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BANACH-SAKS-MAZUR AND KAKUTANI-KY FAN THEOREMS IN SPACES OF MULTIFUNCTIONS AND APPLICATIONS TO SET DIFFERENTIAL INCLUSIONS

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ABSTRACT. Some multivalued versions of the theorems of Banach-Saks-Mazur and Kakutani-Ky Fan are proved. These results are used in existence problems for set differential inclusions.

Keywords and phrases. Multifunction, upper semicontinuity, α-convex metric space, Banach-Saks-Mazur theorem, Kakutani-Ky Fan theorem, set differential inclusion, Hukuhara derivative, Hukuhara integral.

1. INTRODUCTION

It is well known that certain classical results of functional analysis, like the theorems of Banach-Saks-Mazur [3] [24] and Kakutani-Ky Fan [15] [9], have a fundamental role in many areas of mathematics, in particular in the theory of differential inclusions in finite (or infinite) dimensional Euclidean spaces. For a fuller information and bibliography on this subject, see the monographs of Aubin and Cellina [2] and Hu and Papageorgiou [13].

Let $\mathbb{R}^d$ be the real d-dimensional Euclidean space, and let $\mathfrak{X}$ be the hyperspace of all nonempty compact convex subsets of $\mathbb{R}^d$.

In the present paper we establish some multivalued analogues of the theorems of Banach-Saks-Mazur and Kakutani-Ky Fan (see Theorems 3.4, 3.8), which are valid for maps with values in the hyperspace $\mathfrak{X}$. These results turn out to be useful in the investigation of set differential inclusions.

Denote by Φ a multifunction from $I \times \mathfrak{X}$, $I = [0, 1]$, to the space of all nonempty compact convex subsets of $\mathfrak{X}$. For $A \in \mathfrak{X}$, consider the Cauchy problem with Hukuhara derivative, of the form

$$DX(t) \in \Phi(t, X(t)) \qquad X(0) = A \ . \tag{C}$$

Under Carathéodory assumptions on Φ, it will be proved that the Cauchy problem (C) has solutions (Theorem 4.4). In our approach we follow some ideas contained in

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the seminal papers of Lasota and Opial [23] and Lasota and Olech [22]. Moreover, we use an important characterization theorem for integrals of multifunctions, due to Hermes [10], and the multivalued analogues of the theorems of Banach-Saks-Mazur and Kakutani-Ky Fan.

Under stronger assumptions on Φ, an existence theorem for the Cauchy problem (C) has recently been obtained in [6], by using a quite different approach.

It is worth noting that the investigation of set differential equations was started in 1969 by de Blasi and Iervolino [5]. For developments, also in other directions, see Brandão Lopes Pinto, de Blasi and Iervolino [4], Kisielewicz [16] [17], Kisielewicz, Serafin and Sosulski [18], Michta [25] [26], Artstein [1], Plotnikov and Plotnikova [30], Plotnikov and Rashkov [31], Lakshmikantham, Gnana Bhaskar and Vasumdhara Devi [21]. Moreover, evidence of the relations of this kind of differential equations with other areas, as ordinary differential inclusions, control theory, fuzzy and stochastic differential equations, can be found in Diamond and Kloeden [8], Tolstonogov [32], Lakshmikantham [19], Michta [27], Michta and Motyl [28], Lakshmikantham, Leela and Vatsala [20], Plotnikov and Tumbrukaki [29].

The present paper consists of 4 sections, with the introduction. Section 2 contains notation and preliminaries. Section 3 contains the multivalued versions of the theorems of Banach-Saks-Mazur and Kakutani-Ky Fan. Section 4 contains an existence theorem for the Cauchy problem (C).

2. NOTATION AND PRELIMINARIES

Let M be a metric space with distance ρ and let $\mathcal{P}(M)$ be the space of all nonempty bounded subsets of M.

If $a \in M$ and $\phi \neq X \subset M$, put $d(a, X) = \inf_{x\in X} \rho(a, x)$. For X, $Y \in \mathcal{P}(M)$ let

$$e(X, Y) = \sup_{x\in X} d(x, Y) \qquad e(Y, X) = \sup_{y\in Y} d(y, X)$$

and define

$$h(X, Y) = \max\{e(X, Y), e(Y, X)\} .$$

h is a semidistance on $\mathcal{P}(M)$. It is a distance, said the Pompeiu-Hausdorff metric, when it is restricted to the space of all nonempty compact subsets of M. The latter space is complete and separable if so is M.

By $B_M(a, r)$ and $B_M[a, r]$ we mean an open and a closed ball in M with centre a and radius r. If $\phi \neq A \subset M$ and $r \geq 0$, we set $N_M[A, r] = \{x \in M | d(x, A) \leq r\}$.

Let T be a metric space. A map $\Psi : T \to \mathcal{P}(M)$ is *upper semicontinuous* (*u.s.c.*) (resp. *lower semicontinuous* (*l.s.c.*), *continuous*) at $x_0 \in T$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in B_T(x_0, \delta)$ implies $e(\Psi(x), \Psi(x_0)) < \varepsilon$ (resp.

$e(\Psi(x_0), \Psi(x)) < \varepsilon$, $h(\Psi(x), \Psi(x_0)) < \varepsilon$) or, equivalently, $\Psi(x) \subset N_M[\Psi(x_0), \varepsilon]$ (resp. $\Psi(x_0) \subset N_M[\Psi(x), \varepsilon]$, $h(\Psi(x), \Psi(x_0)) < \varepsilon$).

Let $\mathbb{R}^d$ be the usual d-dimensional real Euclidean space with norm $|\cdot|$ induced by the inner product $\langle \cdot, \cdot \rangle$. Set

$$\mathfrak{x} = \{A \subset \mathbb{R}^d | A \text{ is nonempty compact convex}\} .$$

$\mathfrak{x}$ is endowed with the Pompeiu-Hausdorff distance h, under which it is a complete and separable metric space. We introduce in $\mathfrak{x}$ the Minkowski operations of *addition* and *multiplication* by nonnegative scalars, given by

$$A + B = \{a + b \in \mathbb{R}^d | a \in A\ ,\ b \in B\}\ , \quad \lambda A = \{\lambda a \in \mathbb{R}^d | a \in A\}$$

where A, $B \in \mathfrak{x}$ and $\lambda \geq 0$.

Remark 2.1. $\mathfrak{x}$ is a semilinear space, i.e. $\mathfrak{x}$ is closed under the above operations and, moreover, for arbitrary A, B, $C \in \mathfrak{x}$ and λ, $\mu \geq 0$ the following properties are satisfied: (i) $A + \{0\} = A$ (0 the zero of $\mathbb{R}^d$); (ii) $A + B = B + A$; (iii) $A + (B + C) = (A + B) + C$; (iv) $1 \cdot A = A$; (v) $\lambda(\mu A) = (\lambda\mu)A$; (vi) $\lambda(A + B) = \lambda A + \lambda B$; (vii) $(\lambda + \mu)A = \lambda A + \mu A$.

For convenience we put $|A| = \sup\{|a|\ |a \in A\}$ if $A \subset \mathbb{R}^d$ is non empty and bounded.

A set $\mathcal{A} \subset \mathfrak{x}$ is *convex* if $(1 - \lambda)A + \lambda B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$ and $\lambda \in [0, 1]$.

Set

$$\mathbb{K}(\mathfrak{x}) = \{\mathcal{A} \subset \mathfrak{x} | \mathcal{A} \text{ is nonempty compact convex}\} .$$

$\mathbb{K}(\mathfrak{x})$ is endowed with the Pompeiu-Hausdorff distance $h_{\mathbb{K}}$ under which it is a complete and separable metric space. We introduce in $\mathbb{K}(\mathfrak{x})$ the Minkowski operations of *addition* and *multiplication* by nonnegative scalars, given by

$$\mathcal{A} + \mathcal{B} = \{A + B \in \mathfrak{x} | A \in \mathcal{A}, B \in \mathcal{B}\}\ , \quad \lambda\mathcal{A} = \{\lambda A \in \mathfrak{x} | A \in \mathcal{A}\}\ ,$$

where $\mathcal{A}$, $\mathcal{B} \in \mathbb{K}(\mathfrak{x})$ and $\lambda \geq 0$.

Remark 2.2. $\mathbb{K}(\mathfrak{x})$ is a semilinear space. Moreover, if $\mathcal{A} \in \mathbb{K}(\mathfrak{x})$ and $\varepsilon > 0$, then $N_{\mathfrak{x}}[\mathcal{A}, \varepsilon] \in \mathbb{K}(\mathfrak{x})$.

Let J be a bounded and measurable subset of $\mathbb{R}$.

A map $U : J \to \mathfrak{x}$ (resp. $\Psi : J \to \mathbb{K}(\mathfrak{x})$) is *measurable* if, for each closed $C \subset \mathbb{R}^d$ (resp. $C \subset \mathfrak{x}$) the set $\{t \in J | U(t) \cap C \neq \phi\}$ (resp. $\{t \in J | \Psi(t) \cap C \neq \phi\}$) is Lebesgue measurable. If $U : J \to \mathfrak{x}$ is measurable and integrably bounded, i.e.

$\int_J |U(t)|dt < +\infty$, then the *Hukuhara integral* [14] of U on J exists and is denoted by $\int_J U(t)dt$.

For any $A \in \mathfrak{X}$ and $u \in \mathbb{R}^d$, set

$$\sigma(u, A) = \max_{a \in A} \langle u, a \rangle \; . \tag{2.1}$$

The map $\sigma(\cdot, A) : \mathbb{R}^d \to \mathbb{R}$ defined by (2.1) for each $u \in \mathbb{R}^d$, is called the *support function* of A. For details see Hörmander [12] and Hu and Papageorgiou [13].

Some elementary properties of the support function are collected in the following

Remark 2.3. Let A, $B \in \mathfrak{X}$, $u \in \mathbb{R}^d$, and $\alpha, \beta \geq 0$. Then, (i) $\sigma(\alpha u, A) = \alpha\, \sigma(u, A)$; (ii) $\sigma(u, \alpha A + \beta B) = \alpha \sigma(u, A) + \beta \sigma(u, B)$; (iii) $A \subset B$ if and only if $\sigma(u, A) \leq \sigma(u, B)$ for every $u \in \mathbb{R}^d$; (iv) if $u_n \to u$ and $A_n \to A$, where u_n, $u \in \mathbb{R}^d$ and A_n, $A \in \mathfrak{X}$, then $\sigma(u_n, A_n) \to \sigma(u, A)$; (v) $h(A, B) = \max_{|u|=1} |\sigma(u, A) - \sigma(u, B)|$.

Remark 2.4. If $U : I \to \mathfrak{X}$ is Hukuhara integrable, then

$$\sigma\left(u, \int_I U(t)dt\right) = \int_I \sigma(u, U(t))dt \quad \text{for each } u \in \mathbb{R}^d \; .$$

The space $I \times \mathfrak{X}$, where $I = [0, 1]$, is endowed with the metric $\max\{|t - t'|, h(X, X')\}$, (t, X), $(t', X') \in I \times \mathfrak{X}$.

Given $\Phi : I \times \mathfrak{X} \to \mathbb{K}(\mathfrak{X})$ and $A \in \mathfrak{X}$, consider the Cauchy problem (C). For Φ we shall use the following assumptions:

(h_1) the map $t \to \Phi(t, X)$ is measurable, for each $X \in \mathfrak{X}$;

(h_2) the map $X \to \Phi(t, X)$ is u.s.c. for each $t \in I$;

(h_3) $\Phi(t, X) \subset B_{\mathfrak{X}}[0, M]$, $M > 0$, for each $(t, X) \in I \times \mathfrak{X}$.

A map $X : I \to \mathfrak{X}$ is *solution* of the Cauchy problem (C), if there exists a measurable map $U : I \to \mathfrak{X}$ such that:

$$X(t) = A + \int_0^t U(s)ds \quad \text{for each } t \in I \; ,$$

$$U(t) \in \Phi(t, X(t)) \quad \text{for } t \in I \text{ a.e.}$$

Here the integral is in the sense of Hukuhara.

By virtue of [14], [5], if X is solution of the Cauchy problem (C), then X is continuous on I, has Hukuhara derivative DX a.e. in I, and $DX(t) = U(t)$ for $t \in I$ a.e.

3. BANACH-SAKS-MAZUR AND KAKUTANI-KY FAN THEOREMS FOR MULTIFUNCTIONS

In this section we present some multivalued analogues of the theorems of Banach-Saks-Mazur and Kakutani-Ky Fan for maps taking their values in the hyperspace $\mathfrak{X}$.

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We write $y_n \xrightarrow{s} y$ (resp. $y_n \xrightarrow{w} y$) to mean that y_n converges strongly (resp. weakly) to y in $\mathbb{H}$, as $n \to +\infty$.

The following geometric lemma is probably known, yet the author is unable to furnish any reference. Therefore the proof is included.

Lemma 3.1. *Let $\{y_n\} \subset \mathbb{H}$ be a sequence which converges weakly to $y \in \mathbb{H}$. Then there exists a subsequence $\{y_{n_k}\}$ such that, for each of its subsequences, the corresponding sequence of the arithmetic means converges strongly to y.*

Proof. Without loss of generality we suppose that $y_n \xrightarrow{w} 0$. Then $\|y_n\| \le M$, $n \in \mathbb{N}$, for some $M \ge 0$.

Set $y_{n_1} = y_1$. As $\langle y_{n_1}, y_n\rangle \to 0$ as $n \to \infty$, there exists $n_2 > n_1$ such that

$$|\langle y_{n_1}, y_n\rangle| \le \frac{1}{2} \quad \text{for every } n \ge n_2 \ .$$

Analogously, $\langle y_{n_1}, y_n\rangle \to 0$ and $\langle y_{n_2}, y_n\rangle \to 0$ as $n \to +\infty$, and thus there exists $n_3 > n_2$ such that

$$|\langle y_{n_1}, y_n\rangle| \le \frac{1}{2^2}\,, \quad |\langle y_{n_2}\ ,\ y_n\rangle| < \frac{1}{2^2} \quad \text{for every } n \ge n_3 \ .$$

Then, by induction, one can construct a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that, for each $k \in \mathbb{N}$, one has

$$|\langle y_{n_1}, y_n\rangle| \le \frac{1}{2^k}\ ,\ |\langle y_{n_2}, y_n\rangle| \le \frac{1}{2^k}\,, \ldots, |\langle y_{n_k}, y_n\rangle| \le \frac{1}{2^k} \quad \text{for every } n \ge n_{k+1} \ .$$

Set, for simplicity, $z_k = y_{n_k}$. Then the sequence $\{z_k\}$ satisfies for each $k \ge 2$ the following property

$$|\langle z_i, z_k\rangle| \le \frac{1}{2^{k-1}}\ ,\ i = 1, 2, \ldots, k-1 \ . \tag{3.1}$$

Let $\{z_{k_n}\}$ be an arbitrary subsequence of $\{z_k\}$. For $n \ge 2$ we have

$$\|z_{k_1} + z_{k_2} + \cdots + z_{k_n}\|^2 = \sum_{i=1}^{n} \|z_{k_i}\|^2 + 2[\langle z_{k_1}, z_{k_2}\rangle + (\langle z_{k_1}, z_{k_3}\rangle + \langle z_{k_2}, z_{k_3}\rangle) + \cdots + \sum_{i=1}^{n-1} \langle z_{k_i}, z_{k_n}\rangle].$$

In view of (3.1), $|\langle z_{k_i}, z_{k_n}\rangle| \le 1/2^{k_n-1} \le 1/2^{k_{n-1}} \le 1/2^{n-1}$, for $i = 1, 2, \ldots, k_{n-1}$, and thus

$$\|z_{k_1} + z_{k_2} + \cdots + z_{k_n}\|^2 \le nM^2 + 2\left[\frac{1}{2} + 2\frac{1}{2^2} + \cdots + (n-1)\frac{1}{2^{n-1}}\right] .$$

Hence

$$\left\|\frac{z_{k_1} + z_{k_2} + \cdots + z_{k_n}}{n}\right\|^2 < \frac{nM^2 + 4}{n^2} ,$$

which implies that the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n} z_{k_i}\right\}$ converges strongly to 0, as $n \to +\infty$. This completes the proof.

Set $S = \{x \in \mathbb{R}^d |\ |x| = 1\}$.

Lemma 3.2. *Let A, $B \in \mathfrak{X}$. For $\varepsilon > 0$, let $\{u_i\}_{i=1}^N \subset S$ be an ε-net of S. Then,*

$$\left|h(A, B) - \max_{1\le i\le N} |\sigma(u_i, A) - \sigma(u_i, B)|\right| \le \varepsilon(|A| + |B|) . \tag{3.2}$$

Proof. Let $u \in S$. Take u_i, for some $1 \le i \le N$, such that $|u_i - u| < \varepsilon$. It is evident that

$$|\sigma(u, A) - \sigma(u_i, A)| \le \varepsilon|A|, \quad |\sigma(u, B) - \sigma(u_i, B)| \le \varepsilon|B|.$$

From

$$\begin{aligned} |\sigma(u, A) - \sigma(u, B)| &\le |\sigma(u, A) - \sigma(u_i, A)| + |\sigma(u_i, A) - \sigma(u_i, B)| \\ &\quad + |\sigma(u_i, B) - \sigma(u, B)| \\ &\le \max_{1\le i<N} |\sigma(u_i, A) - \sigma(u_i, B)| + \varepsilon(|A| + |B|), \end{aligned}$$

as $u \in S$ is arbitrary, one has

$$h(A, B) \le \max_{1\le i\le N} |\sigma(u_i, A) - \sigma(u_i, B)| + \varepsilon(|A| + |B|) .$$

Since, on the other hand, $\max_{1\le i\le N} |\sigma(u_i, A) - \sigma(u_i, B)| \le h(A, B)$, then (3.2) holds. This completes the proof.

We denote by $\mathbb{L}^2(I, \mathfrak{X})$ the space of all measurable multifunctions $U : I \to \mathfrak{X}$ such that $\int_I |U(t)|^2 dt < +\infty$.

The following is a weaker version of the theorem of Banach and Saks, valid for multifunctions.

Theorem 3.3. *Let $\{U_n\}$ be a sequence of multifunctions $U_n \in \mathbb{L}^2(I, \mathfrak{X})$ and let $U \in \mathbb{L}^2(I, \mathfrak{X})$, with $|U_n(t)| \le m(t)$, $|U(t)| \le m(t)$, $t \in I$ a.e., where $m \in L^2(I, \mathbb{R})$. Suppose that*

$$\lim_{n\to+\infty} \int_0^t U_n(s)ds = \int_0^t U(s)ds \quad \textit{for each } t \in I . \tag{3.3}$$

Then, for every $\varepsilon > 0$ there exists a subsequence $\{U_{n_k}\}$ (depending on ε) such that

$$\limsup_{k\to+\infty} \int_I h^2\left(\frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t), U(t)\right) dt \le \varepsilon \ . \tag{3.4}$$

Proof. Let ε be arbitrary, with

$$0 < \varepsilon < \frac{1}{16(K+1)} \qquad \text{where } K = \int_I m^2(t)dt \ . \tag{3.5}$$

Denote by $\{u_i\}_{i=1}^N \subset S$ an ε-net of the unit sphere S of $\mathbb{R}^d$. Define $\varphi_n : I \to \mathbb{R}^N$, $n \in \mathbb{N}$, and $\varphi : I \to \mathbb{R}^N$ by

$$\varphi_n(t) = (\sigma(u_1, U_n(t)), \sigma(u_2, U_n(t)), \dots, \sigma(u_N, U_n(t))) \quad t \in I$$

$$\varphi(t) = (\sigma(u_1, U(t)), \sigma(u_2, U(t)), \dots, \sigma(u_N, U(t))) \quad t \in I \ ,$$

and observe that φ_n, $\varphi \in L^2(I, \mathbb{R}^N)$ since all coordinates of φ_n and φ are in $L^2(I, \mathbb{R})$.

We have

$$\varphi_n \xrightarrow{w} \varphi \quad \text{ in } L^2(I, \mathbb{R}^N) \ . \tag{3.6}$$

It is sufficient to show that for $i = 1, \dots, N$ one has $\sigma(u_i, U_n(\cdot)) \xrightarrow{w} \sigma(u_i, U(\cdot))$ in $L^2(I, \mathbb{R})$ or, equivalently,

$$\lim_{n\to+\infty} \int_0^t \sigma(u_i, U_n(s))ds = \int_0^t \sigma(u_i, U(s))ds \quad \text{ for each } t \in I \ . \tag{3.7}$$

Indeed, by Remark 2.4, for $i = 1, \dots, N$ and each $t \in I$ we have

$$\int_0^t \sigma(u_i, U_n(s))ds = \sigma\left(u_i, \int_0^t U_n(s)ds\right),$$
$$\int_0^t \sigma(u_i, U(s))ds = \sigma\left(u_i, \int_0^t U(s)ds\right),$$

from which (3.7) follows, by virtue of (3.3) and the continuity of $\sigma(u_i, \cdot)$. Hence (3.6) holds.

By Lemma 3.1, in view of (3.6), the sequence $\{\varphi_n\}$ contains a subsequence $\{\varphi_{n_k}\}$ such that the sequence of its arithmetic means converges strongly to φ, i.e.

$$\frac{1}{k}\sum_{j=1}^{k} \varphi_{n_j} \xrightarrow{s} \varphi \quad \text{ in } L^2(I, \mathbb{R}^N) \ . \tag{3.8}$$

Clearly,

$$\begin{aligned}
\frac{1}{k}\sum_{j=1}^{k} \varphi_{n_j}(t) &= \left(\frac{1}{k}\sum_{j=1}^{k} \sigma(u_1, U_{n_j}(t)), \dots, \frac{1}{k}\sum_{j=1}^{k} \sigma(u_N, U_{n_j}(t))\right) \\
&= \left(\sigma(u_1, \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t)), \dots, \sigma(u_N, \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t))\right) \ .
\end{aligned}$$

Combining the latter with (3.8) gives, for $i = 1, \dots, N$,

$$\int_I |\sigma\left(u_i, \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t)\right) - \sigma(u_i, U(t))|^2 dt \to 0 \ , \ \text{as } k \to +\infty \ .$$

Therefore, there exists $k_0 \in \mathbb{N}$ such that

$$\int_I [\max_{1\le i\le N} |\sigma(u_i, \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t)) - \sigma(u_i, U(t))|]^2 dt < \frac{\varepsilon}{4} \quad \text{for each } k \ge k_0 \ . \tag{3.9}$$

Let $k \ge k_0$ be arbitrary. Since $\{u_i\}_{i=1}^N$ is an ε-net of S then, by Lemma 3.2, setting $r_k(t) = \left|(1/k)\sum_{j=1}^{k} U_{n_j}(t)\right| + |U(t)|$, one has

$$h\left(\frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t), U(t)\right) \le \max_{1\le n\le N} |\sigma(u_i \ , \ \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t)) - \sigma(u_i, U(t))| + \varepsilon r_k(t) \ ,$$

and thus

$$\int_I h^2\left(\frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t), U(t)\right) dt \le 2\int_I [\max_{1\le i\le N} |\sigma(u_i, \frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t)) - \sigma(u_i, U(t))|]^2 dt + +2\varepsilon^2 \int_I r_k^2(t) dt.$$

On the right hand side, the first integral is less than $\varepsilon/4$, by (3.9), while the second one is less than $4K$, for $r_k(t) \le 2m(t)$, $t \in I$ a.e. Since $8\varepsilon^2 K < \varepsilon/2$, by (3.5), and $k \ge k_0$ is arbitrary, it follows

$$\sup_{k\ge k_0} \int_I h^2\left(\frac{1}{k}\sum_{j=1}^{k} U_{n_j}(t), U(t)\right) dt \le \varepsilon \ ,$$

and thus (3.4) holds. This completes the proof.

The following is a multivalued analogue of the Banach-Mazur theorem.

Theorem 3.4. *Let U_n, $U \in \mathbb{L}^2(I, \mathfrak{X})$, $n \in \mathbb{N}$, satisfy the assumptions of Theorem* 3.3. *Then, there exists a strictly increasing sequence $\{m_k\} \subset \mathbb{N}$ and, for each $k \in \mathbb{N}$, there exist an $r_k \in \mathbb{N}$ and $r_k + 1$ constants $\lambda_i \ge 0$, with $\lambda_{m_k} + \lambda_{m_k+1} + \cdots + \lambda_{m_k+r_k} = 1$, such that*

$$\lim_{k\to+\infty} \int_I h^2\left(\sum_{i=m_k}^{m_k+r_k} \lambda_i U_i(t), U(t)\right) dt = 0 \ . \tag{3.10}$$

Proof. Let $\varepsilon > 0$. By Theorem 3.3, there exist a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ and an integer $k_0 \ge 2$ such that

$$\int_I h^2\left(\frac{1}{k}\sum_{i=1}^{k} U_{n_i}(t), U(t)\right) dt < \frac{\varepsilon}{4} \quad \text{for every } k \ge k_0 \ . \tag{3.11}$$

Claim. For each integer $p \geq k_0$ there exists an $r \in \mathbb{N}$ such that

$$\int_I h^2 \left(\frac{1}{r+1} \sum_{i=p}^{p+r} U_{n_i}(t), U(t) \right) dt < \varepsilon \ . \tag{3.12}$$

Let $p \geq k_0$. Fix an $r \in \mathbb{N}$ sufficiently large so that

$$8K \left(\frac{p-1}{r+1} \right)^2 < \frac{\varepsilon}{2} \quad \text{where} \quad K = \int_I m^2(t) dt \ . \tag{3.13}$$

Setting $S_m(t) = (1/m) \sum_{i=1}^m U_{n_i}(t)$, we have

$$\begin{aligned} h\left(\frac{1}{r+1} \sum_{i=p}^{p+r} U_{n_i}(t), U(t) \right) &= h\left(\frac{p+r}{r+1} S_{p+r}(t), U(t) + \frac{p-1}{r+1} S_{p-1}(t) \right) \\ \leq h\left(\frac{p+r}{r+1} S_{p+r}(t), S_{p+r}(t) \right) &+ h(S_{p+r}(t), U(t)) + \frac{p-1}{r+1} |S_{p-1}(t)| \\ \leq h(S_{p+r}(t), U(t)) + 2\frac{p-1}{r+1} m(t),& \ \ t \in I \text{ a.e.}, \end{aligned}$$

since $|S_{p-1}(t)| \leq m(t)$ and $h\left(\frac{p+r}{r+1} S_{p+r}(t), S_{p+r}(t)\right) \leq \frac{p-1}{r+1}|S_{p+r}(t)| \leq \frac{p-1}{r+1} m(t)$. Hence,

$$\begin{aligned} \int_I h^2 \left(\frac{1}{r+1} \sum_{i=p}^{p+r} U_{n_i}(t), U(t) \right) dt \leq 2 \int_I h^2(S_{p+r}(t), U(t)) dt \\ + 8 \left(\frac{p-1}{r+1} \right)^2 \int_I m^2(t) dt, \end{aligned}$$

from which (3.12) follows, by virtue of (3.11) and (3.13).

Set $\varepsilon_k = 1/2^k$, $[a,b] = \{n \in \mathbb{N} | a \leq n \leq b\}$, where $a, b \in \mathbb{N}$. In view of Theorem 3.3 and the Claim above, given ε_1, there exist a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ and two integers p_1, $s_1 \in \mathbb{N}$ such that

$$\int_I h^2 \left(\frac{1}{s_1+1} \sum_{i=p_1}^{p_1+s_1} U_{n_i}(t), U(t) \right) dt < \varepsilon_1 \ . \tag{3.14}$$

Put $m_1 = n_{p_1}$, $m_1 + r_1 = n_{p_1+s_1}$, and define $\lambda_i = \frac{1}{s_1+1}$ if $i \in \{n_{p_1}, n_{p_1+1}, \dots, n_{p_1+s_1}\}$, $\lambda_i = 0$ if $i \in [m_1, m_1+r_1] \backslash \{n_{p_1}, n_{p_1+1}, \dots, n_{p_1+s_1}\}$. Then (3.14) can be written in the form

$$\int_I h^2 \left(\sum_{i=m_1}^{m_1+r_1} \lambda_i U_i(t), U(t) \right) dt < \varepsilon_1 \ ,$$

where the $r_1 + 1$ constants $\lambda_i \geq 0$ satisfy $\lambda_{m_1} + \lambda_{m_1+1} + \cdots + \lambda_{m_1+r_1} = 1$.

Similarly, given ε_2, there exist a subsequence, say $\{U_{n_k}\}$, of $\{U_n\}_{n \geq m_1+r_1}$, and two integers p_2, $s_2 \in \mathbb{N}$, such that

$$\int_I h^2 \left(\frac{1}{s_2+1} \sum_{i=p_2}^{p_2+s_2} U_{n_i}(t), U(t) \right) dt < \varepsilon_2 \ . \tag{3.15}$$

Setting $m_2 = n_{p_2}$, $m_2 + r_2 = n_{p_2+s_2}$ and defining λ_i accordingly as before, then (3.15) can be written in the form

$$\int_I h^2 \left(\sum_{i=m_2}^{m_2+r_2} \lambda_i U_i(t), U(t) \right) dt < \varepsilon_2 \ ,$$

where the $r_2 + 1$ constants $\lambda_i \geq 0$ satisfy $\lambda_{m_2} + \lambda_{m_2+1} + \cdots + \lambda_{m_2+r_2} = 1$, and $m_2 > m_1$. In this way, by a simple induction argument, one can construct a strictly increasing sequence $\{m_k\} \subset \mathbb{N}$ and, for each $k \in \mathbb{N}$, $r_k + 1$ constants $\lambda_i \geq 0$, with $\lambda_{m_k} + \lambda_{m_k+1} + \cdots + \lambda_{m_k+r_k} = 1$, such that

$$\int_I h^2 \left(\sum_{i=m_k}^{m_k+r_k} \lambda_i U_i(t), U(t) \right) dt < \varepsilon_k \ .$$

From this, letting $k \to +\infty$, (3.10) follows. This completes the proof.

Let $\mathbb{C}(I, \mathfrak{X})$ be the space of all continuous maps $X : I \to \mathfrak{X}$, with distance

$$h_{\mathbb{C}}(X, Y) = \max_{t \in I} h(X(t), Y(t)) \qquad X, Y \in \mathbb{C}(I, \mathfrak{X}) \ .$$

$\mathbb{C}(I, \mathfrak{X})$ is a complete metric space. Moreover, $\mathbb{C}(I, \mathfrak{X})$ is a semilinear space if endowed with the operations of *addition* and *multiplication* by nonnegative scalars induced by the corresponding operations in $\mathfrak{X}$.

Occasionally, we write $\mathbb{C}$ instead of $\mathbb{C}(I, \mathfrak{X})$ when no confusion can arise.

Define $\alpha : \mathbb{C} \times \mathbb{C} \times [0, 1] \to \mathbb{C}$ by

$$\alpha(X, Y, \lambda) = (1 - \lambda)X + \lambda Y, \quad \text{for each } (X, Y, \lambda) \in \mathbb{C} \times \mathbb{C} \times [0, 1] \ .$$

A set $\mathcal{A} \subset \mathbb{C}$ is *convex* if $\alpha(X, Y, \lambda) \in \mathcal{A}$ for every $X, Y \in \mathcal{A}$ and $\lambda \in [0, 1]$.

Remark 3.5. $\mathbb{C}(I, \mathfrak{X})$ equipped with the map α is an α-convex metric space (see [7], Definition 3.1). In fact α is continuous and, moreover, (i) $\alpha(X, X, \lambda) = X$ for every $X \in \mathbb{C}$ and $\lambda \in [0, 1]$; (ii) $\alpha(X, Y, 0) = X$, $\alpha(X, Y, 1) = Y$, for every $(X, Y) \in \mathbb{C} \times \mathbb{C}$; (iii) $h_{\mathbb{C}}(\alpha(X, Y, \lambda), \alpha(\overline{X}, \overline{Y}, \lambda)) \leq \max\{h_{\mathbb{C}}(X, \overline{X}), h_{\mathbb{C}}(Y, \overline{Y})\}$, for every (X, Y), $(\overline{X}, \overline{Y}) \in \mathbb{C} \times \mathbb{C}$ and $\lambda \in [0, 1]$. Hence, by [7] Proposition 3.1, $\mathbb{C}(I, \mathfrak{X})$ is an α-convex metric space.

Set

$$\mathbb{K}(\mathbb{C}) = \{\mathcal{A} \subset \mathbb{C}(I, \mathfrak{X}) | \mathcal{A} \text{ is nonempty compact convex}\}.$$

The space $\mathbb{K}(\mathbb{C})$ is semilinear if endowed with the operations of *addition*, $\mathcal{A} + \mathcal{B} = \{X + Y \in \mathbb{C}(I, \mathfrak{X}) | X \in \mathcal{A}, Y \in \mathcal{B}\}$, and *multiplication* by scalars $\lambda \geq 0$, $\lambda\mathcal{A} = \{\lambda X \in \mathbb{C}(I, \mathfrak{X}) | X \in \mathcal{A}\}$, where $\mathcal{A}$, $\mathcal{B} \in \mathbb{K}(\mathbb{C})$. We equip $\mathbb{K}(\mathbb{C})$ with the Pompeiu-Hausdorff metric H induced by the metric $h_{\mathbb{C}}$ of $\mathbb{C}(I, \mathfrak{X})$, i.e.

$$H(\mathcal{A}, \mathcal{B}) = \max\{e(\mathcal{A}, \mathcal{B}), e(\mathcal{B}, \mathcal{A})\},$$

where $e(\mathcal{A}, \mathcal{B}) = \sup_{X \in \mathcal{A}} d(X, \mathcal{B})$, $d(X, \mathcal{B}) = \inf_{Y \in \mathcal{B}} h_{\mathbb{C}}(X, Y)$, and similarly for $e(\mathcal{B}, \mathcal{A})$.

The *convex hull* $co\mathcal{A}$ of a set $\mathcal{A} \subset \mathbb{C}(I, \mathfrak{x})$ is given by

$$co\mathcal{A} = \{Y \in \mathbb{C}(I, \mathfrak{x}) | Y = \sum_{i=1}^{n} \lambda_i X_i,$$

$$\text{for some } X_i \in \mathcal{A} \text{ and } \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1 \ , \ n \in \mathbb{N}\}.$$

The *closed convex hull* $cl_{\mathbb{C}}(co\mathcal{A})$ of $\mathcal{A}$ is the closure of $co\mathcal{A}$ in $\mathbb{C}(I, \mathfrak{x})$.

Remark 3.6. Let $\mathcal{A}$, $\mathcal{B} \subset \mathbb{C}(I, \mathfrak{x})$. Then, (i) $co\mathcal{A}$ and $cl_{\mathbb{C}}(co\mathcal{A})$ are convex; (ii) if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ is closed and convex, then $cl_{\mathbb{C}}(co\mathcal{A}) \subset \mathcal{B}$; if $\mathcal{A} \in \mathbb{K}(\mathbb{C})$ and $\varepsilon > 0$, then $N_{\mathbb{C}}[\mathcal{A}, \varepsilon]$ is closed and convex.

The following is a Mazur type theorem in the space $\mathbb{C}(I, \mathfrak{x})$.

Theorem 3.7. *The closed convex hull $cl_{\mathbb{C}}(co\mathcal{A})$ of a compact set $\mathcal{A} \subset \mathbb{C}(I, \mathfrak{x})$ is compact and convex.*

Proof. Let $\varepsilon > 0$. Let $\{X_1, \ldots, X_k\}$ be an $\frac{\varepsilon}{2}$-net of $\mathcal{A}$. For $i = 1, \ldots, k$ the sets $\mathcal{V}_i = \{Y \in \mathbb{C}(I, \mathfrak{x}) | Y = \alpha X_i, \ \alpha \in [0, 1]\}$ are in $\mathbb{K}(\mathbb{C})$, hence also the set $\mathcal{V} = \mathcal{V}_1 + \cdots + \mathcal{V}_k$ is in $\mathbb{K}(\mathbb{C})$. Let $\{Y_i, \ldots, Y_n\}$ be an $\frac{\varepsilon}{2}$-net of $\mathcal{V}$. Then

$$\mathcal{A} \subset N_{\mathbb{C}}\left[\{X_1, \ldots, X_k\}, \varepsilon/2\right] \subset N_{\mathbb{C}}[\mathcal{V}, \varepsilon/2] \subset N_{\mathbb{C}}[\{Y_1, \ldots, Y_n\}, \varepsilon] \ .$$

$N_{\mathbb{C}}[\mathcal{V}, \varepsilon/2]$ is closed and convex, hence $cl_{\mathbb{C}}(co\mathcal{A}) \subset N_{\mathbb{C}}[\mathcal{V}, \varepsilon/2]$, and thus

$$cl_{\mathbb{C}}(co\mathcal{A}) \subset N_{\mathbb{C}}[\{Y_1, \ldots, Y_n\}, \ \varepsilon] \ .$$

Therefore $cl_{\mathbb{C}}(co\mathcal{A})$ is totally bounded and a fortiori compact, for $\mathbb{C}(I, \mathfrak{x})$ is complete. The convexity of $cl_{\mathbb{C}}(co\mathcal{A})$ is obvious. This completes the proof.

A multifunction $\Gamma : M \to \mathbb{K}(\mathbb{C})$ is *compact* if the set $\mathcal{R} = cl_{\mathbb{C}}\left(\bigcup_{x \in M} \Gamma(x)\right)$ is compact in $\mathbb{C}(I, \mathfrak{x})$.

The following theorem is a version of the fixed point theorem of Kakutani-Ky Fan [15], [9], for multifunctions with values contained in $\mathbb{C}(I, \mathfrak{x})$.

Theorem 3.8. *Let Ω be a nonempty closed convex subset of $\mathbb{C}(I, \mathfrak{x})$. Let $\Gamma : \Omega \to \mathbb{K}(\mathbb{C})$ be an u.s.c. and compact multifunction with values $\Gamma(X) \subset \Omega$, for every $X \in \Omega$. Then, there exists at least one $X \in \Omega$ such that $X \in \Gamma(X)$.*

Proof. By Remark 3.5, $\mathbb{C}(I, \mathfrak{x})$ is an α-convex complete metric space. Since $\mathcal{R} = cl_{\mathbb{C}}(\bigcup_{X \in \Omega} \Gamma(X))$ is compact, then so is $\mathcal{R}_o = cl_{\mathbb{C}}(co\mathcal{R})$, by Theorem 3.7. Moreover, $\mathcal{R}_0 \subset \Omega$, because $\mathcal{R} \subset \Omega$ and Ω is closed and convex in $\mathbb{C}(I, \mathfrak{x})$.

Define $\Gamma_0 : \mathcal{R}_0 \to \mathbb{K}(\mathbb{C})$ by $\Gamma_0(X) = \Gamma(X)$ for each $X \in \mathcal{R}_0$. Since Γ_0 is u.s.c. on $\mathcal{R}_0$, a compact convex subset of $\mathbb{C}(I, \mathfrak{x})$, and takes values $\Gamma(X) \subset \mathcal{R}_0$ then, by virtue of [7] Proposition 7.7, there exists an $X \in \mathcal{R}_0$ such that $X \in \Gamma_0(X)$. As $\Gamma_0(X) = \Gamma(X)$, the statement follows, completing the proof.

4. APPLICATION TO SET DIFFERENTIAL INCLUSIONS

In this section, using the previous results, we establish an existence theorem for the Cauchy problem (C).

We first prove some lemmas. The following one is a variant (see [6]) of an important theorem due to Hermes [10].

Lemma 4.1. *Let* $\{U_n\}$ *be a sequence of measurable maps* $U_n\colon\ I \to \mathfrak{x}$ *satisfying* $|U_n(t)| \le M$, $t \in I$, M *a constant, and suppose that the corresponding sequence* $\{Y_n\}$ *of continuous maps* $Y_n : I \to \mathfrak{x}$, *given by*

$$Y_n(t) = \int_0^t U_n(s)ds \quad \textit{for each } t \in I \ , \tag{4.1}$$

converges uniformly to $Y : I \to \mathfrak{x}$. *Then there exists a measurable map* $U : I \to \mathfrak{x}$, *with* $|U(t)| \le M$, $t \in I$, *such that*

$$Y(t) = \int_0^t U(s)ds \quad \textit{for each } t \in I \ . \tag{4.2}$$

Lemma 4.2. *Let* $\Phi : I \times \mathfrak{x} \to \mathbb{K}(\mathfrak{x})$ *satisfy the assumption* (h_1), (h_2), (h_3). *Then, for each* $X \in \mathbb{C}(I, \mathfrak{x})$, *the set*

$$\mathcal{U}(X) = \{U : I \to \mathfrak{x} | U \textit{ is a measurable selection of } \Phi(\cdot, X(\cdot)) \textit{ on } I\} \tag{4.3}$$

is nonempty and convex.

Proof. $\mathcal{U}(X)$ is non empty. To see this, for $n \in \mathbb{N}$, set $t_i = i/2^n$, $i = 0, 1, \ldots, 2^n$, $I_i = [t_{i-1}, t_i)$, $i = 1, \ldots, 2^n - 1$, $I_{2^n} = [t_{2^n-1}, 1]$, and define $S_n : I \to \mathfrak{x}$, by

$$S_n(t) = \sum_{i=1}^{2^n} X(t_{i-1})\chi_{I_i}(t) \quad \text{for each } t \in I \ ,$$

where χ_{I_i} stands for the characteristic function of I_i. It is evident that $S_n \to X$ uniformly on I.

For each $n \in \mathbb{N}$, consider the multifunction $t \to \Phi(t, S_n(t))$, $t \in I$. Clearly

$$\Phi(t, S_n(t)) = \sum_{i=1}^{2^n} \Phi(t, X(t_{i-1}))\chi_{I_i}(t) \quad \text{for each } t \in I \ .$$

By (h_1), each $\Phi(\cdot, X(t_{i-1}))$ restricted to I_i, $i = 1, \ldots, 2^n$, is measurable and hence it admits a measurable selection (see Himmelberg [11]). Therefore there exists a measurable map $U : I \to \mathfrak{x}$ satisfying

$$U_n(t) \in \Phi(t, S_n(t)) \qquad \text{for each } t \in I \ , \tag{4.4}$$

where, by (h_3), $|U_n(t)| \le M$ for each $t \in I$. For $n \in \mathbb{N}$, let $Y_n : I \to \mathfrak{x}$ be given by (4.1). By Arzelà-Ascoli's theorem, the sequence $\{Y_n\} \subset \mathbb{C}(I, \mathfrak{x})$ contains a subsequence, say $\{Y_n\}$, which converges uniformly to some $Y \in \mathbb{C}(I, \mathfrak{x})$. An application of Lemma 4.1

gives a measurable $U : I \to \mathfrak{X}$, with $|U(t)| \leq M$ for each $t \in I$, so that (4.2) is valid. Since (3.3) trivially holds then, by virtue of Theorem 3.4, there exists a strictly increasing sequence $\{m_k\} \subset \mathbb{N}$ and, for each $k \in \mathbb{N}$, there are $r_k + 1$ constants $\lambda_i \geq 0$, with $\sum_{i=m_k}^{m_k+r_k} \lambda_i = 1$, such that, setting $W_{m_k}(t) = \sum_{i=m_k}^{m_k+r_k} \lambda_i U_i(t)$, $t \in I$, we have

$$\lim_{k \to +\infty} \int_I h^2(W_{m_k}(t), U(t))dt = 0 \ .$$

It follows that a subsequence $\{W_{m_{k_j}}\}$ of $\{W_{m_k}\}$ satisfies

$$\lim_{j \to +\infty} h(W_{m_{k_j}}(t), U(t)) = 0 \quad \text{for each} \quad t \in I \backslash I_0 \ , \tag{4.5}$$

where I_0 is a subset of I of measure zero.

We claim that

$$U(t) \in \Phi(t, X(t)) \quad \text{for each} \quad t \in I \backslash I_0 \ . \tag{4.6}$$

In fact, let $t \in I \backslash I_0$ and $\varepsilon > 0$. By (h_2), $\Phi(t, \cdot)$ is u.s.c. at $X(t)$, and thus there exists $\delta > 0$ such that

$$\Phi(t, Z) \subset N_{\mathfrak{X}}[\Phi(t, X(t)), \varepsilon] \quad \text{for each} \quad Z \in B_{\mathfrak{X}}(X(t), \delta) \ . \tag{4.7}$$

Fix $j_0 \in \mathbb{N}$ so that $S_{m_{k_j}}(t) \in B_{\mathfrak{X}}(X(t), \delta)$, for all $j \geq j_0$. Then, in view of (4.4) and (4.7), we have

$$\begin{aligned} W_{m_{k_j}}(t) &\in \sum_{i=m_{k_j}}^{m_{k_j}+r_{k_j}} \lambda_i \Phi(t, S_i(t)) \\ &\subset \sum_{i=m_{k_j}}^{m_{k_j}+r_{k_j}} \lambda_i N_{\mathfrak{X}}[\Phi(t, X(t)), \varepsilon] = N_{\mathfrak{X}}[\Phi(t, X(t)), \varepsilon]. \end{aligned}$$

From this and (4.5) it follows that $U(t) \in N_{\mathfrak{X}}[\Phi(t, X(t)), \varepsilon]$. As $\varepsilon > 0$ and $t \in I \backslash I_0$ are arbitrary, (4.6) holds and thus $U \in \mathcal{U}(X)$. The convexity of $\mathcal{U}(X)$ is obvious. This completes the proof.

Lemma 4.3. *Let $\Phi : I \times \mathfrak{X} \to \mathbb{K}(\mathfrak{X})$ satisfy the assumptions (h_1), (h_2), (h_3) and let $A \in \mathfrak{X}$. For $X \in \mathbb{C}(I, \mathfrak{X})$, set*

$$\begin{aligned} \Gamma(X) = \{Z \in \mathbb{C}(I, \mathfrak{X}) |\ &\textit{there is } U \in \mathcal{U}(X) \textit{ such that} \\ &Z(t) = A + \int_0^t U(s)ds \textit{ for each } t \in I\} \ , \end{aligned} \tag{4.8}$$

where $\mathcal{U}(X)$ is given by (4.3). Then, $\Gamma(X) \in \mathbb{K}(\mathbb{C})$. Moreover, the map $\Gamma : \mathbb{C}(I, \mathfrak{X}) \to \mathbb{K}(\mathbb{C})$ defined by (4.8) is u.s.c.

Proof. $\Gamma(X) \in \mathbb{K}(\mathbb{C})$. In fact $\Gamma(X)$ is nonempty and convex, for so is $\mathcal{U}(X)$, by Lemma 4.2. It remains to prove that $\Gamma(X)$ is compact. Let $\{Z_n\} \subset \Gamma(X)$. Then, for some $U_n \in \mathcal{U}(X)$,

$$Z_n(t) = A + \int_0^t U_n(s)ds \quad \text{for each } t \in I \ . \tag{4.9}$$

Consider the sequence $\{Y_n\} \subset \mathbb{C}(I, \mathfrak{X})$, where Y_n is given by (4.1). By Arzelà-Ascoli's theorem a subsequence, say $\{Y_n\}$, converges uniformly to some $Y \in \mathbb{C}(I, \mathfrak{X})$. Hence $\{Z_n\}$ converges uniformly to $Z = A + Y$. An application of Lemma 4.1 gives a measurable $U : I \to \mathfrak{X}$ for which (4.2) is valid. Then, as in the proof Lemma 4.2, retaining the same notation, one can construct a sequence $\{W_{m_{k_j}}\}$ for which (4.5) holds. Moreover, for each $t \in I \backslash I_0$,

$$W_{m_{k_j}}(t) = \sum_{i=m_{k_j}}^{m_{k_j}+r_{k_j}} \lambda_i U_i(t) \in \sum_{i=m_{k_j}}^{m_{k_j}+r_{k_j}} \lambda_i \Phi(t, X(t)) = \Phi(t, X(t)) \ .$$

From this, letting $j \to +\infty$, it follows that $U(t) \in \Phi(t, X(t))$, $t \in I$ a.e., and thus $U \in \mathcal{U}(X)$. Since, in addition,

$$Z(t) = A + \int_0^t U(s)ds \quad \text{for each } t \in I \ , \tag{4.10}$$

it follows that $Z \in \Gamma(X)$. Hence $\Gamma(X)$ is compact, and thus $\Gamma(X) \in \mathbb{K}(\mathbb{C})$.

Γ is u.s.c. In the contrary case, there exist $X \in \mathbb{C}(I, \mathfrak{X})$, $\varepsilon > 0$, and a sequence $\{X_n\} \subset \mathbb{C}(I, \mathfrak{X})$ converging to X, such that $e(\Gamma(X_n), \Gamma(X)) > \varepsilon$ for every $n \in \mathbb{N}$. In each $\Gamma(X_n)$ take a Z_n such that

$$d(Z_n, \Gamma(X)) > \varepsilon \quad \text{for each} \quad n \in \mathbb{N} \ . \tag{4.11}$$

Then, for some $U_n \in \mathcal{U}(X_n)$, Z_n satisfies (4.9). Consider the sequence $\{Y_n\}$, where Y_n is given by (4.1). By Arzelà-Ascoli's theorem a subsequence, say $\{Y_n\}$, converges uniformly to some $Y \in \mathbb{C}(I, \mathfrak{X})$. Thus, $\{Z_n\}$ converges uniformly to $Z = A + Y$. An application of Lemma 4.1 gives a measurable $U : I \to \mathfrak{X}$ such that (4.2) holds. Likewise in Lemma 4.2, retaining the same notation, one can construct a sequence $\{W_{m_{k_j}}\}$ for which (4.5) is valid. Then, arguing as in Lemma 4.2 (with $S_p(t)$ replaced by $X_p(t)$, $p \in \mathbb{N}$) one can show that $U(t) \in \Phi(t, X(t))$, $t \in I$, a.e. and thus $U \in \mathcal{U}(X)$. Since, in addition, Z satisfies (4.10), it follows that $Z \in \Gamma(X)$. This contradicts (4.11), since $Z_n \to Z$ as $n \to +\infty$. Therefore Γ is u.s.c. This completes the proof.

We now are ready to prove an existence result for the Cauchy problem (C).

Theorem 4.4. *Let $\Phi : I \times \mathfrak{X} \to \mathbb{K}(\mathfrak{X})$ satisfy the assumptions (h_1), (h_2), (h_3) and let $A \in \mathfrak{X}$. Then the set $\mathcal{S}$ of all solutions $X : I \to \mathfrak{X}$ of the Cauchy problem (C) is a nonempty compact subset of $\mathbb{C}(I, \mathfrak{X})$.*

Proof. $\mathcal{S}$ is nonempty. For each $X \in \mathbb{C}(I, \mathfrak{x})$ let $\Gamma(X)$ be given by (4.8). By Lemma 4.3, (4.8) defines an u.s.c. multifunction

$$\Gamma : \mathbb{C}(I, \mathfrak{x}) \to \mathbb{K}(\mathbb{C}) \ .$$

Γ is compact, i.e. the set

$$\mathcal{R} = cl_{\mathbb{C}} \left(\bigcup_{X \in \mathbb{C}(I, \mathfrak{X})} \Gamma(X) \right) \tag{4.12}$$

is compact in $\mathbb{C}(I, \mathfrak{x})$. In fact, let $\{Z_n\} \subset \mathcal{R}$. By (4.12) there exists a sequence $\{\tilde{Z}_n\}$, where $\tilde{Z}_n \in \Gamma(X_n)$ for some $X_n \in \mathbb{C}(I, \mathfrak{x})$, such that $h_{\mathbb{C}}(Z_n, \tilde{Z}_n) < 1/n$, $n \in \mathbb{N}$. As $\tilde{Z}_n \in \Gamma(X_n)$, for some $U_n \in \mathcal{U}(X_n)$ we have

$$\tilde{Z}_n(t) = A + \int_0^t U_n(s) ds \quad \text{for each} \quad t \in I \ .$$

By Arzelà-Ascoli's theorem $\{\tilde{Z}_n\}$ contains a subsequence, say $\{\tilde{Z}_{n_k}\}$, which converges uniformly to a $Z \in \mathcal{R}$. Since $\{Z_{n_k}\}$ converges to Z, it follows that Γ is compact.

By virtue of Theorem 3.8, there exists an $X \in \mathbb{C}(I, \mathfrak{x})$ such that $X \in \Gamma(X)$. Therefore, for some measurable $U : I \to \mathfrak{x}$, we have

$$X(t) = A + \int_0^t U(s) ds \quad \text{for each } t \in I \ ,$$

$$U(t) \in \Phi(t, X(t)) \quad \text{for } t \in I \text{ a.e.} \ ,$$

i.e. $X : I \to \mathfrak{x}$ is a solution of the Cauchy problem (C). Hence $\mathcal{S} \neq \phi$. The compactness of $\mathcal{S}$ in $\mathbb{C}(I, \mathfrak{x})$ can be proved by an easy adaptation of the argument of Lemma 4.3. This completes the proof.

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