. Consider the following equation

$$x^{\Delta}(t) + p\, x(\tau(t)) = 0, \qquad t \geq t_0 \tag{3.1}$$

where $p > 0$ and $\tau(t) = t - (k-1)h$, for any positive integer $k > 1$. Since

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p\ \Delta s = pkh,$$

from Theorem 2.4, we conclude that if

$$pkh > 1,$$

then Eq. (3.1) is oscillatory.

Example 3.2. Assume that in equation (3.1), $p = 1$, $k = 9$, and $h = \frac{1}{10}$. Since $pkh = 9/10$ is not greater than 1, Theorem 2.4 cannot be applied. However, it is easy to see that there exists a positive real number $M \in (\sqrt{2/5}, 9/10)$ so that (2.18) and the condition (2.19)

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s = pkh = \frac{9}{10} > 1 - \frac{M^2}{4}$$

are satisfied. So, by Theorem 2.5, all solutions of Eq. (3.1) are oscillatory.

Example 3.3. Consider the equation on $\mathbb{T} = \mathbb{N}$

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \qquad n \in \mathbb{N}, \tag{3.2}$$

where

$$p_{2n} = \frac{8}{100}, \qquad p_{2n+1} = \frac{8}{100} + \frac{746}{1000}\sin^2\frac{n\pi}{2}, \qquad n \in \mathbb{N}.$$

Then

$$\liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < \left(\frac{3}{4}\right)^4,$$

which means that the condition (2.22) of Corollary 2.7 is not satisfied. However, it is easy to see that there exists a positive real number $M \in (\sqrt{7/5^3}, 24/100)$ such that

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{746}{1000} > 1 - \frac{M^2}{4}.$$

So, by Corollary 2.8, all solutions of Eq. (3.2) are oscillatory.

Example 3.4. Consider the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-1} = 0. \tag{3.3}$$

where

$$p_{2n} = \frac{1}{5}, \qquad p_{2n+1} = \frac{127}{128}, \qquad n \in \mathbb{N}.$$

By Theorem 1.1,

$$\begin{aligned}\alpha &= \limsup_{t\to\infty}\sup_{\lambda\in E}\{\lambda\exp_{-\lambda p}(\tau(t),t)\}\\ &= \limsup_{n\to\infty}\sup_{\lambda\in E}\lambda(1-\lambda p_{n-1})\\ &= \limsup_{n\to\infty}\frac{1}{4p_{n-1}},\end{aligned}$$

because $\lambda(1-\lambda p_{n-1})$ takes its maximum value at $\lambda=\dfrac{1}{2p_{n-1}}$. Thus,

$$\alpha=\limsup_{n\to\infty}\frac{1}{4p_{n-1}}=\frac{5}{4}>1,$$

and therefore, Corollary 1.2 cannot be applied. Also

$$\liminf_{n\to\infty}\sum_{i=n-1}^{n-1}p_i=\frac{1}{5}<\left(\frac{1}{2}\right)^2$$

that is, condition (2.22) is not satisfied and therefore Corollary 2.7 cannot be applied. However

$$\limsup_{n\to\infty}\sum_{i=n-1}^{n-1}p_i=\frac{127}{128},$$

and taking $M\in(1/4\sqrt{2},1/5)$, conditions (2.18) and (2.19) of Theorem 2.5 are satisfied. Therefore, all solutions of Eq. (3.3) are oscillatory.

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