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LEARNING DYNAMICS AND STABILITY IN NETWORKS WITH FUZZY SYNAPSES

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ABSTRACT. A new model of Hopfield-type neural network of neurons with crisp somatic activations which have some fuzzy synaptic modifications is formulated which incorporates a Hebbian-type unsupervised learning algorithm. A set of sufficient conditions are derived for the existence of a globally exponentially stable steady state; the exponential convergence of the learning algorithm is also considered. Our model will reduce to one of fuzzy neural networks considered by others when the learning component is absent; when the fuzzy synapses are absent, our model will reduce to the well known Hopfield-type network

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1. INTRODUCTION

A neural network known as the Hopfield-type and several of its generalizations have been intensively investigated by numerous authors from the areas of mathematics, physics, engineering, computer science etc. Earlier work on neural networks can be traced to the neuron model known as McCulloch-Pitts model (McCalloch and Pitts [19]). Many applications of neural networks have been known in diverse areas such as optimization, signal and image processing, pattern recognition, control system etc. These applications are based on the existence and stability of equilibria of the network models. The neuron models in these networks have been based on the McCalloch-Pitts neuron and those derived form it while the mathematics of the analysis used has been based on usual bivalent logic of set theory. It has been recognized (see Lee and Lee [15]) that neuron models based on the McCalloch-Pitts or those of the Hopfield-type fail to appropriately "reflect the fact that the behaviour of even the simplest type of nerve cell exhibits not only randomness but more importantly a type of imprecision which is associated with the lack of sharp transition from the occurrence of an event to its non-occurrence" (Lee and Lee [15]). Furthermore neural networks, biological or artificial have to analyze, process and resolve certain ambiguities different from those of randomness of probability theory. Phenomena of the type such as associative memories, images, perceptions become targets of certain

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neural networks. The mechanism of fuzzy logic based on multivalence rather than bivalence may be another alternative (see Zadeh [29], Kosko [14], Rao and Rao [22], Chen and Pham [3], Kaufman and Gupta [12]) to build a class of "fuzzy neurons" and associated neural networks to model the phenomena related to human thinking, perception, memory etc. A simplistic characterization of a biological neuron is that it has four major components namely its soma, axon, dendrites and synapses; it is at an immediate neighborhood of the soma that computations are done whose result is transmitted through its axon to its synaptic fan close to the dendrites of other neurons; synapses themselves can do some preprocessing of the information they receive before information carrying signals are delivered by a chemical transfer to the dendrites of other neurons. In this respect we can say that if a neuron does not follow the crisp logic and related computations, then it can become a "fuzzy neuron" in the sense that its somatic computations can become "fuzzy" by following a set of "if-then" rules or the synaptic processes can become "fuzzy" in the sense that the usual process of multiplication and addition are replaced by operations of fuzzy logic namely "min" and "max" operators respectively. Thus neural networks can merge with fuzzy logic leading to the concept of "fuzzy neural networks" (see Kosko [14]) which is believed to have considerable potential for applications in the areas of image processing, medical diagnosis, control system, pattern recognition, anti-lock brake systems, automatic transmission, smart elevators, auto-pilot systems and many other house-hold appliances. It is well known that classical control theory is based on the theory of differential equations; however as it has been noted by Haykin [5] that "intelligent control is largely rule based because the dependencies involved in its development are much too complex to permit an analytical representation. And to deal with such dependencies, it is expedient to use the mathematics of fuzzy systems and neural network". This is largely due to the ability of fuzzy systems in quantifying linguistic inputs and formulate a workable approximation to the complex problem. Neural networks can learn or adapt themselves to new environments and there is a natural "synergy neural networks and fuzzy systems" (see Haykin [5].)

More recently there have been several publications on the theme of neural networks where fuzzy logic is used; Yang and Yang [24, 25] and Yang et al. [26] have proposed a fuzzy cellular neural network to include and analyze the ambiguity or vagueness inherent in the inputs and outputs of neural networks; we can say that in these networks the somatic operations are crisp while some of the the synaptic operations are not crisp and are from fuzzy logic. Further analysis of this type of networks can be found in the works of Yuan et al. [28], Liu and Tang [17], Huang and Zhang [10], Huang [7, 8], Chen and Liao [2]. This author could find only one publication which considers a network of fuzzy neurons which use fuzzy logic in their somatic computations (see Huang et al. [9]). There are many other publications where rule based fuzzy dynamical systems have been considered (see Kosko [13, 14], Johnson et al. [11], Yang et al. [27], Tian and Peng [23]).

The mathematical operations in a network of crisp neurons are defined by means of the usual multiplication and addition of signals at the synapses; the learning and adaptation are also based on the crisp logic of bivalence. In the case of synaptically fuzzy networks, the synaptic computations are carried out by fuzzy operators of "maximum" and "minimum". It is the purpose of this article to propose a system modelling the dynamics of somatically crisp and synaptically fuzzy neurons with a learning component incorporated in the network to learn a set of crisp weights when a crisp external signal is presented to the network. The network can learn a set of crisp synaptic weights. The network is fuzzy in the sense that the operations include those of fuzzy logic namely "max" and "min" in addition to those of the bivalent addition and multiplication. We suppose that the learning component in the network is based on the crisp logic and is similar to the Hebbian based learning rule with a forgetting term included as proposed by Amari [1]. We obtain sufficient conditions for the existence of a unique equilibrium and its global stability in a network of synaptically fuzzy neurons. While various types of time delays can be included, we have not done this aspect in this article.

We recall that the well known and widely studied Hopfield-type neural network can be described by the system of equations

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + J_i,\ i = 1, 2, \cdots, n,\ t > 0$$

in which $x_i(t)$ denotes the state of the neuron i at time t, a_i denotes a resetting time constant, a_{ij}, $(i, j = 1, 2, \cdots, n)$ denotes the synaptic weight associated with the $j'th$ synapse of the $i'th$ neuron; $f_j()$ denotes the activation function of the $j'th$ neuron and J_i denotes the external input directed to the neuron i from outside the network. The literature on neural networks contains a large number of extensions, modifications and variations of the above type of the model system. A predominant theme in the mathematics of the above system and its generalizations is to find conditions for the existence of a unique equilibrium and its stability characteristics.

2. MODEL FORMULATION

It is widely believed that information and knowledge are stored in the synaptic weights and while a neural network learns, the synaptic weights change. We consider a class of networks of somatically crisp neurons with fuzzy synapses which has learnable synaptic weights described by the following system of equations

(2.1)

$$\left.\begin{aligned}\frac{du_i(t)}{dt} =& - a_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + B_i \sum_{j=1}^{n} m_{ij}(t) p_j \\ &+ b_i \bigvee_{j=1}^{n} b_{ij} f_j(u_j(t)) + c_i \bigwedge_{j=1}^{n} c_{ij} f_j(u_j(t)) + J_i \\ \frac{dm_{ij}(t)}{dt} =& -\alpha_i\, m_{ij}(t) + \Big[\beta_i f_i(u_i(t)) + \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij} f_j(u_j(t)) \\ &+ \delta_i \bigvee_{j=1}^{n} \delta_{ij} f_j(u_j(t))\Big] p_j\end{aligned}\right\},\ i = 1, 2, \cdots, n,\ t > 0$$

in which $u_i(t)$ denotes the state of neuron i at time t, a_i denotes the passive negative stabilizing feedback of neuron i, $a_{ij}, b_{ij}, c_{ij}, \gamma_{ij}, \delta_{ij}$ denote the denote the synaptic weights of the various fuzzy and non-fuzzy synapses of neuron i, $B_i, b_i, c_i, \beta_i, \gamma_i, \delta_i$ are disposable constants, m_{ij} denotes a learnable synaptic weight of neuron when it is presented with a constant input signal vector $p = (p_1, p_2, p_3, \cdots, p_n)$; the external bias to the network is denoted by the constant vector $J = (J_1, J_2, \cdots, J_n)$. The operators $\bigvee$ and $\bigwedge$ appearing (2.1) denote respectively the "max" and "min" operators used in fuzzy logic. The learning equation or algorithm in (2.1) is based on the Hebbian-type (Hebb [6]) unsupervised algorithm modified by the introduction of a forgetting term as proposed by Amari [1]; such learning models have been recently discussed by Lemmon and Kumar [16], Lu and He [18], Meyer-Baese et al. [20, 21]. It is well known that neuronal activations are fast in comparison with the synaptic modifications of weights and it may be appropriate to consider the combined dynamics of neuronal activations and synaptic modifications with two different time scales and possibly by singular perturbation methods. In our analysis below, we do not consider the difference in time scales.

The dynamics of neural networks neurons with fuzzy synapses described by (2.1) without the learning component has been discussed by several authors recently (see Yang and Yang [24], Yang et al. [27], Liu and Tang [17], Yuan et al. [28], Huang [7, 8], Chen and Liao [2]). Our model becomes one of fuzzy neural networks in the absence of the learning component; in the absence of fuzzy synapses, our model reduces to the most commonly studied Hopfield-type neural network. To analyze the system (2.1) further we introduce auxiliary variables $v_i, i = 1, 2, \cdots, n$ defined by

$$v_i(t) = \sum_{j=1}^{n} m_{ij}(t) p_j, \qquad i = 1, 2, \cdots, n,\ t \geq 0 \tag{2.2}$$

so that the system (2.1) becomes

(2.3)

$$\left.\begin{aligned}\frac{du_i(t)}{dt} &= -a_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + B_i v_i(t)\\ &\qquad + b_i \bigvee_{j=1}^{n} b_{ij} f_j(u_j(t)) + c_i \bigwedge_{j=1}^{n} c_{ij} f_j(u_j(t)) + J_i\\ \frac{dv_i(t)}{dt} &= -\alpha_i v_i(t) + \Big[\beta_i f_i(u_i(t)) + \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij} f_j(u_j(t))\\ &\qquad + \delta_i \bigvee_{j=1}^{n} f_j(u_j(t))\Big] c\end{aligned}\right\}, i = 1, 2, \cdots, n, t > 0$$

in which $c = \sum_{j=1}^{n} p_j^2$. We are interested in the derivation of sufficient conditions for the existence of a globally stable equilibrium solution of (2.3). The following preliminary result from Yang and Yang [1996] will be useful in our analysis of the system (2.3).

Lemma 2.1. *Suppose* $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ *be any two vectors in* $\mathbb{R}^n$. *Then*

$$\left.\begin{aligned}\Big|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j)\Big| &\le \sum_{j=1}^{n} |\alpha_{ij}|\,|f_j(x_j) - f_j(y_j)|\\ \Big|\bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(y_j)\Big| &\le \sum_{j=1}^{n} |\beta_{ij}|\,|f_j(x_j) - f_j(y_j)|\end{aligned}\right\}, i = 1, 2, \cdots, n. \tag{2.4}$$

Proof. Suppose there exist indices p and q such that

$$\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) = \alpha_{ip}, \qquad \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j) = \alpha_{iq} f_q(y_q).$$

Then it will follow that

$$\begin{aligned}\Big|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j)\Big| &\le \max\Big\{\Big|\alpha_{ip}[f_p(x_p) - f_p(y_p)]\Big|, \Big|\alpha_{iq}[f_q(x_q) - f_q(y_q)]\Big|\Big\}\\ &\le \sum_{j=1}^{n} |\alpha_{ij}|\,|f_j(x_j) - f_j(y_j)|\end{aligned}$$

and hence (2.4) follows. The proof of the second of (2.4) is similar and is omitted. □

3. EQUILIBRIUM EXISTENCE

We will derive sufficient conditions for the existence of a unique equilibrium of the system (2.3). Note that $(x^*, y^*) = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_n^*)$ is an equilibrium of (2.3) if (x^*, y^*) satisfies the system of equations

$$\left.\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + B_i y_i^* + b_i \bigwedge_{j=1}^{n} b_{ij} f_j(x_j^*) \\ &\qquad + c_i \bigvee_{j=1}^{n} c_{ij} f_j(x_j^*) + J_i \\ \alpha_i y_i^* &= \left[\beta_i f_i(x_i^*) + \gamma_i \bigwedge_{k=1}^{n} \gamma_{ik} f_k(x_k^*) + \delta_i \bigvee_{k=1}^{n} \delta_{ik} f_k(x_k^*)\right] c \end{aligned}\right\}, \; i = 1, 2, \cdots, n \tag{3.1}$$

which simplifies to the system of equations

$$\left.\begin{aligned} a_i x_i^* = \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + b_i \bigwedge_{j=1}^{n} b_{ij} f_j(x_j^*) + c_i \bigvee_{j=1}^{n} c_{ij} f_j(x_j^*) + J_i \\ + c\frac{B_i}{\alpha_i}\left[\beta_i f_i(x_i^*) + \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij} f_j(x_j^*) + \delta_i \bigvee_{j=1}^{n} \delta_{ij} f_j(x_j^*)\right] \end{aligned}\right\}, \; i = 1, 2, \cdots, n. \tag{3.2}$$

We can simplify (3.2) further as follows: we let

$$a_i x_i^* = X_i^*, \qquad i = 1, 2, \cdots, n \tag{3.3}$$

so that

$$\left.\begin{aligned} X_i^* = \sum_{j=1}^{n} a_{ij} f_j\left(\frac{X_j^*}{a_j}\right) + b_i \bigwedge_{j=1}^{n} b_{ij} f_j\left(\frac{X_j^*}{a_j}\right) + c_i \bigvee_{j=1}^{n} c_{ij} f_j\left(\frac{X_j^*}{a_j}\right) + J_i \\ + c\frac{B_i}{\alpha_i}\left[\beta_i f_i\left(\frac{X_i^*}{a_i}\right) + \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij} f_j\left(\frac{X_j^*}{a_j}\right) + \delta_i \bigvee_{j=1}^{n} \delta_{ij} f_j\left(\frac{X_j^*}{a_j}\right)\right] \end{aligned}\right\}, \; i = 1, 2, \cdots, n \tag{3.4}$$

Theorem 3.1. *Suppose the activation functions f_j, $j = 1, 2, \cdots, n$ satisfy global Lipschitz conditions so that*

$$|f_j(x) - f_j(y)| \le L_j |x - y|, \qquad x, y \in \mathbb{R},\; j = 1, 2, \cdots, n. \tag{3.5}$$

If

$$\left.\begin{aligned} \max_{1 \le i \le n} \frac{L_i}{a_i}\Bigg[\sum_{j=1}^{n}\Big(|a_{ji}| + |b_j||b_{ji}| + |c_j||c_{ji}|\Big) \\ + c\frac{B_i}{\alpha_i}\Big(|\beta_i| + \sum_{j=1}^{n}\Big\{|\gamma_j||\gamma_{ji}| + |\delta_j||\delta_{ji}|\Big\}\Bigg] < 1 \end{aligned}\right\} \tag{3.6}$$

then the system (3.4) has a unique solution.

Proof. Define a map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ as follows:

(3.7)
$$\left.\begin{aligned} &F(x) = \{F_1(x), F_2(x), \cdots, F_n(x)\}, \qquad x \in \mathbb{R}^n \\ &F_i(x) = \sum_{j=1}^{n} a_{ij} f_j\left(\frac{x_j}{a_j}\right) + b_i \bigwedge_{j=1}^{n} b_{ij} f_j\left(\frac{x_j}{a_j}\right) + c_i \bigvee_{j=1}^{n} c_{ij} f_j\left(\frac{x_j}{a_j}\right) + J_i \\ &\qquad + c\frac{B_i}{\alpha_i}\left[\beta_i f_i\left(\frac{x_i}{a_i}\right) + \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij} f_j\left(\frac{x_j}{a_j}\right) + \delta_i \bigvee_{j=1}^{n} \delta_{ij} f_j\left(\frac{x_j}{a_j}\right)\right] \end{aligned}\right\}, \ i = 1, 2, \cdots, n.$$

We have from (3.7) and for any $x,\ y \in \mathbb{R}^n$ that

$$\left.\begin{aligned} |F_i(x) - F_i(y)| = \Bigg| &\sum_{j=1}^{n} a_{ij}\left[f_j\left(\frac{x_j}{a_j}\right) - f_j\left(\frac{y_j}{a_j}\right)\right] \\ &+ b_i \bigwedge_{j=1}^{n} b_{ij}\left[f_j\left(\frac{x_j}{a_j}\right) - f_j\left(\frac{y_j}{a_j}\right)\right] \\ &+ c_i \bigvee_{j=1}^{n} c_{ij}\left[f_j\left(\frac{x_j}{a_j}\right) - f_j\left(\frac{y_j}{a_j}\right)\right] \\ &+ c\frac{B_i}{\alpha_i}\Bigg\{\beta_i\left[f_i\left(\frac{x_i}{a_i}\right) - f_i\left(\frac{y_i}{a_i}\right)\right] \\ &+ \gamma_i \bigwedge_{j=1}^{n} \gamma_{ij}\left[f_j\left(\frac{x_j}{a_j}\right) - f_j\left(\frac{y_j}{a_j}\right)\right] \\ &+ \delta_i \bigvee_{j=1}^{n} \delta_{ij}\left[f_j\left(\frac{x_j}{a_j}\right) - f_j\left(\frac{y_j}{a_j}\right)\right]\Bigg\}\Bigg| \end{aligned}\right\}, i = 1, 2, \cdots, n. \tag{3.8}$$

We can estimate the right side of (3.8) by using (3.5) and Lemma 2.1 so that

(3.9)
$$\left.\begin{aligned} |F_i(x) - F_i(y)| \le &\sum_{j=1}^{n} a_{ij}\frac{L_j}{a_j}|x_j - y_j| + |b_i|\sum_{j=1}^{n}|b_{ij}|\frac{L_j}{a_j}|x_j - y_j| \\ &+ |c_i|\sum_{j=1}^{n}|c_{ij}|\frac{L_j}{a_j}|x_j - y_j| + c\frac{|B_i|}{\alpha_i}\Bigg\{|\beta_i|\frac{L_i}{a_i}|x_i - y_i| \\ &+ |\gamma_i|\sum_{j=1}^{n}|\gamma_{ij}|\frac{L_j}{a_j}|x_j - y_j| + |\delta_i|\sum_{j=1}^{n}|\delta_{ij}|\frac{L_j}{a_j}|x_j - y_j|\Bigg\}; \end{aligned}\right\}, \ i = 1, 2, \cdots, n;$$

we have directly from the above,

$$\left.\begin{aligned} \sum_{i=1}^{n}|F_i(x)-F_i(y)| \leq & \sum_{i=1}^{n}\Bigg[\sum_{j=1}^{n}|a_{ji}|+\sum_{j=1}^{n}|b_j||b_{ji}|+\sum_{j=1}^{n}|c_j||c_{ji}| \\ & + c\frac{|B_i|}{\alpha_i}\Big\{|\beta_i|+\sum_{j=1}^{n}|\gamma_j||\gamma_{ji}|+\sum_{j=1}^{n}|\delta_j||\delta_{ji}|\Big\}\Bigg]\frac{L_i}{a_i}|x_i-y_i| \\ \leq & \, p\sum_{i=1}^{n}|x_i-y_i| \end{aligned}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned} p = \max_{1\leq i\leq n}\Bigg[\frac{L_i}{a_i}\Bigg(&\sum_{j=1}^{n}\Big\{|a_{ji}|+|b_j||b_{ji}|+|c_j||c_{ji}|\Big\} \\ &+ c\frac{|B_i|}{\alpha_i}\Big\{|\beta_i|+\sum_{j=1}^{n}(|\gamma_j||\gamma_{ji}|+|\delta_j||\delta_{ji}|\Big\}\Bigg)\Bigg] \end{aligned}\right\}. \tag{3.11}$$

By hypothesis $p < 1$ and therefore it follows from (3.10) that the map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a contraction and by the contraction mapping principle, there exists a unique fixed point of the map F say $u^* = (u_1^*, u_2^*, \cdots, u_n^*)$ so that $u^* = F(u^*)$ or equivalently that (3.12)

$$\left.\begin{aligned} u_i^* = &\sum_{j=1}^{n}a_{ij}f_j\left(\frac{u_j^*}{a_j}\right)+b_i\bigwedge_{j=1}^{n}b_{ij}f_j\left(\frac{u_j^*}{a_j}\right)+c_i\bigvee_{j=1}^{n}c_{ij}f_j\left(\frac{u_j^*}{a_j}\right)+J_i \\ &+c\frac{B_i}{\alpha_i}\Big\{\beta_i f_i\left(\frac{u_i^*}{a_i}\right)+\gamma_i\bigwedge_{j=1}^{n}\gamma_{ij}f_j\left(\frac{u_j^*}{a_j}\right)+\delta_i\bigvee_{j=1}^{n}\delta_{ij}f_j\left(\frac{u_j^*}{a_j}\right)\Big] \end{aligned}\right\}, i = 1, 2, \cdots, n.$$

If we now define

$$\alpha_i v_i^* = c\Bigg[\beta_i f_i\left(\frac{u_i^*}{a_i}\right)+\gamma_i\bigwedge_{j=1}^{n}\gamma_{ij}f_j\left(\frac{u_i^*}{a_i}\right)+\delta_i\bigvee_{j=1}^{n}\delta_{ij}f_j\left(\frac{u_i^*}{a_i}\right)\Bigg]$$

then $(a_1u_1^*, a_2u_2^*, \cdots, a_nu_n^*, v_1^*, \cdots, v_n^*)$ is an unique equilibrium solution of (3.1) and this completes the proof. □

4. EXPONENTIAL STABILITY

If $(u_1^*, u_2^*, \cdots, u_n^*, v_1^*, v_2^*, \cdots, v_n^*)$ is an equilibrium of the system (2.3) then we have from (2.3)

(4.1)

$$\left.\begin{aligned}\frac{d}{dt}\big[u_i(t)-u_i^*\big] &= -\,a_i\big[u_i(t)-u_i^*\big]+\sum_{j=1}^{n}a_{ij}\big[f_j(u_j(t))-f_j(u_j^*)\big]\\ &\quad+B_i\big[v_i(t)-v_i^*\big]+b_i\bigwedge_{j=1}^{n}b_{ij}\big[f_j(u_j(t))-f_j(u_j^*)\big]\\ &\quad+c_i\bigvee_{j=1}^{n}c_{ij}\big[f_j(u_j(t))-f_j(u_j^*)\big]\\ \frac{d}{dt}\big[v_i(t)-v_i^*\big] &= -\,\alpha_i\big[v_i(t)-v_i^*\big]+\Big\{\beta_i\big[f_i(u_i(t))-f_i(u_i^*)\big]\\ &\quad+\gamma_i\bigwedge_{j=1}^{n}\gamma_{ij}\big[f_j(u_j(t))-f_j(u_j^*)\big]+\delta_i\bigvee_{j=1}^{n}\delta_{ij}\big[f_j(u_j(t))-f_j(u_j^*)\big]\Big\}c\end{aligned}\right\},$$

where $i=1,2,\cdots,n,\ t>0$. By using the upper right derivatives, the Lipschitzian nature of the activationss $f_j, j=1,2,\cdots,n$ and the Lemma 2.1 it is not difficult to derive that

(4.2)

$$\left.\begin{aligned}\frac{d}{dt}\Big|u_i(t)-u_i^*\Big| &\le -\,a_i\,|u_i(t)-u_i^*|+\sum_{j=1}^{n}|a_{ij}|L_j|u_j(t)-u_j^*|+|B_i|L_i|v_i(t)-v_i^*|\\ &\quad+|b_i|\sum_{j=1}^{n}|b_{ij}|L_j|u_j(t)-u_j^*|+|c_i|\sum_{j=1}^{n}|c_{ij}|L_j|u_j(t)-u_j^*|;\\ \frac{d}{dt}\Big|v_i(t)-v_i^*\Big| &\le -\,\alpha_i\,|v_i(t)-v_i^*|+|\beta_i|cL_i|u_i(t)-u_i^*|\\ &\quad+|\gamma_i|c\sum_{j=1}^{n}|\gamma_{ij}|L_j|u_j(t)-u_j^*|+|\delta_i|c\sum_{j=1}^{n}|\delta_{ij}|L_j|u_j(t)-u_j^*|\end{aligned}\right\},$$

where $i=1,2,\cdots,n,\ t>0$. We can now formulate our result on the exponential stability of the equilibrium $(u_i^*,v_i^*),\ i=1,2,\cdots,n$.

Theorem 4.1. *Suppose the activation functions f_j, $j=1,2,\cdots,n$ are globally Lipschitzian with constants $L_j, j=1,2,\cdots,n$ respectively. If furthermore,*

(4.3)

$$\left.\begin{aligned}&a>L_i\Big\{\sum_{j=1}^{n}\Big(|a_{ji}|+|b_j||b_{ji}|+|c_j||c_{ji}|\Big)\ +\Big[|\beta_i|+\sum_{j=1}^{n}\Big(|\gamma_j||\gamma_{ji}|+|\delta_j||\delta_{ji}\Big)\Big]c\Big\}\\ &\alpha_i>|B_i|\end{aligned}\right\},$$

where $i=1,2,\cdots,n$; then the system (2.3) has a unique equilibrium (u_i^*,v_i^*) which is globally exponentially stable in the sense that if $(u_i(t),v_i(t)$ is any solution of (2.3) *then there exist positive numbers K and μ such that*

$$\sum_{i=1}^{n}\Big(|u_i(t)-u_i^*|+|v_i(t)-v_i^*|\Big)\le Ke^{-\mu t},\quad t\ge 0. \tag{4.4}$$

Proof. The inequalities in (4.3) together imply that

$$\left.\begin{aligned} a_i > L_i\Big\{&\sum_{j=1}^{n}\Big(|a_{ji}|+b_j||b_{ji}+|c_j||c_{ji}|\Big)\\ &+\sum_{j=1}^{n}\Big(|\gamma_j||\gamma_{ji}|+|\delta_j||\delta_{ji}|\Big)c+\frac{|\beta_i||B_i|}{\alpha_i}\Big\}\end{aligned}\right\},\ i=1,2,\cdots,n \tag{4.5}$$

and hence it follows from Theorem 3.1 that the system (2.3) has a unique equilibrium. Let

$$\begin{aligned} p=\min_{1\le i\le n}\Big[a_i-L_i\Big\{&\sum_{j=1}^{n}\Big(|a_{ji}|+|b_j||b_{ji}|+|c_j||c_{ji}|\Big)\\ &+c|\beta_i|+\sum_{j=1}^{n}\Big(|\gamma_j||\gamma_{ji}|+|\delta_j||\delta_{ji}|\Big)c\Big\}\Big]\end{aligned} \tag{4.6}$$

$$q=\min_{1\le i\le n}\big[\alpha_i-|B_i|\big]. \tag{4.7}$$

Let

$$r=\min\{p,q\} \tag{4.8}$$

and note that there is a positive number say λ such that $0<\lambda<r$. We consider a Lyapunov function $V(t)=V(x,y)(t)$ defined by

$$V(t)=e^{\lambda t}\sum_{i=1}^{n}\Big[|u_i(t)-u_i^*|+|v_i(t)-v_i^*|\Big]. \tag{4.9}$$

One can calculate the upper right derivative of V along the solutions of (2.3) and estimate it by using (4.2) to obtain

$$\begin{aligned}\frac{dV(t)}{dt}&=\lambda V(t)+e^{\lambda t}\sum_{i=1}^{n}\Big[\frac{d}{dt}|u_i(t)-u_i^*|+\frac{d}{dt}|v_i(t)-v_i^*|\Big]\\ &\le \lambda V(t)+e^{\lambda t}\sum_{i=1}^{n}\Big[-a_i+L_i\sum_{j=1}^{n}\Big(|a_{ji}|+|b_j||b_{ji}|+|c_j||c_{ji}|\Big)\Big]|u_i(t)-u_i^*|\\ &\quad +|B_i||v_i(t)-v_i^*|e^{\lambda t}\end{aligned} \tag{4.10}$$

$$+e^{\lambda t}\sum_{i=1}^{n}\Big[-\alpha_i|v_i(t)-v_i^*|+\Big\{|\beta_i|L_i+\sum_{j=1}^{n}|\gamma_j||\gamma_{ji}|+|\delta_j||\delta_{ji}|\Big\}L_i|u_i(t)-u_i^*|\Big] \tag{4.11}$$

$$\le \lambda V(t)-re^{\lambda t}\sum_{i=1}^{n}\Big[|u_i(t)-u_i^*|+|v_i(t)-v_i^*|\Big]$$

$$\leq -(r-\lambda)V(t) \leq 0. \tag{4.12}$$

It follows from the above that

$$V(t) \leq V(0)$$

and hence

$$\sum_{i=1}^{n}\Big[|u_i(t)-u_i^*| + |v_i(t)-v_i^*|\Big] \leq V(0)e^{-\lambda t}, \qquad t \geq 0 \tag{4.13}$$

and this completes the proof. □

We consider briefly the asymptotic behaviour of the learning dynamics governed by the equation

$$\begin{aligned}\frac{dm_{ij}(t)}{dt} = -\alpha_i m_{ij}(t) + \Big[&\beta_i f_i(u_i(t)) + \gamma_i \bigwedge_{k=1}^{n} \beta_{ik} f_k(u_k(t)) \\ &+ \delta_i \bigvee_{k=1}^{n} \gamma_{ik} f_k(u_k(t))\Big] p_j, \quad i,j = 1,2,\cdots,n,\ t > 0.\end{aligned} \tag{4.14}$$

Under the conditions of the previous theorem, the equation (4.14) has a unique equilibrium m_{ij}^* given by

$$m_{ij}^* = \Big[\beta_i f_i(u_i^*) + \gamma_i \bigwedge_{k=1}^{n} \beta_{ik} f_k(u_k^*) + \delta_i \bigvee_{k=1}^{n} \gamma_{ik} f_j(u_k^*)\Big] p_j,\ i,j = 1,2,\cdots,n. \tag{4.15}$$

We have from (4.14) and (4.15) that

$$\begin{aligned}\frac{d}{dt}\Big[m_{ij}(t) - m_{ij}^*\Big] = -\alpha_i\Big[m_{ij}(t) - m_{ij}^*\Big] + p_j\Big[&\beta_i\Big(f_i(u_i(t)) - f_i(u_i^*)\Big) \\ &+ \gamma_i \bigwedge_{k=1}^{n} \beta_{ik}\Big(f_k(u_k(t)) - f_k(u_k^*)\Big) \\ &+ \delta_i \bigvee_{k=1}^{n} \delta_{ik}\Big(f_k(u_k(t)) - f_k(u_k^*)\Big)\Big], \qquad i,j = 1,2,\cdots,n,\ t > 0.\end{aligned} \tag{4.16}$$

It is found from (4.16) that

$$\begin{aligned}\frac{d}{dt}\Big|m_{ij}(t) - m_{ij}^*\Big| &\leq -\alpha_i|m_{ij}(t) - m_{ij}^*| + |p_j||\beta_i|L_i|u_i(t) - u_i^*| \\ &\quad + |\gamma_i||p_j|\sum_{k=1}^{n}|\beta_{ik}|L_k|u_k(t) - u_k^*| + |\delta_i||p_j|\sum_{k=1}^{n}|\delta_{ik}|L_k|u_k(t) - u_k^*| \qquad (4.17) \\ &\leq -\alpha_i|m_{ij}(t) - m_{ij}^*| \\ &\quad + |p_j|\Big(|\beta_i|L_i + |\gamma_i|\sum_{k=1}^{n}|\beta_{ik}|L_k + |\delta_i|\sum_{k=1}^{n}|\delta_{ik}|L_k\Big)V(0)e^{-\lambda t},\ t > 0. \qquad (4.18)\end{aligned}$$

One can put the above equation in the form

$$\frac{d}{dt}|m_{ij}(t) - m^*_{ij}| \le -\alpha_i|m_{ij}(t) - m^*_{ij}| + Q_{ij}V(0)e^{-\lambda t},\ t > 0,\ i, j = 1, 2, \cdots, n \tag{4.19}$$

for some suitable positive constants $Q_{ij},\ i, j = 1, 2, \cdots, n$. We note that $0 < \lambda < r < \alpha_i$ and hence it follows that

$$\begin{aligned} |m_{ij}(t) - m^*_{ij}| &\le |m_{ij}(0) - m^*_{ij}|e^{-\alpha_i t} + \int_0^t e^{-\alpha_i(t-s)}Q_{ij}V(0)e^{-\lambda s}\,ds \\ &\le |m_{ij}(0) - m^*_{ij}|e^{-\alpha_i t} + Q_{ij}V(0)e^{-\lambda t},\ i, j = 1, 2, \cdots, n,\ t > 0 \\ &\le \Big[|m_{ij}(0) - m^*_{ij}| + Q_{ij}V(0)\Big]e^{-\lambda t}, i, j = 1, 2, \cdots, n,\ t > 0 \qquad (4.20) \end{aligned}$$

Thus we can conclude that the time dependent and learnable synaptic weights converge exponentially to the time independent weights m^*_{ij} encoding the signal vector $p = (p_1, p_2, \cdots, n)$ in the sense that

$$\lim_{t\to\infty} m_{ij}(t) = m^*_{ij} = \frac{p_j}{\alpha_i}\Big[\beta_i f_i(u^*_i) + \gamma_i \bigwedge_{k=1}^{n} \gamma_{ik} f_k(u^*_k) + \delta_i \bigvee_{k=1}^{n} \delta_{ik} f_k(u^*_k)\Big],\ i, j = 1, 2, \cdots, n.$$

5. CONCLUDING REMARKS

We have proposed a network of somatically crisp and synaptically fuzzy neurons which learn an externally input signal vector by means of an unsupervised Hebbian-type learning algorithm incorporating a forgetting term. The neurons are somatically crisp in the sense that their intrinsic parameters such as the decay rate of its state and its output are crisp while some of their synapses on receiving crisp signals produce and send fuzzily computed signals to the neurons' soma for somatic processing. The other possibility is for a neuron to be somatically fuzzy so that their parameters are rule based leading to rule based fuzzy systems. Such systems are considered recently by Huang et al. [2005] where the relevance of the premise variables to the model is not clear.

In our analysis we have derived sufficient conditions for the existence of a unique exponentially stable equilibrium of the combined system of neuronal activations and synaptic modifications. Our model is a generalization of other existing models in the sense that if the learning component is removed, it will reduce to a fuzzy network and if the fuzziness is removed, it will reduce to a non-fuzzy network. One can further generalize our model by the incorporation of discrete or continuous time delays and this can be done in routine way by using Lyapunov functions or functionals. A physiological interpretation of our sufficient condition is that the self-regulating and stabilizing effect of the neurons have to dominate other effects in the dynamics of the neuron and the forgetting coefficient should be strong enough to quickly learn (or converge) to an equilibrium and a failure to do so will imply inability to learn or slow learning. One can incorporate some time delays in our model and if there are no time delays in the stabilizing negative feedback terms, then our stability conditions should be robust to provide stability under time delays in processing and transmission.

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