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DIFFERENTIAL INCLUSIONS OPERATOR VALUED MEASURES AND OPTIMAL CONTROL

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ABSTRACT. The objective of this paper is to briefly summarize some recent results on Differential Inclusions and their optimal control. Then, using vector measures as controls, we present some new results on the necessary conditions of optimality. Further, we consider systems having structural perturbation modeled by operator valued measures. The paper is concluded indicating some open problems.

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1. INTRODUCTION

Differential inclusions are abstract representations of control systems, deterministic variational inequalities and the so called "uncertain systems". For example an uncertain system may be governed by a differential inclusion given by

$$\dot{x} \in Ax + G(t, x), x(0) = \xi \tag{1}$$

where A is the infinitesimal generator of a C_0-semi group of operators $S(t), t \geq 0$, in a Banach space E and $G : I \times E \longrightarrow 2^E \setminus \emptyset$ is a suitable multi valued map determined by a single valued map g dependent on an unknown parameter α taking values from a known subset Λ of a topological space Σ. The multifunction G may be given by

$$G(t, x) \equiv \{g(t, x, \alpha), \alpha \in \Lambda\} \equiv g(t, x, \Lambda)$$

where Λ is the set of uncertainty. In other words, the range of the system parameters is known but its actual values in force are unknown. This is known as parametric uncertainty.

Similarly for a control system

$$G(t, x) \equiv \{g(t, x, v), v \in U(t, x)\} \equiv g(t, x, U(t, x))$$

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where $U(t,\xi)$ is a suitable multi function from $I \times E$ to $2^U \setminus \emptyset$ where U is a metric space representing the space of controls and $g : I \times E \times U \longrightarrow E$ is a Borel measurable map.

Another interesting problem arises when the solution is required to satisfy certain algebraic constraints. Let E, F be any two Banach spaces. Consider the differential-algebraic system given by

$$\dot{x} = Ax + f(x,y), \;\; h(x,y) = 0 \tag{2}$$

where $f : E \times F \longrightarrow E, \;\; h : E \times F \longrightarrow F$. Define the multifunctions

$$H(x) \equiv \{y \in F : h(x,y) = 0\} \neq \emptyset \text{ and } G(x) \equiv f(x, H(x)).$$

Then the differential-algebraic system given above is equivalent to the system governed by the differential inclusion

$$\dot{x} \in Ax + G(x). \tag{3}$$

Similarly, systems governed by parabolic variational inequalities in Hilbert spaces have equivalent formulation as differential inclusions. For example, a parabolic variational inequality is described by

$$(\dot{x}(t) - Ax(t) - f(t), x(t) - y)_{V^*,V} \leq \Phi(y) - \Phi(x(t)), a.e\ t \in I, \forall y \in V \tag{4}$$

where $A \in \mathcal{L}(V, V^*)$ is coercive and $V \hookrightarrow H \hookrightarrow V^*$ is the Gelfand triple with continuous and dense embeddings, and Φ is a proper lower semi continuous convex function defined on V taking values from the extended real number system. This variational formulation is equivalent to the differential inclusion

$$\dot{x}(t) \in -Ax(t) + G(t, x(t)), t \in I, \tag{5}$$

on the Hilbert space H where

$$G(t,\xi) \equiv f(t) - \partial\Phi(\xi)$$

with $\partial\Phi$ denoting the subdifferential of Φ.

Another interesting example comes from time optimal control problems for linear systems where the control may turn out to be a discontinuous function of the state. For example, consider the system

$$\dot{x} = Ax + Bu, x(0) = x_0$$

where A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, in a Hilbert space H and B is a bounded linear operator, $B \in \mathcal{L}(R^n, H)$, where the controls are finite dimensional. Given that u must take values from the unit cube $U \equiv \{v \in R^n : |v_i| \leq 1, i = 1, 2, \cdots, n\}$ and that the pair $\{x_0, 0\} \in H$ are controllable, the problem is to find a control that transfers the system from state x_0 to the state 0 in minimum time. The optimal control may turn out to be a bang-bang control (well known in the finite dimensional context) and the corresponding feedback control may turn out to be a discontinuous function of the state, $u^o(t) = g(x^o(t))$ with $g : H \longrightarrow U$. In this case we have a feedback system

$$\dot{x} = Ax + Bg(x)$$

with discontinuous righthand side. By lifting g from single valued discontinuous map to an equivalent multifunction G we arrive at a differential inclusion

$$\dot{x} \in Ax + BG(x). \tag{6}$$

These examples illustrate very well the generality of differential inclusions. Because of its generality and hence the prospect of applicability in diverse fields of physical and social sciences, great many workers in the field have contributed much to bring it to the present level of depth and understanding. Both deterministic and stochastic systems on finite and infinite dimensional spaces have been considerably developed including application to systems and control theory. For deterministic systems and their optimal control see [1,3,5,6,10,13,14,16,31] and for stochastic systems see [2,12,24,25,26,27,28,29,30]. For controllability problems see [26] and stochastic viability problems see [30] and the extensive references therein.

The rest of the paper is organized as follows. In section 2, some basic notions and notations, used throughout the paper, are presented. In section 3, questions of existence of solutions for systems governed by Differential Equations and Inclusions are considered. Regularity properties of solutions are also discussed. Using the results of sections 3, questions of existence of optimal controls are treated in section 4. Here two main existence results are presented. In section 5 we present some new results on the necessary conditions of optimality for systems governed by differential inclusions. In the concluding section we discuss some open problems in regards to structural controls where operator valued measures are the controls.

2. PRELIMINARIES

Function Spaces: Let D be an arbitrary nonempty set furnished with a σ-algebra $\mathcal{B}_D \equiv \mathcal{B}$ of subsets of the set D and suppose that E is a separable Banach space. Let $B(D, E)$ denote the space of bounded Borel measurable functions on D with values in E. Furnished with the sup norm topology, this is a Banach space. If D is also a metric space, then $C(D, E)$, denoting the space of bounded continuous functions on D with values in E and furnished with the sup norm topology, is again a Banach space. Note that $C(D, E)$ can not be dense in $B(D, E)$. Similarly let $L_1(I, E)$ denote the Banach space of all Lebesgue-Bochner integrable functions on I with values in E.

Multifunctions: Let $(\Omega, \mathcal{B})$ be an arbitrary measurable space and $\mathcal{Z}$ a Polish space. A multi function $G : \Omega \longrightarrow 2^{\mathcal{Z}} \backslash \emptyset$ is said to be measurable (weakly measurable) if for every closed (open) set $C \subset \mathcal{Z}$ the set

$$G^{-1}(C) \equiv \{\omega \in \Omega : G(\omega) \cap C \neq \emptyset\} \in \mathcal{B}.$$

Since $\mathcal{Z}$ is a Polish space, it is metrizable by a metric d with respect to which it is a complete separable metric space. It is known that measurability of the multi function G is equivalent to the measurability of the scalar valued function $\omega \rightarrow d(x, G(\omega))$ for every $x \in \mathcal{Z}$. Even more, it is also equivalent to the graph measurability of G in the sense that

$$\{(x, \omega) \in \mathcal{Z} \times \Omega : x \in G(\omega)\} \in \mathcal{B}(\mathcal{Z}) \times \mathcal{B}$$

where $\mathcal{B}(\mathcal{Z})$ denotes the sigma algebra of Borel sets of $\mathcal{Z}$. Let X, Y be any two topological spaces and $G : X \longrightarrow c(Y)$ be a multi function. G is said to be upper semi continuous (USC) if for each set $C \in c(Y)$

$$G^{-1}(C) \equiv \{x \in X : G(x) \cap C \neq \emptyset\} \in c(X).$$

This is equivalent to the statement : G is upper semi continuous on X if for every $x \in X$ and every open set $V \supset G(x)$ there exists an open set $U \ni x$ in X such that $G(\xi) \subset V$ for all $\xi \in U$. In case G is a single valued map, this is just the definition of continuity.

If Y is a metric space with metric d, we can introduce a metric d_H on $c(Y)$, called the Hausdorff metric, as follows:

$$d_H(K, L) \equiv \max\big\{\sup\{d(k, L), k \in K\}, \sup\{d(K, \ell), \ell \in L\}\big\}$$

where $d(x, K) \equiv \inf\{d(x, y), y \in K\}$ is the distance of x from the set K. If Y is a complete metric space then $(c(Y), d_H)$ is also a complete metric space. We shall use the following notations: $b(Y), c(Y), cc(Y), cbc(Y), k(Y)$ to denote the class of nonempty bounded, closed, closed convex, closed bounded convex, and compact subsets of Y respectively whenever they are defined.

Measures of Noncompactness Let E be any Banach space and $b(E)$ the family of nonempty bounded subsets of E. The map $\alpha : b(E) \longrightarrow [0, \infty]$ defined by

$$\begin{aligned} \alpha(B) \quad &\equiv \inf\{d > 0 : B \text{ admits a finite cover by} \\ &\qquad \text{sets having diameter not exceeding } d\} \end{aligned}$$

is called the Kuratowski measure of noncompactness.

Similarly, the ball (Hausdorff) measure of noncompactness is defined by

$$\beta(B) \equiv \inf\{r > 0 : B \text{ admits a finite cover by balls of radius not exceeding } r\}.$$

It is easy to verify that (1): $\alpha(\Gamma) = 0$ if and only if $\overline{\Gamma}$ is compact. (2): $\alpha(cB) = |c|\alpha(B)$ (3): $\alpha(\Gamma_1 + \Gamma_2) \leq \alpha(\Gamma_1) + \alpha(\Gamma_2)$ (4): $\Gamma_1 \subset \Gamma_2 \Rightarrow \alpha(\Gamma_1) \leq \alpha(\Gamma_2)$ (5) $\alpha(\Gamma) =$

$\alpha(co\Gamma), \alpha(\Gamma) = \alpha(\overline{\Gamma})$ (6): α is continuous with respect to the Hausdorff metric d_H (7): for any $e \in E$, $\alpha(B_r(e)) = 2r, \beta(B_r(e)) = r$.

Definition 2.1. A map $G : b(E) \longrightarrow b(E)$ is said to be an α contraction if there exists a $k \in (0, 1)$ such that

$$\alpha(G(B)) \leq k\alpha(B), \forall\ B \in b(E)$$

and it is said to be α-condensing if

$$\alpha(G(B)) < \alpha(B)\ \forall\ B \in b(E).$$

Vector Measures: Let F be a Banach space and let $\mathcal{M}_c(I, F)$ denote the space of bounded countably additive vector measures on the sigma algebra $\mathcal{B}$ of subsets of the set $I \subset R_0 \equiv [0, \infty)$ with values in the Banach space F. This is furnished with the total variation norm. That is, for each $\mu \in \mathcal{M}_c(I, F)$, we write

$$|\mu| \equiv |\mu|(I) \equiv \sup_{\pi}\{\sum_{\sigma\in\pi} \parallel \mu(\sigma) \parallel_F\}$$

where the supremum is taken over all partitions π of the interval I into a finite number of disjoint members of $\mathcal{B}$. With respect to this topology, $\mathcal{M}_c(I, F)$ is a Banach space. For any $\sigma \in \mathcal{B}$ define the variation of μ on σ by

$$V(\mu)(\sigma) \equiv V(\mu, \sigma) \equiv |\mu|(\sigma).$$

Since μ is countably additive and bounded, this defines a countably additive bounded positive measure on $\mathcal{B}$. In case $F = R$, the real line, we have the space of real valued signed measures. We denote this by simply $\mathcal{M}_c(I)$ in place of $\mathcal{M}_c(I, R)$. Clearly for $\nu \in \mathcal{M}_c(I)$, $V(\nu)$ is also a countably additive bounded positive measure. For uniformity of notation we use λ to denote the Lebesgue measure.

Measurable Selections: Let $H : I \longrightarrow 2^E \setminus \emptyset$ be a multi function from I to the class of nonempty subsets of the Banach space E. We use $\mathcal{S}_H^1$ to denote the class of $L_1(I, E)$ selections of H, that is

$$\mathcal{S}_H^1 \equiv \big\{f \in L_1(I, E) : f(t) \in H(t)\ a.e \text{ on } I\big\}.$$

For any set K, coK denotes the convex hull of K and $clcoK$ denotes the closed convex hull of K. Let E and F be any pair of Banach spaces and $\mathcal{L}(E, F)$ the space of bounded linear operators from E to F.

For the system dynamics, let the Banach spaces E, U denote the state space and the control space respectively. We shall introduce additional notations in the sequel as required.

3. DIFFERENTIAL INCLUSIONS AND THEIR CONTROL

We have seen in the introduction that differential inclusions provide a general frame work for study of a wide variety of dynamical systems and their control. Naturally it has broad application in physical and engineering sciences. We will briefly mention both deterministic and stochastic control and optimization problems where it has been widely used. We must mention that the subject of differential inclusions has been studied by many workers in the field over last four and half decades and still continues to attract interest of both abstract and applied mathematicians. However control theory for such systems is rather recent possibly a little more than two decades. There are many interesting results in this area. It is not possible to present them in any details in a brief account. However it may be useful to some readers to indicate some of the major tools and techniques used in this area.

In addition to standard functional analysis, some of the major tools used are multi functions, upper and lower semi continuity of multi functions, Lipschitz continuity with respect to Hausdorff metric, measurable multi functions, measurable selection, L_p selections, lower closure theorems, Kuratowski and Cesari properties of multi valued maps, Fixed point theorems for multi valued maps, Kakutani-Ky Fan fixed point theorems, Kuratowski's measure of noncompactness and Condensing maps, Inward maps etc.

We will briefly mention only some of the results the author has been intimately familiar with. Here is one classical result.

Consider the system

$$\dot{x} \in F(t,x), t \in I \equiv [0,T], x(0) = \xi. \tag{7}$$

Theorem 3.1. *Let E be any Banach space and suppose $F : I \times E \longrightarrow cc(E)$ measurable in t on I and usc(upper smicontinuous) in x on E satisfying the following properties:*

(F1): $F(\cdot, x)$ has strongly measurable selections.

(F2): $\sup\{\| y \| : y \in F(t,x)\} \leq K(t)(1+ \| x \|), K \in L_1^+(I)$.

(F3): $\alpha(F(t,B)) \leq \ell(t)\alpha(B), \ell \in L_1^+(I) \; \forall \; B \in b(E)$.

Then system equation (7) has at least one solution $x \in AC(I,E)$.

Proof. We present only an outline. First note that by a solution of (7) one means that there is an $x \in AC(I,E)$ and a measurable selection v of the multifunction $t \to F(t,x(t))$, which is Lebesgue-Bochner integrable, such that

$$x(t) = \xi + \int_0^t v(s)ds, t \in I.$$

The proof is based on fixed point theorem for α condensing maps [21,p 506] applied to the multivalued map $\hat{F} : C(I,E) \longrightarrow cbc(C(I,E))$ given by

$$\hat{F}(z) \equiv \{y \in C(I,E) : y(t) = \xi + \int_0^t v(s)ds, v(t) \in F(t, z(t))\}.$$

This completes our outline. □

Remark 1. Note that the above result is free of Lipschitz assumption. In case F is Lipschitz, Banach fixed point theorems for multi valued maps can be used to obtain similar result for even more general systems. Here is an example involving unbounded operators.

Consider the system

$$\dot{x} \in Ax + F(t,x), x(0) = \xi.$$

We can prove the following result relatively easily.

Theorem 3.2. *Suppose the operator $A : D(A) \subset E \longrightarrow E$ is the infinitesimal generator of a C_0-semigroup of operators $S(t), t \geq 0$, in E and the multifunction F satisfy the following assumptions:*

(F1): $F : I \times E \longrightarrow cc(E)$ is measurable in t on I for each fixed $x \in E$, and, for almost all $t \in I$, it is upper semicontinuous (usc) on E

(F2): for every finite positive number r, there exists an $\ell_r \in L_1^+(I)$ such that

$$\inf\{\| z \|, z \in F(t,x)\} \leq \ell_r(t), \ \forall \ t \in I, \ \forall \ x \in B_r(E)$$

(F3): there exists an $\ell \in L_1^+(I)$ such that

$$d_H(F(t,x), F(t,y)) \leq \ell(t) \| x - y \|, \ \forall \ x, y \in E, t \in I.$$

Then, for each initial state $x(0) = \xi \in E$, the system has at least one solution $x \in C(I,E)$.

Proof. The proof is based on generalized Banach fixed point theorem for multivalued contraction maps. First define the affine map $\mathcal{R} : L_1(I,E) \longrightarrow C(I,E)$ by

$$(\mathcal{R}f)(t) \equiv S(t)\xi + \int_0^t S(t-r)f(r)dr, t \in I$$

and construct the multi valued map as follows: for each $f \in L_1(I,E)$ define the set

$$\hat{F}(f) \equiv \{g \in L_1(I,E) : g(t) \in F(t, (\mathcal{R}f)(t)), t \in I\}.$$

One must verify that $f \to \hat{F}(f)$ is a non trivial multifunction. This requires proof of measurability of the multifunction $t \longrightarrow F(t, (\mathcal{R})f(t))$ and then the proof of existence of measurable or more precisely L_1 selections. Here the assumptions (F1) and (F2) are required and theory of measurable selections. Once this is satisfied, one must verify that $\hat{F} : L_1(I,E) \longrightarrow cc(L_1(I,E))$ and that $\hat{F}$ is Lipschitz with respect to Hausdorff metric on $cc(L_1(I,E))$ and that it is a contraction on an equivalent space. The proof then follows from generalized Banach fixed point theorem for multi valued maps. For details see [5] where also impulsive or more generally measure driven systems and their optimal controls have been considered. $\square$

Existence and regularity properties of solutions for time varying controlled evolution inclusions of the form

$$\dot{x} + A(t)x \in F(t, x, \mu_t), x(0) = \xi, \tag{8}$$

where $A(\cdot)$ generates a transition operator of parabolic type and F is a multi valued map dependent on relaxed controls, have been treated in considerable detail in [1] (see also the references therein) proving existence of optimal relaxed controls. Here Kakutani-Ky Fan fixed point theorem was used. In [13,10] questions of existence and regularity properties of solutions for fully nonlinear problems, rather than semi linear problems as indicated above, given by

$$\dot{x} + A(t, x) \in f + G(t, x), x(0) = \xi, \tag{9}$$

was treated in the setting of Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ where the embeddings are continuous. Here the operator A is a nonlinear coercive operator, monotone and hemicontinuous. The multi valued map $G : I \times V \longrightarrow cc(H)$ is assumed to be measurable in $t \in I$ and weakly upper semicontinuous in $x \in V$ with respect to inclusion. Here the tools used are Galerkin approximation, monotonicity and hemicontinuity of A and upper semi continuity and measurable selections for G using Kuratowski-Cesari property. The solutions are contained in $L_\infty(I,H) \cap L_p(I,V)$ with $\dot{x} \in L_q(I,V^*)$ where $(1/p) + (1/q) = 1$, $p \geq 2$. Equivalently, $x \in W_{p,q} \equiv \{z \in L_p(I,V) : \dot{z} \in L_q(I,V^*)\}$. Furnished with the norm topology,

$$\| x \|_{W_{p,q}} \equiv \| x \|_{L_p(I,V)} + \| \dot{x} \|_{L_q(I,V^*)},$$

$W_{p,q}$ is a Banach space. It is known that the embedding $W_{p,q} \hookrightarrow C(I,H)$ is continuous [7]. It is also known that if the embedding $V \hookrightarrow H$ is compact then the embedding $W_{p,q} \hookrightarrow L_2(I,H)$ is also compact [21]. Note that even though G is more regular than the nonlinear single valued monotone operator A, the continuity requirement is rather stringent. Using the embedding facts as stated above and assuming stronger regularity for the multi valued map, $G : I \times \mathcal{V} \longrightarrow cc(H)$, one can relax the assumption of weak upper semi continuity by simple upper semi continuity. Here $V \hookrightarrow \mathcal{V} \hookrightarrow H$ with the embeddings assumed compact.

Questions of existence of solutions and existence of optimal relaxed controls for systems governed by measure driven differential inclusions of the form,

$$dx \in Axdt + f(t, x)\nu(dt) + G(t, x, \mu_t)dt, x(0) = \xi, \tag{10}$$

have been considered in [5] where the questions of existence and regularity properties of solutions are treated using generalized Banach fixed point theorem for multivalued maps mapping $L_1(I,E)$ to $cc(L_1(I,E))$.

Relaxed Controls: Let U be a compact Polish space and $\mathcal{M}_1 \equiv \mathcal{M}_1(U) \subset \mathcal{M}(U)$ the space of probability measures on U where $\mathcal{M}(U)$ is the space of Radom measures on U. Since the weak topology is metrizable, $\mathcal{M}_1$ furnished with the weak topology is also a compact Polish space. For the set of admissible controls we choose

$$\hat{\mathcal{U}}_{ad} \equiv L_\infty^w(I, \mathcal{M}_1)$$

which consists of weakly measurable functions on I with values in $\mathcal{M}_1$.

Now we are prepared to present an existence result that appeared in [5]. For this, the following assumptions were used. Let

$$G : I \times E \times \mathcal{M} \longrightarrow cc(X)$$

be a Borel measurable multifunction, linear in the last argument, satisfying the following assumptions:

(G1): for every $x \in E$ and $\mu \in \hat{\mathcal{U}}_{ad}$,

$$t \longrightarrow \hat{G}_\mu(t,x) \equiv G(t,x,\mu_t)$$

is a measurable set valued map with values from $cc(E)$,

(G2): for almost all $t \in I$, $x \longrightarrow \hat{G}_\mu(t,x)$ is continuous in the Hausdorff metric and there exists a $K \in L_1(I, R_+)$ such that

$$d_H(\hat{G}_\mu(t,x), \hat{G}_\mu(t,y)) \leq K(t) \parallel x - y \parallel_E, \ \forall\ x, y \in E.$$

(G3): there exists a $K \in L_1(I, R_+)$ independent of $\mu \in \hat{\mathcal{U}}_{ad}$ such that

$$\sup\{\parallel z \parallel_E : z \in \hat{G}_\mu(t,x)\} \leq K(t)\{1 + \parallel x \parallel_E\} \ \forall x \in E,$$

(G4): there exists an $h \in L_1(I, R_+)$, possibly dependent on μ, such that

$$\inf\{\parallel z \parallel_E : z \in \hat{G}_\mu(t,x)\} \leq h(t)\{1 + \parallel x \parallel_E\}, \ \forall\ x \in E.$$

Under these assumptions the following existence result was proved in [5].

Theorem 3.3. *Consider the system* (10) *driven by relaxed control* $\mu \in \hat{\mathcal{U}}_{ad}$. *Suppose* A *is the infinitesimal generator of a* C_0*-semigroup of operators on* E *and* $f : I \times E \longrightarrow \mathcal{L}(F,E)$ *and* $\nu \in \mathcal{M}_c(I,F)$ *and the multi function* $\hat{G}_\mu$ *satisfy the assumptions* (G1)-(G4). *Then for every initial state* $x_0 = \xi \in E$, *the evolution inclusion* (10) *has a nonempty set of (right continuous) solutions* $\hat{\mathcal{X}}(\mu)$ *which is a bounded subset of* $B_0(I,E)$.

Proof. See [5]. □

Also in [5] the questions of existence of optimal relaxed controls for several control problems including Lagrange and Bolza problems were considered.

4. STRUCTURALLY PERTURBED INCLUSIONS

Let E and U be separable Banach spaces with E denoting the state space and U the control space. Throughout this section we assume that E is a reflexive Banach space. Consider the system

$$dx \in Axdt + B(dt)x(t) + C(t, x(t))dt + \Gamma(t)u(dt), x(0+) = x_0, \tag{11}$$

where $A \in \mathcal{G}_0(E)$, $C : I \times E \longrightarrow 2^E \setminus \emptyset$ is a multi function, $\Gamma \in BM(I, \mathcal{L}(U, E))$ and B is an operator valued measure mapping

$$B : \Sigma \equiv \mathcal{B}(I) \longrightarrow \mathcal{L}(E)$$

and $u \in \mathcal{U}_{ad} \subset \mathcal{M}_c(I, U)$ where $\mathcal{U}_{ad}$ denotes the class of admissible controls.

We introduce the following assumptions for the multifunction $C : I \times E \longrightarrow cc(E)$:

(C1): for every $x \in E$, $t \longrightarrow C(t, x)$ is a measurable set valued map with values from $cc(E)$

(C2): for almost all $t \in I$, $x \longrightarrow C(t, x)$ is continuous in the Hausdorff metric and there exists a $K \in L_1(I, R_+)$ such that

$$d_H(C(t, x), C(t, y)) \leq K(t) \parallel x - y \parallel_E, \ \forall\, x, y \in E.$$

(C3): there exists a $K \in L_1(I, R_+)$ such that

$$\sup\{\parallel z \parallel_E : z \in C(t, x)\} \leq K(t)\{1+ \parallel x \parallel_E\} \ \forall x \in E,$$

(C4): there exists an $h \in L_1(I, R_+)$ such that

$$\inf\{\parallel z \parallel_E : z \in C(t, x)\} \leq h(t)\{1+ \parallel x \parallel_E\}, \ \forall\, x \in E.$$

Theorem 4.1. *Suppose $A \in \mathcal{G}_0(E)$ generating the semigroup $S(t), t \geq 0$, and $B \in \mathcal{M}_c(I, \mathcal{L}(E))$ a countably additive bounded ($\mathcal{L}(E)$-valued) vector measure having bounded total variation on I. The multifunction C satisfies the assumptions* (C1)-(C4) *and $\Gamma \in BM(I, \mathcal{L}(U, E))$ the space of bounded measurable operator valued functions. Then for every $x_0 \in E$ and $u \in \mathcal{M}_c(I, U)$ the system* (11) *has a nonempty set of mild solutions $X(u) \subset B(I, E)$.*

Proof. First consider the system

$$dy = Aydt + B(dt)y(t-), y(0) = x_0.$$

Under the assumptions on A and the operator valued measure B it follows from [4, Theorem 4.1] that this equation has a unique mild solution $y \in B(I, E)$ given by

$y(t) = U_B(t,0)x_0$ where $U_B(t,s), 0 \leq s \leq t \leq T$ is a strongly measurable evolution operator in E. This follows from the fact that the integral equation

$$y(t) = S(t)x_0 + \int_0^t S(t-s)B(ds)y(s-),\ t \in I$$

has a unique mild solution $y \in B(I,E)$. For any given $u \in \mathcal{U}_{ad}$, define the multifunction $\mathcal{N}_u$ on $L_1(I,E)$ by

$$\mathcal{N}_u(f) \equiv \{g \in L_1(I,E) : g(t) \in C(t,(L_u f)(t)), a.e\}$$

where L_u is the affine map as defined below:

$$(L_u f)(t) \equiv \gamma_u(t) + \int_0^t U_B(t,s)f(s)ds, t \in I$$

with

$$\gamma_u(t) \equiv U_B(t.0)x_0 + \int_0^t U_B(t,s)\Gamma(s)u(ds). \tag{12}$$

We have already seen that the question of existence of (mild) solutions for systems of the form (11) is equivalent to the question of existence of a fixed point of the multifunction $\mathcal{N}_u$. The proof that the set of fixed points of the multi function $\mathcal{N}_u$ is nonempty is precisely the same as in [5, Theorem 3.5] once we prove that the system

$$dx = Axdt + B(dt)x + f(t)dt + \Gamma(t)u(dt), x(0) = x_0 \tag{13}$$

has a unique mild solution in $B(I,E)$ for each $x_0 \in E$ and $f \in L_1(I,E)$. But this follows readily from the well known variation of constants formula giving

$$x(t) = (L_u f)(t) \equiv \gamma_u(t) + \int_0^t U_B(t,s)f(s)ds \tag{14}$$

where $\gamma_u(t)$ is as defined above. Thus under the assumptions (C1)-(C4), it follows from [5, Theorem 3.5] that the multifunction $\mathcal{N}_u$ is nonempty and that the set of fixed points, denoted by

$$Fix\ \mathcal{N}_u \equiv \{f \in L_1(I,E) : f \in \mathcal{N}_u(f)\},$$

is also nonempty. Hence for each $u \in \mathcal{U}_{ad}$, the system (11) has a nonempty set of solutions denoted

$$X(u) \equiv \{L_u(f), f \in Fix\ \mathcal{N}_u\} \subset B(I,E).$$

Since both the control measure $u(\cdot)$ and the operator valued measure $B(\cdot)$ are assumed to be countably additive having bounded total variation on I, they can have at most a countable set of atoms in I and hence the elements $\{x\}$ of the set $X(u)$ are actually piecewise continuous. This ends the brief outline of our proof. $\square$

Remark 2. If the set $\mathcal{U}_{ad} \subset \mathcal{M}_c(I,U)$ is bounded then

$$\mathcal{X} \equiv \bigcup \{X(u), u \in \mathcal{U}_{ad}\}$$

is also bounded. This is easy to verify. Let x^u be any mild solution of equation (13) and define $Z^u(t) \equiv \sup\{\| x^u(s) \|_E, 0 \leq s \leq t\}$. Then using the growth assumption (C3) for the multifunction C and the expressions (13) and (14), it follows from elementary computations that

$$Z^u(t) \leq \alpha(u) + M \int_0^t Z^u(s)\ \nu_B(ds), t \in I \tag{15}$$

where

$$\begin{aligned}
&\alpha(u) \equiv M\big\{\| x_0 \|_E + \| \Gamma \|_0 |u| + \| K \|_{L_1}\big\}, \\
&\nu_B(\sigma) \equiv \mu_B(\sigma) + \int_\sigma K(s)ds, \sigma \in \mathcal{B} \\
&\| \Gamma \|_0 \equiv \sup\{\| \Gamma(s) \|_{\mathcal{L}(U,E)}, s \in I\},
\end{aligned}$$

and $|u|$ is the total variation norm of the vector measure u, μ_B is the measure induced by the variation (in the uniform operator topology) of the operator valued measure $B(\cdot)$ and ν_B is the measure as defined above. Note that this is a countably additive bounded positive measure. Using generalized Gronwall inequality, it follows from (15) that

$$Z^u(t) \leq \alpha(u) \exp\{M\nu_B(I)\}.$$

Since $\mathcal{U}_{ad}$ is bounded $\sup\{\alpha(u) : u \in \mathcal{U}_{ad}\} < \infty$. Hence $\sup\{Z^u(t), u \in \mathcal{U}_{ad}\} < \infty$ for all $t \in I$ and thus $\mathcal{X}$ is a bounded subset of $B(I,E)$.

Theorem 4.2. *Under the assumptions of Theorem 4.1, for each $u \in \mathcal{M}_c(I,U)$, the solution set $X(u)$ is bounded; and further, if the semigroup $S(t), t > 0$, is compact, then this set is also sequentially closed. If in addition, the operator $\Gamma(t)$ is also compact for each $t \in I$ and $\mathcal{U}_{ad}$ is bounded, then the graph of X given by*

$$\mathcal{G}_r(X) \equiv \{(u,x) \in \mathcal{U}_{ad} \times B(I,E) : x \in X(u)\}$$

is closed with respect to the weak topology of vector measures $\mathcal{M}_c(I,U)$ and strong topology of $B(I,E)$.

Proof. As seen in the preceding remark, the set $X(u)$ is a bounded subset of $B(I,E)$. Hence there exists a finite positive number b_u such that $\| x \|_{B(I,E)} \leq b_u$ for all $x \in X(u)$. Let $x_n \in X(u)$ and suppose $x_n \xrightarrow{s} x^o$. We verify that $x^o \in X(u)$. Since $x_n \in X(u)$ there exists a sequence $\{f_n\} \in Fix\mathcal{N}_u$ such that $f_n(t) \in C(t, x_n(t))$ a.e or equivalently

$$\begin{aligned}
x_n(t) = S(t)x_0 &+ \textstyle\int_0^t S(t-s)\Gamma(s)u(ds) + \int_0^t S(t-s)B(ds)x_n(s-) \\
&+ \textstyle\int_0^t S(t-s)f_n(s)ds.
\end{aligned} \tag{16}$$

Again by the assumption (C3) and the boundedness of the set $X(u)$ we have

$$\| f_n(t) \|_E \leq (1+b_u)K(t) \ a.e \ t \in I$$

for all $n \in N$. This implies that the family $\{f_n\}$ is contained in a bounded subset of $L_1(I,E)$ and it is also uniformly integrable. Since E is a reflexive Banach space it follows from Dunford-Pettis theorem that this set is relatively weakly sequentially compact. Hence there exists a subsequence of the sequence $\{f_n\}$, relabeled as $\{f_n\}$, and an element $f^o \in L_1(I,E)$ such that $f_n \xrightarrow{w} f^0$. Thus we have

$$x_n(t) \xrightarrow{s} x^o(t) \quad hboxinE$$
$$f_n \xrightarrow{w} f^o \quad hboxinL_1(I,E).$$

Hence it follows from lower closure theorem [5, Lemma 4.1] that $f^o(t) \in C(t, x^o(t))$, a.e $t \in I$. Using the expression (14) and compactness of the semigroup $S(t)$, $t > 0$, one can easily verify that $x^o = L_u f^o$ and hence $f^o(t) \in C(t, (L_u f^o)(t))$ for almost all $t \in I$. This implies that $x^o \in X(u)$ and hence $X(u)$ is a closed subset of $B(I,E)$. We now prove that the graph of this multifunction is closed. Take any sequence $(u_n, x_n) \in \mathcal{G}_r(X)$, that is, $x_n \in X(u_n)$ for all $n \in N$ and suppose

$$u_n \xrightarrow{w} u^o \text{ in } \mathcal{U}_{ad}$$
$$x_n \xrightarrow{s} x^o \text{ in } B(I,E).$$

We must show that $x^o \in X(u^o)$. By definition of solution, there exists a sequence $f_n \in L_1(I,E)$ such that for almost all $t \in I$ we have

$$f_n(t) \in C(t, x_n(t)) \text{ and } x_n(t) = (L_{u_n} f_n)(t)).$$

Since $\mathcal{U}_{ad}$ is bounded it follows from the preceding remark that $\{x_n\}$ is contained in a bounded subset of $B(I,E)$ and consequently it follows from the assumption (C3) that the set $\{f_n\} \subset L_1(I,E)$ is bounded and uniformly integrable. Hence, again by Dunford-Pettis theorem one can extract a subsequence, relabeled as the original sequence, so that $f_n \longrightarrow f^o$ weakly in $L_1(I,E)$ and by virtue of assumption on compactness of the operator valued function Γ and the semigroup S, one can verify that $x^o = L_{u^o} f^o$. Since, by the lower closure theorem, $f^o(t) \in C(t, x^o(t)), a.e \ t \in I$ we have $x^o \in X(u^o)$ proving that the graph $\mathcal{G}_r(X)$ is closed. □

Control Problem: We wish to consider the following control problem. Define the functional

$$\Upsilon(u,x) \equiv \int_0^T \ell(t, x(t))dt + \Psi(x(T)) + \varphi(u), x \in X(u) \tag{17}$$

where $X(u)$ denotes the family of solutions of the Differential inclusion (11) corresponding to the control $u \in \mathcal{U}_{ad}$. Define the objective functional as

$$J_o(u) \equiv \sup\{\Upsilon(u,x) : x \in X(u)\}. \tag{18}$$

The problem is to find a control that minimizes the functional $J_o(u)$, that is a control $u^o \in \mathcal{U}_{ad}$ such that

$$J_o(u^o) \leq J_o(u) \ \forall \ u \in \mathcal{U}_{ad}. \tag{19}$$

This is equivalent to the games problem: $\inf_{u \in \mathcal{U}_{ad}} \sup_{x \in X(u)} \Upsilon(u,x)$. We call this problem the weak min-max problem if for every $u \in \mathcal{U}_{ad}$, the supremum in (18) is attained at some point $x^u \in X(u)$ reducing the problem to

$$\inf_{u \in \mathcal{U}_{ad}} J_o(u) \equiv \inf_{u \in \mathcal{U}_{ad}} \Upsilon(u, x^u).$$

Now we are prepared to prove the existence of an optimal control.

Theorem 4.3. *Suppose the assumptions of Theorem* 4.2 *hold,* $\mathcal{U}_{ad}$ *is a weakly sequentially compact subset of* $\mathcal{M}_c(I,U)$*, the integrand* ℓ *is measurable in* $t \in I$ *for each fixed* $x \in E$ *and continuous in* x *on* E *for almost all* $t \in I$*,* Ψ *is continuous in* x *on* E *and there exist* $\ell_0, \ell_1 \in L_1(I)$*,* $c_0, c_1 \in R$ *and a pair of nonnegative, nondecreasing, continuous, real valued functions* $\beta_1, \beta_2 : [0,\infty] \longrightarrow [0,\infty]$ *bounded on bounded sets satisfying*

$$\ell_0(t) \leq \ell(t,x) \leq \ell_1(t) + \beta_1(\| x \|), \ c_0 \leq \Psi(x) \leq c_1 + \beta_2(\| x \|) \tag{20}$$

and $u \longrightarrow \varphi(u)$ *is weakly lower semi continuous on* $\mathcal{M}_c(I,U)$ *and bounded on bounded sets. Then, there exists an optimal control.*

Proof. Since $X(u)$ is a bounded subset of $B(I,E)$, it follows from the assumptions on ℓ and Ψ that $J_o(u) \equiv \sup\{\Upsilon(u,x), x \in X(u)\} < \infty$. Let $\{x_n\} \in X(u)$ be a maximizing sequence for the functional $x \longrightarrow \Upsilon(u,x)$. Clearly by definition of solution, there exists a sequence $\{f_n\} \in L_1(I,E)$ such that $f_n(t) \in C(t, x_n(t))$ for almost all $t \in I$. Again by virtue of boundedness of the set $X(u)$ and the property (C3) it follows from similar arguments as in Theorem 4.2, that there exists a subsequence of the sequence $\{x_n\}$, relabeled as the original sequence, and an element $x^* \in X(u)$ such that $x_n \stackrel{s}{\longrightarrow} x^*$ in $B(I,E)$. In fact this also proves that the set $X(u)$ is conditionally sequentially compact. By Theorem 4.2 $X(u)$ is closed and hence $X(u)$ is actually sequentially compact. Since both ℓ and Ψ are continuous in x on E, by virtue of (20) it follows from Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \Upsilon(u, x_n) = \Upsilon(u, x^*).$$

This shows that for every $u \in \mathcal{U}_{ad}$, there exists an $x^* \equiv x^u \in X(u)$ at which the functional $x \to \Upsilon(u,x)$ attains its maximum. Thus the functional

$$J_o(u) \equiv \sup\{\Upsilon(u,x), x \in X(u)\} = \Upsilon(u, x^u)$$

is well defined for every $u \in \mathcal{U}_{ad}$. We show that J_o is weakly lower semi continuous on $\mathcal{U}_{ad}$. For simplicity of notation define

$$\Phi(x) \equiv \int_I \ell(t, x(t))dt + \Psi(x(T)), x \in B(I, E).$$

Let $\{u_n\} \in \mathcal{U}_{ad}$ and $u_n \xrightarrow{w} u_o \in \mathcal{M}_c(I, U)$. Since $\mathcal{U}_{ad}$ is weakly sequentially compact $u_o \in \mathcal{U}_{ad}$. Denote by $x_n = x^{u^n}$ the maximizer of the functional $\Upsilon(u_n, \cdot)$ giving $J_o(u_n) = \Upsilon(u_n, x_n)$. Clearly $(u_n, x_n) \in \mathcal{G}_r(X)$. By virtue of compactness of the operators $\Gamma(t), t \in I$, and the semigroup $S(t), t > 0$, we can again argue that, along a subsequence if necessary, $x_n \xrightarrow{s} x_o$ as $u_n \xrightarrow{w} u_o$. Then it follows from the closure of the graph $\mathcal{G}_r(X)$, as stated in Theorem 4.2, that $(u_o, x_o) \in \mathcal{G}_r(X)$. Thus

$$\begin{aligned} \liminf J_o(u_n) &= \liminf\big\{\Phi(x_n) + \varphi(u_n)\big\} \\ &\geq \liminf \Phi(x_n) + \liminf \varphi(u_n) = \lim \Phi(x_n) + \liminf \varphi(u_n) \\ &\geq \Phi(x_o) + \varphi(u_o) = J_o(u_o). \end{aligned} \tag{21}$$

The last inequality follows from continuity of Φ on $B(I, E)$ and weak lower semi continuity of φ on $\mathcal{M}_c(I, U)$. This proves lower semi continuity of the functional $u \longrightarrow J_o(u)$, and, it is clear from (20) that $J_o > -\infty$. Thus J_o attains its infimum on $\mathcal{U}_{ad}$. This proves the existence of an optimal control. $\square$

5. NECESSARY CONDITIONS OF OPTIMALITY

In this section we present a set of necessary conditions of optimality for the control problem (19) subject to the dynamic system (11). Here we must use differentials of multifunctions. There are several notions of derivatives for multi functions; the most popular ones being those due to Bouligand, Nagumo, and Clerk [15, 16]. The most appropriate one for us is the last one. For transparency we present this in the direct context of the problem in hand. Denote $X \equiv B(I, E), Y \equiv L_1(I, E)$ and assume that the product space $\mathcal{Z} \equiv X \times Y$ is furnished with the product topology $\tau \equiv \tau_s \times \tau_w$ where τ_s denotes the strong topology (supnorm topology) on X and τ_w the weak topology on Y. Define the multifunction $\hat{C} : X \longrightarrow Y$ and its graph $\mathcal{G}_r(\hat{C})$ as follows:

$$\begin{aligned} \hat{C}(x) &\equiv \{f \in Y : f(t) \in C(t, x(t)) \ a.e\ I\}, x \in X \\ \mathcal{G}_r(\hat{C}) &\equiv \{(x, f) \in X \times Y : f \in \hat{C}(x)\} \end{aligned}$$

The Clerk derivative of the multifunction $\hat{C}$ at the point $(x, f) \in \mathcal{G}_r(\hat{C})$ is again a multifunction $D_c\hat{C}(x; f)$ from X to Y. An element (y, g) of $\mathcal{Z}$ is in the graph of $D_c\hat{C}(x; f)$, that is, $g \in D_c\hat{C}(x; f)(y)$ if the following identity holds:

$$\overline{\lim}_{(x^\varepsilon, f^\varepsilon) \overset{\mathcal{G}_r(\hat{C}), \varepsilon\downarrow 0}{\longrightarrow} (x^o, f^o)} \inf_{y^\varepsilon \to y} \big\{d\big(g, \frac{\hat{C}(x^\varepsilon + \varepsilon y^\varepsilon) - f^\varepsilon}{\varepsilon}\big)\big\} = 0, \tag{22}$$

where $d(g, F)$ denotes the distance of $g \in Y$ from the set $F \subset Y$.

Lemma 5.1. *Suppose the assumptions of Theorem* 4.3 *hold, the multifunction* $\hat{C}$ *is differentiable in the sense of Clerk and* $\mathcal{U}_{ad}$ *is a convex subset of* $\mathcal{M}_c(I, U)$. *Let* $u^o, u \in \mathcal{U}_{ad}$ *and* $u^\varepsilon \equiv u^o + \varepsilon(u - u^o)$ *and let* $x^o \in X(u^o)$ *and* $x^\varepsilon \in X(u^\varepsilon)$. *Let* $f^\varepsilon \in \hat{C}(x^\varepsilon)$ *and* $f^o \in \hat{C}(x^o)$. *Then* $y \equiv \lim_{\varepsilon \downarrow 0} y^\varepsilon \equiv (1/\varepsilon)(x^\varepsilon - x^o)$ *exists and it is a mild solution of the variational evolution inclusion*

$$dy = Aydt + B(dt)y + g(t)dt + \Gamma(t)(u - u^o)(dt), y(0) = 0, \tag{23}$$

$$g \in D_c\hat{C}(x^o, f^o)(y). \tag{24}$$

Proof. By definition of mild solution of the evolution inclusion (11) we have

$$x^\varepsilon(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)f^\varepsilon(s)ds + \int_0^t U_B(t,s)\Gamma(s)u^\varepsilon(ds) \tag{25}$$

$$x^o(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)f^o(s)ds + \int_0^t U_B(t,s)\Gamma(s)u^o(ds). \tag{26}$$

Taking the difference $x^\varepsilon - x^o$ and using the Lipschitz property (C2) and Gronwall inequality one can easily verify that

$$\| x^\varepsilon(t) - x^o(t) \|_E \leq \varepsilon \Big(M_B \| \Gamma \|_0 \Big) \| u - u^o \|_v \exp\{M_B \int_0^t K(s)ds\}, t \in I. \tag{27}$$

It is clear from this inequality that as $\varepsilon \downarrow 0$ we have

$$x^\varepsilon \xrightarrow{s} x^o \text{ in } X = B(I, E). \tag{28}$$

Define

$$y^\varepsilon \equiv (1/\varepsilon)x^\varepsilon - x^o) \tag{29}$$

$$g^\varepsilon \equiv (1/\varepsilon)(f^\varepsilon - f^o). \tag{30}$$

Again from inequality (27) we have

$$\overline{\lim}_{\varepsilon \downarrow 0} \| y^\varepsilon \|_X < \infty. \tag{31}$$

Clearly it follows from (25)-(26) and the above definitions that

$$y^\varepsilon(t) = \int_0^t U_B(t,s)g^\varepsilon(s)ds + \int_0^t U_B(t,s)\Gamma(s)(u - u^o)(ds), \quad t \in I. \tag{32}$$

Since by assumptions (C1) and (C3), the multifunction $C(t, \xi)$ is closed convex valued and integrably bounded, it follows from Dunford-Pettis theorem that $f^\varepsilon \xrightarrow{w} f^o$. It follows from (28)-(30) and the Lipschitz property (C2) of the multifunction C that the family $\{g^\varepsilon\}$ is contained in bounded subset of $L_1(I, E)$ and, being the difference of uniformly integrable functions, the set $\{g^\varepsilon\}_{\varepsilon > 0}$ is also uniformly integrable. Since E is a reflexive Banach space, this set is relatively weakly sequentially compact. Thus along a subsequence if necessary, $g^\varepsilon \xrightarrow{w} g$ in $L_1(I, E)$. By virtue of compactness of the semigroup $S(t)$, $t > 0$, or equivalently compactness of the transition operator

$U_B(t,s)$, $0 \le s < t \le T$, it follows from (32) that there exists an $y \in B(I,E)$ so that $y^\varepsilon \xrightarrow{s} y$ giving

$$y(t) = \int_0^t U_B(t,s)g(s)ds + \int_0^t U_B(t,s)\Gamma(s)(u-u^o)(ds), t \in I. \tag{33}$$

Hence it follows from the differentiability of the multifunction $\hat{C}$ that $(y,g) \in \mathcal{G}_r(D_c\hat{C})$ $(x^o; f^o)$. In other words

$$g(t) \in D_cC(t, x^o(t); f^o(t))(y(t)), \ a.e.\ t \in I.$$

From this we conclude that the pair (y,g) is a mild solution of the evolution system (23)-(24). This completes the proof. □

Now we are prepared to prove the necessary conditions of optimality.

Theorem 5.2. *Consider the system* (11) *and the control problem* (19)*. Suppose the assumptions of Lemma* 5.1 *hold,* φ *is sub differentiable on* $\mathcal{U}_{ad}$*,* ℓ *is Borel measurable on* $I \times E$ *with* $x \longrightarrow \ell(t,x)$ *and* $x \longrightarrow \Psi(x)$ *being continuously Frechet differentiable. Then for the pair* $(u^o, x^o) \in \mathcal{U}_{ad} \times B(I,E), x^o \in X(u^o)$ *to be optimal for the problem (19), it is necessary that there exists a* $\psi \in B(I,E^*)$ *which together with* (u^o, x^o) *satisfy the evolution inclusions (34)-(35) (in the mild sense) and the inequality (36):*

$$\begin{aligned}
(34)\ dx^o &= Ax^o dt + B(dt)x^o + f^o(t)dt + \Gamma(t)u^o(dt), x^o(0) = x_0 \\
& \quad f^o(t) \in C(t, x^o(t))\ a.e \\
(35)\ d\psi &= -A^*\psi dt - B^*(dt)\psi - h^o(t)dt - \ell_x(t, x^o(t))dt, \psi(T) = \Psi_x(x^o(T)) \\
& \quad h^o(t) \in (D_cC)^*(x^o, f^o)(\psi(t))
\end{aligned}$$

$$\int_0^T < \Gamma^*(t)\psi(t) + z^o(t), (u-u^o)(dt) >_{U^*,U} \ \ge\ 0,\ z^o \in \partial\varphi(u^o)\ \&\ u \in \mathcal{U}_{ad} \tag{36}$$

where $(D_cC)^*(x,f)$ *denotes adjoint of the closed convex process* D_cC *evaluated at* $(x,f) \in \mathcal{G}_r(C)$ *and* $\partial\varphi(u)$ *is the subdifferential of* φ *given by*

$$\partial\varphi(u) \equiv \{\zeta \in \mathcal{M}_c(I,U)^* :< \zeta, w-u > \le\ \varphi(w) - \varphi(u)\ \forall\ w \in \mathcal{M}_c(I,U)\}.$$

Proof. Suppose the pair $(u^o, x^o) \in \mathcal{U}_{ad} \times B(I,E)$ is optimal for the control problem. Let $u \in \mathcal{U}_{ad}$ and define $u^\varepsilon \equiv u^o + \varepsilon(u-u^o), \varepsilon \in [0,1]$. Since $\mathcal{U}_{ad}$ is convex $u^\varepsilon \in \mathcal{U}_{ad}$. By Theorem 4.3, there exists $x^\varepsilon \in X(u^\varepsilon)$ so that $\Upsilon(u^\varepsilon, x^\varepsilon) = J_o(u^\varepsilon)$, that is $x \longrightarrow \Upsilon(u^\varepsilon, x)$ attains its maximum at x^ε. Thus for the pair (u^o, x^o) to be optimal it is clearly necessary that

$$J_o(u^o) \equiv \Upsilon(u^o, x^o) \le \Upsilon(u^\varepsilon, x^\varepsilon) \equiv J_0(u^\varepsilon)\ \forall\ \varepsilon \in [0,1]. \tag{37}$$

Since x^o and x^ε are solutions of the differential inclusion (11) corresponding to controls u^o and u^ε respectively, there exist $f^o \in L_1(I,E)$ and $f^\varepsilon \in L_1(I,E)$ such that

$$x^o(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)f^o(s)ds + \int_0^t U_B(t,s)\Gamma(s)u^o(ds), t \in I, \tag{38}$$
$$f^o(t) \in C(t, x^o(t))$$
$$x^\varepsilon(t) = U_B(t,0)x_0 + \int_0^t U_B(t,s)f^\varepsilon(s)ds + \int_0^t U_B(t,s)\Gamma(s)u^\varepsilon(ds), t \in I, \tag{39}$$
$$f^\varepsilon(t) \in C(t, x^\varepsilon(t)).$$

Let $dJ_o(u^o, u-u^o)$ denote the sub differential of J_o at u^o in the direction $u-u^o$. Since by assumption φ is sub differentiable and ℓ, Ψ are Frechet differentiable, J_o possesses its sub differential. Hence it follows from (37) that

$$\begin{aligned} dJ_o(u^o, u-u^o) \quad &= \quad \int_0^T < \ell_x(t, x^o(t)), y(t) >_{E^*,E} dt + < \Psi_x(x^o(T)), y(T) > \\ &+ \int_0^T < z^o(t), (u-u^o)(dt) > \geq 0, \forall z^o \in \partial\varphi(u^o) \& \forall\ u \in \mathcal{U}_{ad}, \end{aligned} \tag{40}$$

where, by Lemma 5.1 we have $y \in B(I,E)$ given by the expression (33) with

$$g \in D_c C(x^o, f^o)(y). \tag{41}$$

It is clear from the expression (33) and (41) that the map $\Gamma(u-u^o) \longrightarrow y$ is continuous linear from $\mathcal{M}_c(I,E)$ to $B(I,E)$. It follows from the assumptions on ℓ and Ψ (see (20)) that ℓ^o, given by $\ell^o(t) \equiv \ell_x(t, x^o(t))$, belongs to $L_1(I,E^*)$ and $\Psi_x(x^o(T)) \in E^*$. Since $y \in B(I,E)$ we conclude from this that the map

$$y \longrightarrow L(y) \equiv \int_0^T < \ell_x(t, x^o(t)), y(t) >_{E^*,E} dt + < \Psi_x(x^o(T)), y(T) > \tag{42}$$

is a continuous linear functional on $B(I,E)$. Since $\Gamma \in BM(I, \mathcal{L}(U,E))$ it is clear that for any $u \in \mathcal{M}_c(I,U)$, the measure defined by

$$v(K) \equiv \int_K \Gamma(t)u(dt), K \in \mathcal{B},$$

is also a countably additive E valued vector measure of bounded total variation. Thus the composition map

$$\Gamma(u-u^o) \longrightarrow y \longrightarrow L(y) \equiv \tilde{L}(\Gamma(u-u^o))$$

is a continuous linear functional on $\mathcal{M}_c(I,E)$. Hence there exists an element $\psi \in (\mathcal{M}_c(I,E))^*$, the dual of $\mathcal{M}_c(I,E)$, such that

$$L(y) \equiv \tilde{L}(\Gamma(u-u^o)) = \int_0^T \langle \psi(t), \Gamma(t)(u-u^o)(dt) \rangle_{E^*,E}. \tag{43}$$

Later we will see that $\psi \in B(I, E^*)$. Using (43) in (40) we arrive at the following inequality

$$dJ_o(u^o, u - u^o) = \int_I \langle \Gamma^*(t)\psi(t) + z^o(t), (u - u^o)(dt)\rangle \geq 0, \qquad \forall\, z^o \in \partial\varphi(u^o) \text{ and } u \in \mathcal{U}_{ad}. \tag{44}$$

This is the necessary condition (36). Recall that y given by (33) is the mild solution of the variational evolution equation

$$\begin{aligned} &dy = Aydt + B(dt)y + g(t)dt + \Gamma(u - u^o)(dt), y(0) = 0 \\ &g \in D_cC(x^o, f^o)(y). \end{aligned} \tag{45}$$

Using standard arguments based on Yosida approximation of the generator A, it follows from (45) and integration by parts that (43) can be rewritten as

$$L(y) = \langle \psi(T), y(T)\rangle_{E^*,E} - \int_0^T \langle d\psi + A^*\psi dt + B^*(dt)\psi + h^o(t)dt, y(t)\rangle \tag{46}$$

where

$$h^o \in (D_cC)^*(x^o, f^o)(\psi),$$

with $(D_cC)^*$ being the codifferential or adjoint process. Equating (42) with (46) and setting $\psi(T) = \Psi_x(x^o(T))$ and noting that y is a mild solution corresponding to an arbitrary control $u \in \mathcal{U}_{ad}$, we arrive at the adjoint equation

$$d\psi = -A^*\psi dt - B^*(dt)\psi - h^o(t)dt - \ell_x(t, x^o(T))dt,\ \ \psi(T) = \Psi_x(x^o(T)). \tag{47}$$

Equation (47) and the preceding inclusion give the adjoint inclusion (35). Thus for ψ, whose existence was announced earlier leading to the representation (43), can be taken as the mild solution of the adjoint evolution inclusion (35). Following exactly the same arguments as in Theorem 4.1, we conclude that the adjoint evolution inclusion (35) has a nonempty set of solutions in $B(I, E^*)$. Thus the statement leading to the expression (43) can be modified by stating the existence of a $\psi \in B(I, E^*) \subset (\mathcal{M}_c(I, E))^*$ in the smaller space. This completes the proof. □

Remark 3. It is rather surprising that the adjoint process ψ belongs to the space $B(I, E^*) \supset C(I, E^*)$ even though direct analysis asserts its existence in the space $(\mathcal{M}_c(I, E))^*$ the weak dual of $\mathcal{M}_c(I, E)$. Since E is a reflexive Banach space, $C(I, E^*)$ is the weak star dual (predual) of $\mathcal{M}_c(I, E)$ and note that any $f \in C(I, E^*)$ induces a continuous linear functional on $\mathcal{M}_c(I, E)$ through the functional

$$\ell_f(\mu) \equiv \int_I < f(t), \mu(dt) >_{E^*,E} .$$

Some Examples of Control Cost. Here we present some examples of φ, representing the control cost, which are Gateaux differentiable.

Example (E1): This is an example of a quadratic cost. Let $\mathcal{L}_s(U, U^*)$ denote the class of linear symmetric operators from U to its dual U^* and let $R \in C(I \times I, \mathcal{L}_s(U, U^*))$ and define

$$\varphi(u) \equiv (1/2) \int_{I \times I} \langle R(t,s)u(ds), u(dt) \rangle.$$

Clearly this is Gateaux differentiable and it is given by

$$\partial \varphi(u)(t) = \int_I R(t,s)u(ds) \equiv (\mathcal{R}u)(t).$$

In this case $z^o(t)$ of the necessary condition (36) is given by

$$z^o(t) \equiv \int_I R(t,s)u^o(ds) = (\mathcal{R}u^o)(t).$$

Note that there exist nontrivial kernels R from the class mentioned above for which the corresponding operator $\mathcal{R} \in \mathcal{L}_s^+(\mathcal{M}_c(I,U))$. A simple example of R is a nuclear operator having the representation

$$R(t,s) \equiv \sum \lambda_i \ \xi_i(t) \otimes \xi_i(s)$$

where $\lambda_i \geq 0$, $\sum \lambda_i < \infty$ and $\| \xi_i \|_{C(I,U^*)} = 1$ and the linear span $Span\{\xi_i\}$ is dense in $C(I, U^*)$.

Example (E2): This example is based on a cylindrical function on $\mathcal{M}_c(I,U)$ as follows

$$\varphi(u) \equiv p(\langle \xi_1, u \rangle, \langle \xi_2, u \rangle, \cdots, \langle \xi_n, u \rangle)$$

where $\xi_i \in C(I, U^*)$, $1 \leq i \leq n$, or $B(I, U^*)$, $p : R^n \longrightarrow R$ is continuously differentiable. In this case

$$z^o(t) = \sum \partial_i p^o \ \xi_i(t)$$

where $\partial_i p^o$ denotes the partial of p with respect to the $i-$th variable evaluated at u^o.

Example (E3): (Linear Quadratic Regulator) In case the multifunction $C \equiv 0$, system (11) reduces to a linear evolution equation. Assuming E, U to be suitable Hilbert spaces, and taking $\ell(t,x) \equiv (1/2)(Q(t)x, x)$ with $Q(t)$ a positive self adjoint operator valued function in E; $\Psi(\xi) = (1/2)(M\xi, \xi)$ with M a positive self adjoint operator in E and φ as given in example (E1) one has a quadratic regulator problem. In this case the optimal feedback control can be derived from the necessary conditions of optimality. If $Range(\Gamma^*) \subseteq Range\mathcal{R}$, the optimal feedback control is given by

$$u^o = -\mathcal{R}^{-1}\Gamma^* K x$$

where K is an operator valued measure given by the solution of Differential Riccati equation

$$dK + (KA + A^*K)dt + (KB(dt) + B^*(dt)K) - (K\Gamma)\mathcal{R}^{-1}(\Gamma^*K) = 0, K(\{T\}) = M.$$

Since $\mathcal{R}$ is an integral operator, this is a functional (nonlocal) differential equation and its solution if one exists is an operator valued measure. The author conjectures that if $B \in \mathcal{M}_c(I, \mathcal{L}(E))$ then $K \in \mathcal{M}_c(I, \mathcal{L}_s^+(E))$; alternately if $B \in \mathcal{M}_b(I, \mathcal{L}(E))$ then $K \in \mathcal{M}_b(I, \mathcal{L}_s^+(E))$ where by $\mathcal{M}_b(I, \mathcal{L}(E))$ we mean the space of finitely additive bounded measures with values in $\mathcal{L}(E)$. We leave the analysis of this operator equation as an open problem.

Future Directions. In recent years hybrid systems are occupying the interest of great many researchers on dynamic systems and control. In general the mathematical models used for such systems consist of a number of interconnected and possibly independent dynamic systems which are switched from one configuration to another. The author of this paper prefers to use operator valued and vector valued measures to model such systems. We present here few open problems in this area.

(A): Examples of structural controls are found in mechanics, in particular, robotics and in many other interconnected network of large systems that may be subject to abrupt changes in configuration. Motivated by this we are currently developing general mathematical models for hybrid systems governed by evolution equations or inclusions containing operator valued measures. We develop optimal control theory for such systems using operator valued measures as controls.

(B): In the context of hybrid systems the questions of linear and nonlinear filtering is wide open. To start with one may consider (Kalman or) linear filtering and control of systems of the form

$$dx = Axdt + B(dt)x + C(t)u(dt) + D(t)dW(t) \tag{48}$$

$$dy = Hxdt + C_o(t)dV(t), \tag{49}$$

where $\{W, V\}$ are Brownian motions on a filtered Probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ with values in compatible Hilbert spaces. Here x is the inaccessible state process and y is the observed process. The controls are random vector valued measures $\{u\}$ suitably adapted to the filtration $\mathcal{F}_{t\geq 0}^y \subset \mathcal{F}_{t\geq 0}$.

(C): The questions of nonlinear filtering and control of differential equations and/or inclusions driven by vector and operator valued measures in infinite dimensional spaces is completely open.

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