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FIXED POINT RESULTS FOR MULTIFUNCTIONS IN ORDERED TOPOLOGICAL SPACES WITH APPLICATIONS TO INCLUSION PROBLEMS AND TO GAME THEORY

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ABSTRACT. We prove existence results for minimal and maximal fixed points of multifunctions in ordered topological spaces, and apply the obtained results to study the solvability of inclusion problems and the existence of extremal Nash equilibria for normal-form games.

Keywords: multifunction, fixed point, inclusion problem, solution, maximal, minimal, sup-center, inf-center, normal-form game, Nash equilibrium, ordered topological space

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1. INTRODUCTION

Let P be a nonempty subset of $\mathbb{R}^2$, equipped with coordinatewise ordering. As an introductory result to our study notice that a multifunction $F : P \to 2^P \setminus \emptyset$ has a fixed point, that is, $x \in F(x)$ for some $x \in P$, if the following conditions hold.

(c1) $\sup\{c, y\} \in P$ for some $c \in P$ and for each $y \in F[P] = \bigcup\{F(x) : x \in P\}$.
(c2) If $x \leq y$ in P, then for each $z \in F(x)$ there exists a $w \in F(y)$ such that $z \leq w$, and for each $w \in F(y)$ there exists a $z \in F(x)$ such that $z \leq w$.
(c3) Strictly monotone sequences of $F[P]$ are finite.

For instance, a fixed point of F can be obtained by the following algorithm: Denote $x_0 = c$, and choose y_0 from $F(x_0)$. If x_n and $y_n \in F(x_n)$ are chosen, and if $x_n \neq y_n$, choose $x_{n+1} = y_n$ if $x_n \leq y_n$ or $y_n \leq x_n$, otherwise choose $x_{n+1} = \sup\{c, y_n\}$. If $x_n \leq x_{n+1}$, apply condition (c2) to choose y_{n+1} from $F(x_{n+1})$ such that $y_n \leq y_{n+1}$. If $x_{n+1} \leq x_n$, choose y_{n+1} from $F(x_{n+1})$ such that $y_{n+1} \leq y_n$. Condition (c3) ensures that after a finite number of choices we get the situation where $x_n = y_n \in F(x_n)$, so that x_n is a fixed point of F.

A necessary and sufficient condition for a point $c = (c_1, c_2)$ of P to satisfy (c1) is that whenever a point $y = (y_1, y_2)$ of $F[P]$ and c are unordered, then $(y_1, c_2) \in P$ if $y_2 < c_2$ and $(c_1, y_2) \in P$ if $y_1 < c_1$. No conditions are imposed on other points of $F[P]$.

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In this paper we study first the existence of extremal fixed points of $F : P \to 2^P \backslash \emptyset$ when P is a nonempty subset of an ordered topological space, and when the finiteness of the sequences in (c3) is replaced by their convergence. The obtained results are then used to generalize existence results derived in [4, 6, 7] for inclusion problem $Lu \in Nu$ and to study the existence of extremal Nash equilibria for normal-form games.

A generalized iteration method introduced in [5] is used in the proof of our key result, Lemma 2.6.

2. PRELIMINARIES

Let $X = (X, \leq)$ be an ordered topological space, i.e., for each $a \in X$ the order intervals $[a) = \{x \in X : \ a \leq x\}$ and $(a] = \{x \in X : \ x \leq a\}$ are closed in the topology of X. In what follows P denotes a nonempty subset of X having the following property:

(C) Each chain C of P whose monotone sequences converge in P contains an increasing sequence which converges to $\sup C$ and a decreasing sequence which converges to $\inf C$.

In ordered metric spaces, and in ordered normed spaces equipped with a norm-topology or a weak topology each nonempty subset P has property (C) according to [8], Proposition 1.1.5 and Lemma 1.1.2 and [2], Appendix, Lemma A.3.1 and their duals. If X is an ordered topological space which satisfies the second countability axiom, then each chain of X is separable, whence each nonempty subset P of X has property (C) by [8], Lemma 1.1.7 and its dual.

Definition 2.1. We say that $F : P \to 2^P \setminus \emptyset$ is *increasing upward* if $x, y \in P$, $x \leq y$ and $z \in F(x)$ imply an existence of $w \in F(y)$ such that $z \leq w$. F is *increasing downward* if $x, y \in P$, $x \leq y$ and $w \in F(y)$ imply that $z \leq w$ for some $z \in F(x)$. If F is increasing upward and downward we say that F is *increasing*.

The following Lemma is a consequence of [5], Lemma 2, which in turn is an application of a recursion method introduced in [8], Lemma 1.1.1.

Lemma 2.2. *Given $F : P \to 2^P \setminus \emptyset$, let $G : P \to P$ be a selection function of F, i.e. $G(x) \in F(x)$ for all $x \in P$. Then for each $c \in P$ there is a unique well-ordered chain $C = C(G)$ in P, called a well-ordered (w.o.) chain of cG-iterations, satisfying*

$$x \in C \quad \text{if and only if} \quad x = \sup\{c, G[\{y \in C : \ y < x\}]\}. \tag{2.1}$$

The values of multifunctions are assumed to satisfy the following types of order compactness properties.

Definition 2.3. A nonempty subset A of a subset Y of X is called *sequentially order compact upward* in Y if for each increasing sequence (y_n) of Y the intersection of all the sets $[y_n) \cap A$ is nonempty whenever each $[y_n) \cap A$ is nonempty. If for each decreasing sequence (y_n) of Y the intersection of all the sets $(y_n] \cap A$ is nonempty whenever each $(y_n] \cap A$ is nonempty, we say that A is *sequentially order compact downward* in Y. If both these properties hold, we say that A is *sequentially order compact* in Y. If $Y = A$, we say that A is *sequentially order compact*.

If A has the greatest element (respectively the least element), then A is sequentially order compact upward (respectively downward) in any subset of X which contains A. A sequentially order compact set is not necessarily (topologically) compact, not even closed, as we see by choosing $X = \mathbb{R}^2$, ordered coordinatewise, and $Y = A = \{(x, -x) : \ x \in I\}$, where I is a nonempty open interval of $\mathbb{R}$. On the other hand, each compact or countably compact subset A of X is obviously sequentially order compact in each subset of X which contains A. Moreover, the following results hold.

Lemma 2.4. *(a) If A is a sequentially compact subset of X, then A is sequentially order compact in each subset of X which contains A.*

(b) A subset A of X is sequentially order compact upward (in A) if and only if each increasing sequence of A has an upper bound in A.

Proof. (a) Assume that A is a sequentially compact subset of X, and that $A \subseteq Y \subseteq X$. Let (y_n) be an increasing sequence in Y, and assume that $[y_n) \cap A$ is nonempty for each n. Choose z_n from each $[y_n) \cap A$. Since A is sequentially compact, there exists a subsequence (z_{n_k}) of (y_n) which has a limit z in A. For each fixed n the sequence $(z_{n_k})_{k \geq n}$ is contained in $[y_n)$ which is closed, whence its limit z belongs also to $[y_n)$, and hence to $[y_n) \cap A$. This holds for each n, so that z belongs to the intersection of all $[y_n) \cap A$. This proves that A is sequentially order compact upward in Y. The proof that A is sequentially order compact downward in Y is similar.

(b) Assume that A is sequentially order compact upward, and let (y_n) be an increasing sequence of A. Then $y_n \in [y_n) \cap A$, for each n, whence each $[y_n) \cap A$ is nonempty. Thus their intersection contains at least one point y. In particular, $y \in A$ and $y_n \leq y$ for each n, so that y is an upper bound of (y_n) in A.

Conversely, if $y \in A$ is an upper bound of an increasing sequence (y_n) of A, then y belongs to each $[y_n) \cap A$, and hence also to their intersection. If this holds for every increasing sequence (y_n) of A, then A is sequentially order compact upward. □

Definition 2.5. We say that a subset A of P has a sup-center c in P if $c \in P$ and $\sup\{c, x\}$ exists and belongs to P for each $x \in A$. If $\inf\{c, x\}$ exists and belongs to P for each $x \in A$, we say that c is an inf-center of A in P.

The result of Lemma 2.2 is used in the proof of the next result which plays a key role in the proof of our main fixed point theorem.

Lemma 2.6. *Assume that $F : P \to 2^P \setminus \emptyset$ is increasing upward, that its values are sequentially order compact upward in $F[P]$, that increasing sequences of $F[P]$ have limits in P and the set of these limits has a sup-center in P. Then $(b] \cap F(b) \neq \emptyset$ for a $b \in P$.*

Proof. Let c be a sup-center of the set of limits of increasing sequences of $F[P]$. Denote by

$$\mathcal{G} := \{G : P \to P : \ G(x) \in F(x) \text{ for all } x \in P\}$$

the set of all selections of F. For each $G \in \mathcal{G}$ denote by C_G the longest such an initial segment of the w.o. chain $C(G)$ of cG-iterations that the restriction $G|C_G$ of G to C_G is increasing. Let $\prec$ be a well-ordering of $\mathcal{G}$, and define a transfinite sequence of the elements of $\mathcal{G}$ as follows: Let G_0 be the least element of $\mathcal{G}$. If α is such an ordinal that G_β is chosen for each $\beta < \alpha$, let G_α be the least element of $\mathcal{G}$, if exists, such that C_{G_β} is a proper initial segment of C_{G_α} and $G_\alpha|C_{G_\beta} = G_\beta|C_{G_\beta}$ for each $\beta < \alpha$. Denote $\lambda = \cup\alpha$ and $C = \cup_{\alpha\in\lambda} C_{G_\alpha}$. Since each C_{G_α} is well-ordered, then also C is well-ordered. The above construction implies also that $G = \cup_{\alpha\in\lambda} G_\alpha|C_{G_\alpha}$ is an increasing selection function of the restriction of F to C. Since C is well-ordered and G is increasing, then $G[C]$ is also well-ordered, and it is contained in $F[P]$. This implies by a hypothesis that increasing sequences of $G[C]$ converge. In view of property (C) one of these sequences converge to $w = \sup G[C]$. Moreover, $b = \sup\{c, w\}$ exists in P by a hypothesis. It is easy to see that $b = \sup\{c, G[C]\}$. If $x \in C$, then $x \in C_{G_\alpha}$ for some $\alpha < \lambda$, and hence

$$\begin{aligned} x &= \sup\{c, G_\alpha[\{y \in C_{G_\alpha} : \ y < x\}]\} \\ &= \sup\{c, G[\{y \in C : \ y < x\}]\} \leq \sup\{c, G[C]\} = b. \end{aligned} \tag{2.2}$$

This proves that b is an upper bound of C.

By the above construction there exists an increasing sequence (y_n) in $G[C]$ which converges to $w = \sup G[C]$. Denoting $x_n = \min\{x \in C : \ G(x) = y_n\}$, then $x_n \leq b$. Since $y_n = G(x_n) \in F(x_n)$ and F is increasing upward, there exists a $z_n \in F(b)$ such that $y_n \leq z_n$. This holds for each n, whence the sets $[y_n) \cap F(b)$ are nonempty. Because $F(b)$ is sequentially order compact upward in $F[P]$, then the intersection of the sets $[y_n) \cap F(b)$ is nonempty. Choose z from that intersection. Since z belongs to each $[y_n) \cap F(b)$, then $y_n \leq z$ for each n, whence $w = \lim_n y_n = \sup_n y_n \leq z$.

To show that $b = \max C$, assume on the contrary that b is a strict upper bound of C. Let G_λ be the least element of $\mathcal{G}$ whose restriction to $C \cup \{b\}$ is $G \cup \{(b, z)\}$. Since G is increasing and $G_\lambda(x) = G(x) \leq w \leq z = G_\lambda(b)$ for each $x \in C$, then G_λ is increasing in $C \cup \{b\}$. Moreover,

$$b = \sup\{c, G[C]\} = \sup\{c, G_\lambda[C]\} = \sup\{c, G_\lambda[\{y \in C \cup \{b\} : \ y < b\}]\},$$

whence $C \cup \{b\}$ is an initial segment of the w.o. chain of cG_λ-iterations. Thus C is a proper subset C_{G_λ}. But this is impossible by the construction of C. Consequently, $b = \max C$, whence $b = \sup\{c, G[C]\} = \sup\{c, G(b)\}$ because G is increasing in C. In particular, $G(b) \le b$ and $G(b) \in F(b)$, so that $G(b)$ belongs to the set $(b] \cap F(b)$. □

The next result is the dual to that of Lemma 2.6.

Lemma 2.7. *Assume that $F : P \to 2^P \setminus \emptyset$ increasing downward, that its values are sequentially order compact downward in $F[P]$, and that decreasing sequences of $F[P]$ have limits in P and the set of these limits has an inf-center in P. Then $[a) \cap F(a) \neq \emptyset$ for some $a \in P$.*

3. FIXED POINT RESULTS

Throughout this section we assume that X is an ordered topological space and P is a nonempty subset of X having property (C). The following result is proved in [3].

Lemma 3.1. *Let $F : P \to 2^P \setminus \emptyset$ satisfy the following hypothesis.*

(F1) *If $y_n \in [x_n) \cap F(x_n)$, $n \in \mathbb{N}$, and if (y_n) is increasing, then $x = \lim_n y_n$ exists in P and $[x) \cap F(x) \neq \emptyset$.*

If $[a) \cap F(a) \neq \emptyset$ for some $a \in P$, then F has a maximal fixed point x_+, which is also a maximal element of those $x \in P$ for which $[x) \cap F(x) \neq \emptyset$.

As an application of Lemma 3.1 we prove the following result.

Proposition 3.2. *Assume that $F : P \to 2^P \setminus \emptyset$ is increasing upward, that its values are sequentially order compact upward in $F[P]$, and that increasing sequences of $F[P]$ converge in P. If $[a) \cap F(a) \neq \emptyset$ for some $a \in P$, then F has a maximal fixed point.*

Proof. It suffices to show that the hypothesis (F1) of Lemma 3.1 holds. Assume that $y_n \in [x_n) \cap F(x_n)$, $n \in \mathbb{N}$, and that (y_n) is increasing. Since the increasing sequences of $F[P]$ converge in P, then $x = \lim_n y_n = \sup_n y_n$ exists in P. Because F is increasing upward, then $[y_n) \cap F(x) \neq \emptyset$ for each $n \in \mathbb{N}$. Because $F(x)$ is sequentially order compact upward in $F[P]$, there exists $y \in \cap\{[y_n) \cap F(x) : \ n \in \mathbb{N}\}$. In particular, $y_n \le y$ for each $n \in \mathbb{N}$, whence y is an upper bound of (y_n). Since $x = \sup_n y_n$, then $x \le y$. Moreover, $y \in F(x)$ so that $y \in [x) \cap F(x)$. Thus (F1) is valid. □

The next result is dual to that of Proposition 3.2.

Proposition 3.3. *Assume that $F : P \to 2^P \setminus \emptyset$ is increasing downward, that the values of F are sequentially order compact downward in $F[P]$, and that decreasing sequences of $F[P]$ converge in P. If $(b] \cap F(b) \neq \emptyset$ for some $b \in P$, then F has a minimal fixed point.*

Now we are ready to prove our main fixed point result.

Theorem 3.4. *Assume that $F : P \to 2^P \setminus \emptyset$ is increasing, that its values are sequentially order compact in $F[P]$, and that monotone sequences of $F[P]$ converge in P.*

(a) If the set of limits of increasing sequences of $F[P]$ has a sup-center in P, then F has a minimal fixed point.

(b) If the set of limits of decreasing sequences of $F[P]$ has an inf-center in P, then F has a maximal fixed point.

Proof. (a) The hypotheses of Lemma 2.6 are valid, whence there exists a $b \in P$ such that $(b] \cap F(b) \neq \emptyset$. Thus the hypotheses of Proposition 3.3 hold, which implies the assertion.

(b) The hypotheses of Lemma 2.7 are valid. Thus there exists an $a \in P$ such that $[a) \cap F(a) \neq \emptyset$. The hypotheses of Proposition 3.2 are then valid, which implies the assertion. □

Remark 3.5. Classical fixed point theorems in ordered spaces (cf., e.g., [1, 10, 11, 12, 13, 14, 17, 18]) don't provide tools to prove the results of Theorem 3.4.

Propositions 3.2 and 3.3 and Theorem 3.4 generalize the related fixed point results derived in [4, 5, 6, 7, 10] for increasing multifunctions in ordered topological vector spaces.

4. APPLICATIONS TO AN INCLUSION PROBLEM

In this section we apply Theorem 3.4 to prove a existence results for the inclusion problem

$$Lu \in Nu \tag{4.1}$$

in the case when $L : V \to P$ and $N : V \to 2^P \setminus \emptyset$, where V is a nonempty set and P is a subset of an ordered topological space X having property (C).

Theorem 4.1. *Assume that $L : V \to P$ is bijective, that the values of $N : V \to 2^P \setminus \emptyset$ are sequentially order compact in $N[V]$, that monotone sequences of N[V] converge in P, and that $N \circ L^{-1}$ is increasing. If P has a sup-center or an inf-center, then the inclusion problem 4.1 has a solution.*

Proof. We shall show that the multifunction $F := N \circ L^{-1} : P \to 2^P \setminus \emptyset$ satisfies the hypotheses of Theorem 3.4. Notice first that F is increasing by a hypothesis, and that $F[P] = NL^{-1}[P] = N[V]$. Moreover, if $x \in P$, then denoting $u = L^{-1}x$ we have $F(x) = NL^{-1}x = Nu$, whence $F(x)$ is sequentially order compact in $F[P] = N[V]$ because Nu is. Since the monotone sequences of $N[V] = F[P]$ converge by a hypothesis, then F satisfies the hypotheses of Theorem 3.4. Because P has a sup-center or an inf-center, then F has by Theorem 3.4 a fixed point x. Denoting $u = L^{-1}x$, then $Lu = x \in F(x) = NL^{-1}x = Nu$, whence u is a solution of 4.1. □

In the next Theorem we assume that V is a partially ordered set (poset).

Theorem 4.2. *Assume that V is a poset, and that $L : V \to P$ and $N : V \to 2^P \setminus \emptyset$ satisfy the following hypotheses*

(L) *The greatest solutions of $Lu = y$ exist and are increasing in $y \in P$.*
(N) *N is increasing, its values are sequentially order compact in $N[V]$, and monotone sequences of $N[V]$ converge in P.*

If P has an inf-center, then 4.1 has a solution u_+ which is maximal in the sense that if $u \in V$ is any solution of 4.1 such that $u_+ \le u$ and $Lu_+ \le Lu$, then $u_+ = u$.

Proof. The hypotheses (L) and (N) imply that the relations

$$(4.2)\qquad \begin{aligned} &V_+ = \max\{u \in V : \ Lu = y, \ y \in P\}, \\ &L_+ = L|V_+, \quad \text{and} \quad N_+ = N|V_+ \end{aligned}$$

define mappings $L_+ : V_+ \to P$ and $N_+ : V_+ \to 2^P \setminus \emptyset$ which have the following properties.

(i) L_+ is a bijection and its inverse is increasing.
(ii) N_+ is increasing, its values are sequentially order compact in $N_+[V_+]$ and monotone sequences of $N_+[V_+]$ converge in P.

Because L_+^{-1} and N_+ are increasing, then a mapping $F = N_+ \circ L_+^{-1} : P \to 2^P \setminus \emptyset$ is increasing. If $x \in P$, then denoting $u = L_+^{-1}x$ we have $F(x) = N_+L_+^{-1}x = N_+u = Nu$, whence $F(x)$ is sequentially order compact in $F[P] = N_+[V] \subseteq N[V]$ because Nu is. Since the monotone sequences of $N[V]$ converge in P by (N), and $F[P] \subseteq N[V]$ then monotone sequences of $F[P]$ converge in P. Consequently, if P has an inf-center, then F has by Theorem 3.4 a maximal fixed point x_+. Denoting $u_+ = L_+^{-1}x_+$, then $Lu_+ = L_+u_+ = x \in F(x) = N_+L_+^{-1}x_+ = N_+u_+ = Nu_+$, whence u_+ is a solution of 4.1. Maximality of u_+ can be proved as in [4], Proposition 2.1 applying the last conclusion of Lemma 3.1. □

Corollary 4.3. *Let P be a bounded and closed ball in a reflexive lattice-ordered Banach space X having the following property.*

(C+) $\|x^+\| \le \|x\|$ *for each* $x \in X$, *where* $x^+ = \sup\{0, x\}$.

(a) If $L : V \to P$ *is a bijection, if* $N : V \to 2^P \setminus \emptyset$ *has weakly sequentially closed values, and if* $N \circ L^{-1}$ *is increasing, then 4.1 has a solution.*

(b) If V *is a poset, if* $L : V \to P$ *satisfies the hypothesis (L) of Theorem 4.2, and if* $N : V \to 2^P \setminus \emptyset$ *is increasing and has weakly sequentially closed values, then 4.1 has a maximal solution.*

Proof. The hypothesis (C+) ensures that the geometrical center of P is also its inf-center. Since X is reflexive and P is bounded, then all monotone sequences of P have weak limits in P, and all weakly sequentially closed subsets of P are weakly sequentially compact, and hence sequentially order compact. Thus the hypotheses of Theorem 4.1 hold in (a) and the hypotheses of Theorem 4.2 are valid in (b). □

Remark 4.4. All reflexive Banach lattices are lattice-ordered and reflexive Banach spaces with property (C+) required in Corollary 4.3. Each of the following spaces have also these properties when $1 < p < \infty$.

- $L^p(\Omega)$, ordered a.e. pointwise, where $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space.
- $W^{1,p}(\Omega)$, and $W_0^{1,p}(\Omega)$, ordered a.e. pointwise, where Ω is a domain in $\mathbb{R}^N$.
- l^p, ordered coordinatewise and normed by the p-norm.
- $\mathbb{R}^N$, ordered coordinatewise and normed by the p-norm.

Theorems 4.1 and 4.2 and Corollary 4.3 generalize the corresponding results derived in [4, 6, 7] for the inclusion problem 4.1. For instance, in Theorems 4.1 and 4.2 the space X is not necessarily a vector space, and in the hypotheses the sequential compactness is replaced by the sequential order compactness.

Applying the result Lemma 2.6 one can also show that conditions on the convergence of sequences (x_n^+) can be dropped from the hypotheses used in [4, 6, 7] to derive existence results for

- inclusion problems $x \in F(x)$ and $Lu \in Nu$ in ordered topological vector spaces,

- equation $u = H(u, u)$, where $H : X \times X \to X$, X being an ordered Banach space,

- inclusion problem $\Lambda u \in F(u)$, where Λ is a mapping from a partially ordered set W to an ordered Banach space X and $F : X \to 2^X \setminus \emptyset$,

- implicit inclusion problem $\Lambda u = H(u, \Lambda u)$, where Λ, W and X are as above and $H : W \times X \to 2^X \setminus \emptyset$.

5. APPLICATIONS TO GAME THEORY

In this section we apply Propositions 3.2 and 3.3 and Theorem 3.4 to derive results on the existence of extremal Nash equilibria for a normal-form game, defined as follows.

Definition 5.1. We say that $\Gamma = \{S_1, \dots, S_N, u_1, \dots, u_N\}$ is a *normal-form game* of players i, $i = 1, \dots, N$, if each S_i, called a *strategy set for player* i, is a nonempty subset of a poset $X_i = (X_i, \leq_i)$ and the *utility function* u_i of each player i is a mapping from the product space $S_1 \times \cdots \times S_N$ to a poset $Y_i = (Y_i, \preceq_i)$.

Unless otherwise stated we assume that all the posets X_i and Y_i are ordered topological spaces and that all the sets S_i and Y_i have property (C).

We also use notations $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ and $s = (s_1, \dots, s_N) = (s_i, s_{-i}),\ i = 1, \dots, N$.

Definition 5.2. We say that strategies $s_1^*, \dots, s_N^*$ form a *Nash equilibrium* for Γ if

$$u_i(s_i^*, s_{-i}^*) = \operatorname{argmax} u_i(\cdot, s_{-i}^*) := \max\{u_i(s_i, s_{-i}^*) : \ s_i \in S_i\} \tag{5.1}$$

for each $i = 1, \dots, N$.

The next Lemma gives conditions under which maximization of utilities is possible.

Lemma 5.3. *Assume that*

(H0) *The set $R_i(s_{-i}) = \{u_i(s_i, s_{-i}) : \ s_i \in S_i\}$ is sequentially order compact upward and directed upward, and increasing sequences of $R_i(s_{-i})$ converge in Y_i for all $i = 1, \dots, N$ and $s_{-i} \in S_{-i}$.*

Then the set $R_i(s_{-i})$ has a greatest element for all $i = 1, \dots, N$ and $s_{-i} \in S_{-i}$.

Proof. It suffices to show that $R_i(s_{-i})$ has a maximal element because $R_i(s_{-i})$ is directed upward. If C is a chain in $R_i(s_{-i})$, the hypothesis (H0) and property (C) imply that an increasing sequence (y_n) of C converges to $\sup C$ in Y_i. Because $R_i(s_{-i})$ is sequentially order compact upward, then it contains by Lemma 2.4 an upper bound y of (y_n), whence $\sup C = \lim_n y_n = \sup_n y_n \leq y$. Thus C has an upper bound in $R_i(s_{-i})$, so that $R_i(s_{-i})$ has a maximal element by Zorn's Lemma. □

Denote $P = S_1 \times \cdots \times S_N$ and $S_{-i} = S_1 \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_N$, and assume that all these sets are ordered oomponentwise, and that P is topologized with the product topology. If (H0) holds, we can define a mapping $F : P \to 2^P \setminus \emptyset$ by

$$\begin{cases} F(s) := F_1(s_{-1}) \times \cdots \times F_N(s_{-N}),\ s = (s_1, \dots, s_N) \in P, \\ \text{where } F_i(s_{-i}) := \operatorname{argmax} u_i(\cdot, s_{-i}),\ i = 1, \dots, N. \end{cases} \tag{5.2}$$

It is easy to see that the components of $s^* = (s_1^*, \dots, s_N^*)$ form a Nash equilibrium for Γ if and only if $s^* \in F(s^*)$, i.e., s^* is a fixed point of F.

As an application of Proposition 3.2 we obtain the following result.

Proposition 5.4. *Assume that the hypothesis (H0) holds, that for each $i = 1, \dots, N$ the multifunction $s_{-i} \mapsto F_i(s_{-i})$ is increasing upward and the sets $F_i(s_{-i})$, $s_{-i} \in S_{-i}$, are sequentially order compact upward in $F_i[S_{-i}]$, and increasing sequences of $F_i[S_{-i}]$ converge in S_i. If $[a) \cap F(a) \neq \emptyset$ for some $a \in P = S_1 \times \cdots \times S_N$, then Γ has a maximal Nash equilibrium.*

Proof. We shall show that the mapping $F : P \to 2^P \setminus \emptyset$ defined by 5.2 satisfies the hypotheses of Proposition 3.2. Assume that $s = (s_1, \dots, s_N) \leq \overline{s} = (\overline{s}_1, \dots, \overline{s}_N)$ in P, and let $y = (y_1, \dots, y_N)$ be chosen from $F(s)$. Given $i = 1, \dots, N$, we have $y_i \in F_i(s_{-i})$, and $s_{-i} \leq \overline{s}_{-i}$ in S_{-i}. Since $s_{-i} \mapsto F_i(s_{-i})$ is increasing upward, there exists a $\overline{y}_i \in F_i(\overline{s}_{-i})$ such that $y_i \leq \overline{y}_i$ in S_i. This holds for each $i = 1, \dots, N$, whence $\overline{y} = (\overline{y}_1, \dots, \overline{y}_N) \in F(\overline{s})$, and $y \leq \overline{y}$ in P. This proves that F is increasing upward.

Because of product topologies and componentwise orderings the hypotheses imposed on F_i and the definition 5.2 of F imply that the values of F are sequentially order compact upward in $F[P]$, and that increasing sequences of $F[P]$ converge in P. Moreover, $P = S_1 \times \cdots \times S_N$ has property (C) because each S_i has that property.

The above proof shows that F satisfies the hypotheses of Proposition 3.2, whence F a maximal fixed point s^*, and its components form a maximal Nash equilibrium for Γ. □

By a similar reasoning we obtain the following consequence of Proposition 3.3.

Proposition 5.5. *Assume that the hypothesis (H0) holds, that for each $i = 1, \dots, N$ the multifunction $s_{-i} \mapsto F_i(s_{-i})$ is increasing downward and the sets $F_i(s_{-i})$, $s_{-i} \in S_{-i}$, are sequentially order compact downward in $F_i[S_{-i}]$, and decreasing sequences of $F_i[S_{-i}]$ converge in S_i. If $(b] \cap F(b) \neq \emptyset$ for some $b \in P = S_1 \times \cdots \times S_N$, then Γ has a minimal Nash equilibrium.*

Our main result on the existence of extremal Nash equilibria for Γ is a consequence of Theorem 3.4.

Theorem 5.6. *Let the hypothesis (H0) and the following hypotheses hold.*

(H1) *For each $i = 1, \dots, N$ the multifunction $s_{-i} \mapsto F_i(s_{-i})$ is increasing.*
(H2) *For each $i = 1, \dots, N$ the sets $F_i(s_{-i})$, $s_{-i} \in S_{-i}$, are sequentially order compact, in $F_i[S_{-i}]$ and monotone sequences of $F_i[S_{-i}]$ converge in S_i.*

(a) If the set of limits of increasing sequences of $F_i[S_{-i}]$ has a sup-center in S_i for each $i = i, \dots, N$, then Γ has a minimal Nash equilibrium.

(b) If the set of limits of decreasing sequences of $F_i[S_{-i}]$ has an inf-center in S_i for each $i = i, \dots, N$, then Γ has a maximal Nash equilibrium.

Proof. The proof of Proposition 5.4 and its dual show that the mapping $F : P \to 2^P \setminus \emptyset$ defined by 5.2 satisfies the hypotheses of Theorem 3.4. If the set of limits of increasing sequences of $F_i[S_{-i}]$ has a sup-center c_i in S_i for each $i = i, \dots, N$, then $c = (c_1, \dots, c_N)$ is a sup-center of the limits of increasing sequences of $F[P]$. Thus F has by Theorem 3.4 (a) a minimal fixed point s_*, and its components form a minimal Nash equilibrium for Γ. Similarly, if the set of limits of decreasing sequences of $F_i[S_{-i}]$ has an inf-center c_i in S_i for each $i = i, \dots, N$, then $c = (c_1, \dots, c_N)$ is an inf-center of the limits of decreasing sequences of $F[P]$, so that F has by Theorem 3.4 (b) a maximal fixed point s^*, and its components form a maximal Nash equilibrium for Γ. □

To modify the hypotheses so that they refer only to the strategies and their values we shall prove an auxiliary result for an upper semi-closed function defined as follows.

Definition 5.7. A mapping $f : S_i \to Y_i$ is called *upper semi-closed* if $x_n \to x$ in S_i, $f(x_n) \to y$ in Y_i and $(f(x_n))$ is increasing imply that $y \preceq_i f(x)$.

Each continuous mapping $f : S_i \to Y_i$ is upper semi-closed.

Lemma 5.8. *Assume that S_i is sequentially compact. If a mapping $f : S_i \to Y_i$ is upper semi-closed, and if its range $f[S_i]$ is separable and directed upward and its increasing sequences converge, then the set argmax f of the maximum points is nonempty and sequentially compact.*

Proof. Since $f[S_i]$ is separable, there exists a countable subset $B = \{z_n\}_{n=1}^m$, $1 \le m \le \infty$, of $f[S_i]$ such that the closure $\overline{B}$ of B in Y_i contains $f[S_i]$. Denote $y_1 := z_1$, and when y_k is chosen, let z_{n_k} be the first element of the sequence $(z_n)_{n=1}^m$, if exists, such that $z_{n_k} \not\preceq_i y_k$, and let $y_{k+1} \in f[S_i]$ be an upper bound of $\{z_{n_k}, y_k\}$. The so obtained sequence (y_k) is increasing, whence it has either maximum or a limit y in Y_i. In the former case choose $x \in S_i$ such that $y = f(x)$. In the latter case choose a sequence $(x_k)_{k=1}^\infty$ from S such that $y_k = f(x_k)$, $k = 1, 2, \dots$. The above construction shows that $(f(x_k))_{k=1}^\infty$ is an increasing sequence in $f[S_i]$, whence $y = \lim_k f(x_k)$ exists in Y_i by a hypothesis. Because S_i is compact, the sequence $(x_k)_{k=1}^\infty$ has a convergent subsequence $(x_{k_j})_{j=1}^\infty$. Denote $x = \lim_j x_{k_j}$. Since $\lim_j f(x_{k_j}) = y$, the sequence $(f(x_{k_j}))_{j=1}^\infty$ is increasing and f is upper semi-closed, then $y \preceq_i f(x)$. The above construction and [8], Proposition 1.1.3 imply that y is in both cases an upper bound of B. Since $(y]$ is closed, then $f[S] \subseteq \overline{B} \subseteq (y] \subseteq (f(x)]$, so that $y = \max f$, and x is a maximum point of f.

To prove that the set argmax f of the maximum points of f is sequentially compact, let (x_n) be a sequence in argmax f. Because S_i is sequentially compact, then (x_n) has a subsequence (x_{n_k}) which converges to a point x of S_i. Since $f(x_{n_k}) = y$

for each k, then $\lim_k f(x_{n_k}) = y$. Since f is upper semi-closed, then $y \preceq_i f(x)$. On the other hand, $f(x) \preceq_i \max f = y$, whence $f(x) = y$, and $x \in \operatorname{argmax} f$. □

Theorem 5.9. *Let $\Gamma = (S_1, \dots, S_N, u_1, \dots, u_N)$ be a normal-form game, where each strategy set S_i is sequentially compact, and the utilities u_i satisfy the following hypotheses for all $i = 1, \dots, N$.*

(h0) *For each $s_{-i} \in S_{-i}$ the function $u_i(\cdot, s_{-i})$ is upper semi-closed, and its values form a separable and upward directed set whose increasing sequences converge.*
(h1) *$u_i(\hat{s}_i, s_{-i}) \preceq_i u_i(s_i, s_{-i})$ implies $u_i(\hat{s}_i, \hat{s}_{-i}) \preceq_i u_i(s_i, \hat{s}_{-i})$ whenever $s_i \not\preceq_i \hat{s}_i$ in S_i and $s_{-i} < \hat{s}_{-i}$ in S_{-i}.*
(h2) *$u_i(s_i, \hat{s}_{-i}) \preceq_i u_i(\hat{s}_i, \hat{s}_{-i})$ implies $u_i(s_i, s_{-i}) \preceq_i u_i(\hat{s}_i, s_{-i})$ whenever $s_i \not\preceq_i \hat{s}_i$ in S_i and $s_{-i} < \hat{s}_{-i}$ in S_{-i}.*

(a) If each S_i has a sup-center, then Γ has a minimal Nash equilibrium.

(b) If each S_i has an inf-center, then Γ has a maximal Nash equilibrium.

Proof. Let $i \in \{1, \dots, N\}$ and $s_{-i} \in S_{-i}$ be fixed. Since S_i is sequentially compact, the hypothesis (h0) implies by Lemma 5.8 that $F_i(s_{-i}) = \operatorname{argmax} u_i(\cdot, s_{-i})$ is nonempty and sequentially compact subset of S_i. Thus $F_i(s_{-i})$ is order compact in $F_i[S_{-i}]$ by Lemma 2.4. Moreover, all the monotone sequences of a subset $F_i[S_{-i}]$ of a sequentially compact set S_i converge.

The above proof shows that the hypotheses (H0) and (H2) of Theorem 5.6 hold. To prove that the hypothesis (H1) holds, let $i \in \{1, \dots, N\}$ be fixed. We shall first show that the multifunction $s_{-i} \mapsto F_i(s_{-i})$ is increasing upward. Assume on the contrary the existence of s_{-i}, $\hat{s}_{-i} \in S_{-i}$, $s_{-i} < \hat{s}_{-i}$, and $s_i \in F_i(s_{-i})$ such that

$$s_i \not\preceq_i \hat{s}_i \text{ for all } \hat{s}_i \in F_i(\hat{s}_{-i}). \tag{5.3}$$

Let $\hat{s}_i \in F_i(\hat{s}_{-i})$ be given. It follows from 5.2 that

$$u_i(\hat{s}_i, s_{-i}) \preceq_i u_i(s_i, s_{-i}).$$

This inequality, the hypothesis (h1) and the inequalities $s_i \not\preceq_i \hat{s}_i$ and $s_{-i} < \hat{s}_{-i}$ imply that

$$u_i(\hat{s}_i, \hat{s}_{-i}) \preceq_i u_i(s_i, \hat{s}_{-i}).$$

But then $s_i \in F_i(\hat{s}_{-i})$, which contradicts with 5.3. This shows that $s_{-i} \mapsto F_i(s_{-i})$ is increasing upward.

To prove that $s_{-i} \mapsto F_i(s_{-i})$ is increasing downward, assume on the contrary the existence of s_{-i}, $\hat{s}_{-i} \in S_{-i}$, $s_{-i} < \hat{s}_{-i}$, and $\hat{s}_i \in F_i(\hat{s}_{-i})$ such that

$$s_i \not\preceq_i \hat{s}_i \text{ for all } s_i \in F_i(s_{-i}). \tag{5.4}$$

Given an $s_i \in F_i(s_{-i})$, it follows from 5.2 that

$$u_i(s_i, \hat{s}_{-i}) \preceq_i u_i(\hat{s}_i, \hat{s}_{-i}).$$

This inequality and the inequalities $s_i \not\leq_i \hat{s}_i$ and $s_{-i} < \hat{s}_{-i}$ imply by the hypothesis (h2) that

$$u_i(s_i, s_{-i}) \preceq_i u_i(\hat{s}_i, s_{-i}).$$

But then $\hat{s}_i \in F_i(s_{-i})$, which contradicts with 5.4. Thus $s_{-i} \mapsto F_i(s_{-i})$ is also increasing downward.

The above proof shows that the hypotheses (H0)–(H2) of Theorem 5.6 are valid, whence its results imply that the conclusions (a) and (b) hold. □

In view of the proof of Theorem 5.9 we obtain the following consequences of Propositions 5.4 and 5.5.

Proposition 5.10. *Assume that each S_i is sequentially compact.*

(a) If the hypotheses (h0) and (h1) of Theorem 5.9 hold for each $i = 1, \dots, N$, and if $[a) \cap F(a) \neq \emptyset$ for some $a \in P$, then Γ has a maximal Nash equilibrium.

(b) If the hypotheses (h0) and (h2) of Theorem 5.9 hold for each $i = 1, \dots, N$, and if $(b] \cap F(b) \neq \emptyset$ for some $b \in P$, then Γ has a minimal Nash equilibrium.

Referring to considerations of the Introduction we can drop all the convergence hypotheses of monotone sequences and order compactness hypotheses if all strictly increasing sequences of the values of each $u_i(\cdot, s_{-i})$ and all strictly monotone sequences of each S_i are finite. In particular, the following result holds.

Proposition 5.11. *Assume that for all $i = 1, \dots, N$ and $s_{-i} \in S_{-i}$ the values of $u_i(\cdot, s_{-i})$ are form a directed set and their strictly increasing sequences are finite, that strictly monotone sequence of each S_i are finite, and that the hypotheses (h1) and (h2) of Theorem 5.9 hold.*

(a) If each S_i has a sup-center, the game Γ has a minimal Nash equilibrium.

(b) If each S_i has an inf-center, the game Γ has a maximal Nash equilibrium.

Example 5.12. Assume that for each $i = 1, \dots, N$ the strategy spaces S_i are closed and bounded balls in lattice-ordered reflexive Banach spaces X_i equipped with weak topologies, that the spaces Y_i are ordered second countable topological vector spaces, and that the utilities are of the form

$$u_i(s_i, s_{-i}) = f_i(s_i)g_i(s_{-i}) + h_i(s_{-i}), \quad s_i \in S_i,\ s_{-i} \in S_{-i}, \tag{5.5}$$

where $f_i : S_i \to \mathbb{R}_+$ is bounded and upper semi-closed, $g_i, h_i : S_{-i} \to Y_i$, and $0 \prec_i g_i(s_{-i})$ for all $s_{-i} \in S_{-i}$. The hypotheses imposed on S_i and Y_i imply that they satisfy condition (C), and that each S_i is weakly sequentially compact. Since each f_i is upper semi-closed, bounded and real-valued, and $0 \prec_i g_i(s_{-i})$ for all $s_{-i} \in S_{-i}$, it follows from 5.5 that each $u_i(\cdot, s_{-i})$ satisfies the hypothesis (h0), and that the hypotheses (h1) and (h2) can be reduced to the tautologies: $f_i(\hat{s}_i) \leq f_i(s_i)$ implies

$f_i(\hat{s}_i) \le f_i(s_i)$, and $f_i(s_i) \le f_i(\hat{s}_i)$ implies $f_i(s_i) \le f_i(\hat{s}_i)$, if $s_i \not\le_i \hat{s}_i$ and $s_{-i} < \hat{s}_{-i}$. Thus the hypotheses (h0), (h1) and (h2) are valid. Assume moreover that the spaces X_i have property (C+), i.e., $\|\sup\{0, x_i\}\| \le \|x_i\|$ for all $x_i \in X_i$, $i = 1, \dots, N$ (such spaces are listed in Remark 4.4). Then the geometrical center of each S_i is both a sup-center and an inf-center of S_i. It then follows from Theorem 5.6 that $\Gamma = \{S_1, \dots, S_N, u_1, \dots, u_N\}$ has minimal and maximal Nash equilibria.

Remark 5.13. If Y_i is an ordered vector space, or even an ordered semigroup, the utility function u_i satisfies the hypotheses (h1) and (h2) of Theorem 5.9 if the following condition holds.

(h3) $u_i(\hat{s}_i, s_{-i}) - u_i(\hat{s}_i, \hat{s}_{-i}) \preceq_i u_i(s_i, s_{-i}) - u_i(s_i, \hat{s}_{-i})$ if $s_i \not\le_i \hat{s}_i$ and $s_{-i} < \hat{s}_{-i}$.

Condition (h3) is however stronger than the hypotheses (h1) and (h2). For instance, if the spaces X_i and Y_i are as in Example 5.12 and the utilities are defined by 5.5, then (h3) is reduced to the form

$0 \preceq_i (f_i(\hat{s}_i) - f_i(s_i))(g_i(s_{-i}) - g_i(\hat{s}_{-i}))$ whenever $s_i \not\le_i \hat{s}_i$ and $s_{-i} < \hat{s}_{-i}$.

The validity of the above condition requires monotony properties for f_i and g_i, whereas $0 \prec_i g_i(s_{-i})$ for all $s_{-i} \in S_{-i}$ is the only condition for the utilities given by 5.5 to satisfy the hypotheses (h1) and (h2).

The only difference between condition (h3) and the property of increasing differences, defined in [15], p. 42 is that $\hat{s}_i <_i s_i$ is replaced by $s_i \not\le_i \hat{s}_i$. These two relations are equivalent if the strategy spaces S_i are chains. The hypothesis (h1) resembles the single crossing property defined in [15], p. 59.

No lattice properties are imposed on the strategy sets S_i. However, if S_i is a lattice, as usually assumed (cf. e.g. [15] and the references therein), then each point of S_i is both a sup-center and an inf-center of S_i. If each $F_i(s_{-i})$ is a lattice or directed, then maximal and minimal Nash equilibria for Γ are its least and greatest Nash equilibria. Moreover, the values of utilities can be in ordered topological spaces or in ordered topological vector spaces, which generalizes the usual assumption that the utilities are real-valued.

In Example 5.12 the balls S_i can be replaced by the following nonconvex sets:

$$S_i = \{(x_1, \dots, x_{m_i}) \in \mathbb{R}^{m_i} : \sum_{j=1}^{m_i} |x_i - c_{ij}|^{p_i} \le r_i^{p_i}\},$$

where $p_i \in (0,1)$ and $r_i > 0$. To show this, notice first that each $\mathbb{R}^{m_i}$, ordered coordinatewise, and normed by any norm, is a reflexive lattice-ordered Banach space. It is elementary to verify that $c_i = (c_{i1}, \dots, c_{im_i})$ is both a sup-center and an inf-center of S_i. Moreover, each S_i is a closed and bounded subset of $\mathbb{R}^m$, whence it is sequentially compact. Thus all the hypotheses imposed on S_i in Theorem 5.9 are valid.

For the sake of simplicity the above considerations are restricted to normal-form games with finite number of players. The results corresponding to those derived above can be obtained also for games of more general types, for instance, for those considered in [9, 16].

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