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SET VALUED STRATONOVICH INTEGRAL AND STRATONOVICH TYPE STOCHASTIC INCLUSION

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ABSTRACT. In the paper we discuss the construction of a set-valued stochastic integral of the Stratonovich type driven by a semimartingale. It allows to consider the stochastic inclusion of a type of Stratonovich. The existence of strong solutions to such inclusion with upper separated set-valued functions is investigated.

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1. PRELIMINARIES

Investigation of stochastic controlled dynamical systems by methods of multivalued analysis requires an appropriate kind of regularity of their multivalued structure. The properties of Lipschitz continuity, lower or upper semicontinuity and monotonicity are often considered. In the paper, we consider a different class of set-valued functions called "upper separated". The upper separatity of a set-valued function F is necessary and sufficient for the existence of a convex selection of F. We are able to define a stochastic set-valued integral of a Stratonovich type in a proper way. This enables us to investigate a new class of stochastic inclusions, and therefore to consider control problems driven by stochastic Stratonovich equations. As a consequence, we deduce the existence of solutions to stochastic inclusions with right sides taken from the above class of multifunctions.

The paper is organized as follows. In the Section 2 we define the new type of a set-valued stochastic integral. We discuss there the existence and properties of the set-valued Stratonovich integral $\int_0^t R_s \circ dz_s$, where R is a set-valued semimartingale while z denotes a single-valued semimartingale. In the first part of the Section 3 we discuss selection properties of upper separated set-valued functions. This enable to define the Stratonovich type stochastic inclusion in a proper way. Next we discuss the existence of exploding and nonexploding solutions to such inclusion. Some examples of both situations are presented also.

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2. SET-VALUED STRATONOVICH INTEGRAL

Let $I = [0, \infty)$ and let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathbf{F}_t\}_{t\in I}\}$ is an increasing and right continuous family of σ-subalgebras of $\mathbf{F}$ and $\mathbf{F}_0$ contains all P-null sets. Let $x_t : (\Omega, \mathbf{F}, P) \to \mathbf{R}^n,\ t \in I$ be a random variable. The stochastic process $x = (x_t)_{t\in I}$ is said to be adapted if x_t is $\mathbf{F}_t$-measurable for each $t \in I$. A stochastic process x is called cádlág if it a.s. has sample paths which are right continuous, with left limits. Similarly, a stochastic process x is said to be cáglád if it a.s. has sample paths which are left continuous, with right limits. The family of all adapted cádlág (cáglád) processes is denoted by D $[L]$.

Let $\mathcal{P}(\mathbf{F}_t)$ denote the smallest σ-algebra on $I \times \Omega$ with respect to which every cáglád adapted process is measurable in (t, ω), i.e., $\mathcal{P}(\mathbf{F}_t) = \sigma(L)$. A stochastic process x is said to be predictable if x is $\mathcal{P}(\mathbf{F}_t)$-measurable. One has $\mathcal{P}(\mathbf{F}_t) \subset \beta \otimes \mathbf{F}$, where β denotes the Borel σ-algebra on $[0, \infty)$.

By $H^p,\ p \geq 1$, we denote the normed space of semimartingales with finite H^p-norm, i.e. $\|x\|_{H^p} = \inf_{x=n+a} j_p(n, a)$, where $j_p(n, a) = \|[n, n]_\infty^{\frac{1}{2}} + \int_0^\infty |da_s|\|_{L^p}$, and $|a_t| = \int_0^t |da_s|$ represents the total variation on $[0, t]$ of the measure induced by the paths of the $FV-$process a (see [14] for details).

Let $R = (R_t)_{t\in I}$ be a set-valued stochastic process with values in $Comp(\mathbf{R}^n)$, the space of all compact subsets of $\mathbf{R}^n$ considered with a Hausdorff metric $h(\cdot, \cdot)$, i.e., a family of $\mathbf{F}$ measurable set-valued mappings $R_t : \Omega \to Comp(\mathbf{R}^n)$, each $t \in I$. We call R measurable if it is $\beta \otimes \mathbf{F}$ measurable in the sense of set-valued functions (see e.g.: [1]). Similarly, R is $\{\mathbf{F}_t\}_{t\in I}$-adapted if R_t is $\mathbf{F}_t$-measurable for each $t \in I$. We call R predictable if R is $\mathcal{P}(\mathbf{F}_t)$ -measurable. A set-valued stochastic process $R = (R_t)_{t\geq 0}$ is z-integrably bounded if there exists a predictable and z-integrable process m such that $\|\sup_{t\geq 0} |m_t|\|_{L^p(\Omega)} < \infty$ and such that the Hausdorff distance $h(|R_t|, \{0\}) \leq |m_t|$ a.s., each $t \in I$.

A set-valued process $R = (R_t)_{t\in I}$ is a set-valued martingale if $R_s = \mathbf{E}(R_t|\mathbf{F}_s)$ for $s \leq t$, where $\mathbf{E}$ denotes the set-valued conditional expectation of R relative to $\{\mathbf{F}_t\}_{t\in I}$, [4]. R has a finite variation on compacts (is an FV-process) if

$$\sup_{\pi_L} \sum_{i=0}^{m-1} h(R_{t_i}, R_{t_{i+1}}) < \infty,$$

for each partition $\pi_L : 0 = t_0 < t_1 < \cdots < t_m = L$ on intervals $[0, L],\ L > 0$.

Definition 2.1. (i) Given a predictable set-valued process $R = (R_t)_{t\in I}$ and a semimartingale z let us denote

$$S(R, z) := \{r \in \mathcal{P}(\mathbf{F}_t) : r_t \in R_t \text{ for each } t \in I \ a.e. \text{ and } r \text{ is } z-\text{integrable}\}.$$

For conditions of z-integrability see Chapter IV of [14].

(ii) A predictable set-valued process R is integrable with respect to a semimartingale z, or simply z-integrable if $S(R,z)$ is a nonempty set.

We define a set-valued Itô type stochastic integral by the formula

$$\int R(t)dz(t) := \{\int r(t)dz(t) : r \in S(R,z)\}.$$

It follows immediately from the properties of stochastic integrals with respect to semimartingales (see Theorem 3.2 of [4]) and Kuratowski and Ryll-Nardzewski measurable selection theorem, that every z-integrably bounded and predictable set-valued stochastic process R is z-integrable (see e.g.: [6, 7]).

By $[x,z]$ we denote a quadratic covariation process $[x,z]$ of semimartingales x and z ($x_0 = z_0 = 0$):

$$[x,z] = xz - \int x_- dx + \int x_- dz.$$

For the detailed discussion on the properties of a quadratic covariation process see e.g.: [14].

The definition of a single valued Stratonovich stochastic integral with respect to a semimartingale can be found in [8, 14]. Namely, for semimartingales x and z.

$$\int_0^t x_{s-} \circ dz_s := \int_0^t x_{s-} dz_s + \frac{1}{2}[x,z]_t^c,$$

where $[x,z]^c$ denotes the path by path continuous part of $[x,z]$.

Now we introduce formally the notion of a Stratonovich set-valued stochastic integral.

Definition 2.2. Consider a predictable set-valued function R taking values in compact subsets of $\mathbf{R}^n$ and a continuous one dimensional semimartingale z.

(i) By a set-valued quadratic covariation process $[R,z]$ we mean the set

$$[R,z] = \{[r,z] : r \in R \text{ provided } [r,z] \text{ is finite}\}.$$

A set-valued Stratonovich stochastic integral of a predictable set-valued process R is defined by

$$\int_0^t R_s \circ dz_s := \int_0^t R_s dz_s + \frac{1}{2}[R,z]_t^c,$$

provided both two sets on the right side are nonempty.

For the nonemptiness of the set $\int R_s dz_s$ it is enough to assume that R is z-integrably bounded. To deduce the nonemptiness of the set $[R,z]$ we need the existence of appropriate regular selections of a set-valued function R. And it is the main problem in defining properly the set-valued Stratonovich integral. If we know that R admits at least one selection x being a semimartingale we are done, because in such a case the set $[R, z]$ contains an element $[x, z] = xz - \int x_- dz - \int z_- dx$. The integration by parts formula for semimartingales justifies the existence of a quadratic covariation $[x, z]$ ([14]). Therefore, the set $[R, z]$ is nonempty.

The class of set-valued semimartingales is regular enough for the existence of set-valued Stratonovich integral ([9]).

Definition 2.3. A set-valued process R is a set-valued semimartingale, if R can be decomposed into a sum $R = N + A$, where N and A are $\mathcal{F}_t$-adapted set-valued processes with nonempty closed values, A being an FV process, while N a set-valued local martingale. A set-valued semimartingale R is H^2 bounded if there exists a semimartingale m in H^2 such that the Hausdorff distance $h(|R_t|, \{0\}) \leq |m_t|$ a.s., each $t \in I$.

Proposition 2.4. (see [9]) Let R be a set-valued semimartingale which can be decomposed into a sum $R = N + A$, with N and A having convex and compact values. Then there exists a semimartingale selection x of R such that $x = n + a$, where "n" is a local martingale selection of N and "a" is a process of finite variation being a selection of A.

Below we present some properties of set-valued Stratonovich integral.

For a set-valued semimartingale R, let $Sel(R)$ denote the set of all its semimartingale selections. Let (x^n) be a sequence of one dimensional semimartingales defined on filtered probability spaces $(\Omega^n, \mathbf{F}^n, \{\mathbf{F}^n_t\}_{t\geq 0}, P^n)$.

Definition 2.5. (see [18]). A sequence (x^n) satisfies the (UT) condition if for every $q \in R^+$ the family of random variables

$$\{\int_0^q u^n_s dx^n_s : u^n \in U^n_q, n \in N\} \text{ is tight in } \mathbf{R},$$

where U^n_q denotes the family of predictable processes of the shape

$$u^n_s = \sum_{i=0}^{k} u^n_i I_{\{t_i < s \leq t_{i+1}\}}, \text{ for } 0 = t_0 < t_1 < \cdots < t_k = q,$$

and every u^n_i being an $\mathbf{F}^n_{t_i}$-measurable random variable such that $|u^n_i| \leq 1$, for every $i \in N \cup \{0\}, k, n \in N$.

A family $\mathcal{H}$ of semimartingales satisfies the (UT) condition if every sequence $(x^n) \subset \mathcal{H}$ possesses this property. Consequently a set-valued semimartingale R is said to satisfy (UT) if a family $Sel(R)$ satisfies (UT).

Proposition 2.6. Let R be a set-valued semimartingale with convex and compact values. Then

(i) if R is H^2-bounded, then the sets $Sel(R)$, and $\int R_{s-} dz_s$ satisfy (UT).

(ii) the set $[R, z]$ is nonempty and convex. Moreover, it satisfies (UT) and it is relatively weakly compact in the topology of weak convergence of probability measures on $D(\mathbf{R}^+, \mathbf{R})$.

Proof. (i). Let M =sup$\{||x||_{H^2} : x \in Sel(R)\}$ and let $\{u^n\}$ be a sequence of predictable processes such that $u_q^{n*} = \sup_{t \le q} |u_t^n| \le 1$, for each $q \in R^+, n \in N$. Let $\{x^n\} \subset Sel(R)$ be chosen arbitrarily. Then from Khinthine and Emery‘s inequalities [14] we get the following estimation

$$\begin{aligned} P\{|\int_0^q u_s^n dx_s^n| &> k\} \le \frac{1}{k^2} E\{\sup_{t \le q} |\int_0^t u_s^n dx_s^n|^2\} \le \frac{c_2^2}{k^2} || \int_0^\cdot u_s^n dx_s^n ||_{H_q^2}^2 \\ &\le \frac{c_2^2}{k^2} ||u_q^{n*}||_{L^\infty}^2 ||x^n||_{H_q^2}^2 \le \frac{c_2^2 M^2}{k^2}, \end{aligned}$$

for every $k, q \in \mathbf{R}^+$, and $n \in N$, where c_2 is a constant which appears in Emery‘s inequality. Hence the sequence $\{x^n\}$ satisfies (UT).

Let us assume now that $\{g^n\} \subset \{\int x_- dz : x \in Sel(R)\}$. Then $g^n = \int x_-^n dz$ for some $x^n \in Sel(R)$ and $n = 1, 2, \dots$ But $\{x^n\}$ satisfies (UT). Hence due to Theorem B1 of [17] the sequence $\{\sup_{t \le q} |x_t^n|\}$ is tight for every $q \in \mathbf{R}^+$. Then applying Corollary 1.2 of [17], we deduce that the sequence $\{g^n = \int_0^\cdot x_{s-}^n dz_s\}$ satisfies the condition (UT) as well.

(ii). The convexity of $[R, z]$ follows from the linearity property of quadratic covariation. We prove the last statement. Let $\{v^n\} \subset [R, z]$ be an arbitrary sequence. Then there exists a sequence $\{x^n\} \subset Sel(R)$ such that $v^n = [x^n, z]$, for $n = 1, 2, \dots$ Similarly as before the sequences $\{x^n\}$ and $\{\int_0^\cdot x_{s-}^n dz_s\}$ satisfy the condition (UT). A semimartingale z satisfies the condition (UT) and its distribution $\mathcal{L}(z)$ is tight in $D(\mathbf{R}^+, \mathbf{R})$. Hence, by Lemma 4.3 in [17] we get

$$\{(\int_0^\cdot x_{s-}^n dz_s, z)\} \text{ is tight in } D(\mathbf{R}^+, \mathbf{R}^2). \tag{2.1}$$

By the integration by parts formula we have

$$\int_0^\cdot x_{s-}^n dz_s = x^n z - \int_0^\cdot z_{s-} dx_s^n - v^n. \tag{2.2}$$

Similarly as in the first part of the proof we claim that the sequence $\{\int_0^\cdot z_{s-} dx_s^n\}$ satisfies (UT) as well. Therefore by the formula (2.2) it follows that $\{v^n\}$ also satisfies (UT). Moreover, by Theorem 1.1 of [17] and (UT) condition for $\{x^n\}$ we get

$$\limsup_{n \to \infty} P\{\sup_{t \le q} |\int_0^t z_{s-} dx_s^n| \ge \epsilon\} = 0, \tag{2.3}$$

for every $\epsilon > 0, q \in \mathbf{R}^+$. Applying Theorem B.1 once again, we claim that $\{[x^n, x^n]_q\}$ is tight, for every $q \in \mathbf{R}^+$. Hence $\{v^n\}$ is tight in $D(\mathbf{R}^+, \mathbf{R})$ because of properties (2.1)-(2.2). Finally using the Prohorov‘s Theorem (see [3]) we deduce that the sequence of distributions $\{\mathcal{L}(v^n)\}$ is relatively compact. □

Corollary 2.7. *Let R be an H^2 bounded set-valued semimartingale with convex values and let z be a semimartingale. Then $\int R_{s-} \circ dz_s$ is convex and satisfies (UT).*

Remark 2.8. Since the quadratic covariation process $[x, z]$ is symmetric we are able to describe also a set-valued Stratonovich integral driven by a set-valued semimartingale X. Namely we put

$$\int_0^t z_{s-} \circ dX_s := \int_0^t z_{s-} dX_s + \frac{1}{2}[z, X]_t^c$$

where

$$\int_0^t z_{s-} dX_s := \{\int_0^t z_{s-} dx_s : x \in Sel(X)\}.$$

Such an integral admits properties investigated by the Proposition 2.6 and Corollary 2.7.

3. STOCHASTIC INCLUSION WITH UPPER SEPARATED SET-VALUED MAPS

In the Section we present existence results for the Stratonovich stochastic inclusion related to the set-valued stochastic integrals defined above. Consider set-valued functions $F, G : \mathbf{R^1} \to Conv(\mathbf{R^1})$, a continuous semimartingale z and a continuous process of finite variation a. For a stochastic process x, by $F \bullet x$ we denote a set-valued process $(F \bullet x)_t := F(x_t)$. We define $G \bullet x$ in a similar way.

Definition 3.1. By a Stratonovich stochastic inclusion we mean the relation

$$x_t \in x_0 + \int_0^t F(x_s) \circ dz_s + \int_0^t G(x_s) da_s \tag{3.1}$$

or equivalently

$$x_t \in x_0 + \int_0^t F(x_s) dz_s + \frac{1}{2}[F \bullet x, z]_t + \int_0^t G(x_s) da_s \tag{3.2}$$

where

$$\int_0^t F(x_s) dz_s = \{\int_0^t f_s dz_s : f \in S(F \bullet x, z)\},$$

$$\int_0^t G(x_s) da_s = \{\int_0^t g_s da_s : g \in S(G \bullet x, a)\}$$

and

$$[F \bullet x, z]_t = \{v_t : v \in [F \bullet x, z]\},$$

where $S(F \bullet x, z)$ and $S(G \bullet x, a)$ are meant in the sense of Definition 2.1.

A continuous semimartingale x is a strong solution to Stratonovich inclusion (3.1) if there exist $\mathbf{F}_t$-adapted stochastic processes $u_t, v_t \in F(x_t)$ and $r_t \in G(x_t)$ such that the relation

$$x_t = x_0 + \int_0^t u_s dz_s + \frac{1}{2}[v, z]_t + \int_0^t r_s da_s$$

holds for every $t \in I$ and a.e. $\omega \in \Omega$.

Of course the inclusion (3.1) is well defined if the set from the right side of (5) is nonempty. It can be checked that the Itô type set-valued integral $\int_0^t F(x_s)dz_s$ is nonempty for each set-valued function F such that $F \bullet x$ is predictable and z-integrably bounded for every continuous semimartingale x. As it was mentioned in the previous section the set $[F \bullet x, z]$ is nonempty if a set-valued process $F \bullet x$ admits at least one selection being a semimartingale.

Now we introduce the class of upper separated set-valued functions F for which a set-valued process $[F \bullet x, z]$ is well defined.

Let X be a Banach space while $\bar{\mathbb{R}}$ denotes the extended set of reals, i.e., $\bar{\mathbb{R}} = \mathbf{R^1} \cup \infty$. Let us consider an extended function $f : X \to \bar{\mathbb{R}}$. We define its epigraph $Epi(f)$ by the formula

$$Epi(f) = \{(x, a) \in X \times \mathbf{R^1} : \ f(x) \le a\}$$

An extended function f is proper if it is not a constant function equal everywhere to infinity. A function f is convex if and only if its epigraph is a convex set in $X \times \mathbf{R^1}$. It is lower semicontinuous if $f(x) \le \liminf_{x_n \to x} f(x_n)$. For the properties of extended functions and their epigraphs see, e.g.: [12, 15].

Definition 3.2. Let F be a set-valued function from a Banach space X into nonempty subsets of $\bar{\mathbb{R}}$. We define upper and lower bounds of F by formulas

$$V_F : X \to \bar{\mathbb{R}}, \ \ V_F(x) = sup\{a : a \in F(x)\}$$

$$W_F : X \to \bar{\mathbb{R}}, \ \ W_F(x) = inf\{b : b \in F(x)\}.$$

We say that F is upper separated if for every $x \in DomF$ and $\epsilon > 0$ there exists a hyperplane $H_{x,\epsilon}$ strongly separating a point $(x, W_F(x) - \epsilon)$ from the set $Epi(V_F)$.

The following properties of upper separated set-valued functions have been obtained in [11].

Proposition 3.3. (i) Let F be a proper set-valued function from $\mathbb{R}^n$ into nonempty subsets of $\bar{\mathbb{R}}$. If F admits a convex selection f, then F is upper separated.

(ii) Let F be a proper set-valued function from a Banach space X into nonempty subsets of $\bar{\mathbb{R}}$. If F admits a convex and lower semicontinuous selection f, then F is upper separated.

(iii) Let F be a proper set-valued function from a Banach space X into subsets of $\bar{\mathbb{R}}$. If F has a closed graph and admits a convex selection f, then F is upper separated.

The following result is crucial for the existence of set-valued Stratonovich integral.

Proposition 3.4. Let F be a proper set-valued function from a Banach space X into closed and convex subsets of $\bar{\mathbb{R}}$. If F is upper separated then it admits a convex and lower semicontinuous selection.

Proof. Let V_F^{**} denote the second conjugate function of V_F i.e.

$$V_F^{**} = \sup_{p \in X^*} \{p(x) - \sup_{x \in X}(p(x) - V_F(x))\}.$$

For the definition and properties of conjugate functions see e.g. [15]. We will prove that for every x, $V_F^{**}(x) \in F(x)$. Let us take an arbitrary $x \in DomV_F$ (i.e., such that $V_F(x) < \infty$) and $\epsilon > 0$. Since F is upper separated, then there exists a continuous linear functional $x^*_{x,\epsilon}$ strictly separating $(x, W_F(x) - \epsilon)$ from the set $Epi(V_F)$. Let $x^*_{x,\epsilon}$ be represented by the pair $(p, a) \in X^* \times R$. Then there exists $\delta > 0$ such that for every $y \in DomV_F$ and each $b \geq 0$

$$(p, a)((y, V_F(y) + b)) \leq (p, a)((x, W_F(x) - \epsilon)) - \delta. \tag{3.3}$$

Then we get

$$p(y) + aV_F(y) + ab \leq p(x) + aW_F(x) - a\epsilon - \delta. \tag{3.4}$$

Taking the supremum over b we deduce that $a \leq 0$.

Assume first, that $a < 0$. Then dividing by $-a$ and denoting $-p/a = q$ we get

$$q(y) - V_F(y) \leq q(x) - W_F(x) + \epsilon + \delta/a. \tag{3.5}$$

Taking the supremum over $y \in DomV_F$ we obtain

$$V_F^*(q) = \sup_{y \in DomV_F} (q(y) - V_F(y)) \leq q(x) - W_F(x) + \epsilon + \delta/a. \tag{3.6}$$

Then

$$\begin{aligned} W_F(x) - \epsilon &\leq q(x) - V_F^*(q) + \delta/a \leq q(x) - V_F^*(q) \\ &\leq \sup_q (q(x) - V_F^*(q)) = V_F^{**}(x). \end{aligned} \tag{3.7}$$

By letting ϵ convergent to 0 we obtain

$$W_F(x) \leq V_F^{**}(x) \tag{3.8}$$

for every $x \in DomV_F$.

Now assume $a = 0$. Then $x \notin DomV_F$. Really, for every $x \in DomV_F$ taking $y = x$ we have

$$p(x) + aV_F(x) \leq p(x) + aW_F(x) - a\epsilon - \delta \tag{3.9}$$

and therefore

$$a(V_F(x) - W_F(x) - \epsilon) \leq -\delta, \tag{3.10}$$

so for each $x \in DomV_F$ a cannot be equal to 0.

However, taking $a = 0$ and $x \notin DomV_F$ we get for every $y \in DomV_F$

$$p(y) \leq p(x) - \delta \tag{3.11}$$

Let us take $r \in DomV_F^*$. By the definition of V_F^* we deduce that

$$r(y) - V_F(y) \leq V_F^*(r) \tag{3.12}$$

Adding this to the inequality (3.11) multiplied by $n > 0$, we obtain

$$(np + r)(y) - V_F(y) \leq np(x) - n\delta + V_F^*(r) \tag{3.13}$$

Taking the supremum over $y \in DomV_F$ we have

$$V_F^*(np + r) \leq np(x) - n\delta + V_F^*(r). \tag{3.14}$$

Hence

$$r(x) + n\delta - V_F^*(r) \leq (np + r)(x) - V_F^*(np + r). \tag{3.15}$$

By the definition of V_F^{**} we get

$$r(x) + n\delta - V_F^*(r) \leq V_F^{**}(x) \tag{3.16}$$

for every n and $x \notin DomV_F$. Taking $n \to \infty$ we deduce $V_F^{**}(x) = \infty$ and therefore,

$$W_F(x) \leq V_F^{**}(x) \tag{3.17}$$

for every $x \in X$. Since F admits closed convex values and $V_F^{**}(x) \leq V_F(x)$ by Theorem 11.1 of [15], then we deduce that V_F^{**} is a proper lower semicontinuous and convex selection of F and the proof is complete. □

Remark 3.5. Let a set-valued function $F : \mathbf{R^1} \to Conv\mathbf{R^1}$ be upper separated. Then by Proposition 3.4 it admits a convex selection f. If x is a semimartingale, then it follows by Theorem IV.47 of [14] that $f(x)$ is a semimartingale too. Using Corollary II.6.1 p. 60 of [14] we deduce that $[f(x), z]$ is finite and therefore, $[F \bullet x, z]$ is well defined.

Upper separated set valued mappings need not satisfy the linear growth condition, hence one can expect solutions of stochastic Stratonovich inclusion that not exist globally. In other words there may exist solutions that have explosions. Recall, a stopping time S is an explosion time for a solution process x if x is a solution to Stratonovich inclusion on $[0, S)$, $x_S = +\infty$ P.1 on $\{S < \infty\}$ and $S = \lim S_n$, where

$$S_n := \inf\{t > 0 : |x_t| > n\}, \text{ for } n \geq 1.$$

The case $P\{S = \infty\} = 1$ refers to a nonexploding solution.

The following result on strong solutions to Stratonovich stochastic inclusion with explosions holds true.

Theorem 3.6. *Let z be a continuous and $\mathbf{F}_t$-adapted semimartingale and let a be a continuous, $\mathbf{F}_t$-adapted process of finite variation. Let $F, G : \mathbf{R^1} \to Conv\mathbf{R^1}$ be upper separated set-valued functions. Then there exists a strong solution up to explosion time to a Stratonovich stochastic inclusion* (3.1).

Proof. By Proposition 3.4 there exist convex and continuous selections f and g for set-valued mappings F and G respectively. By Proposition 1.6 of [12] these selections are also locally Lipschitzean at any point and by Theorem 1.16 of [12], they have nondecreasing derivatives f' and g' for all points, except at most countably quantity of points of $\mathbf{R^1}$. Hence, in particular the right derivative f'_+ possesses a cádlág version. By Lemma 2.2 of [16] a function f belongs to the class of antiderivatives of the space D of cádlág functions (i.e.: $f \in \mathcal{A}D$), see [8, 16] for details. Using Theorem 3.6 of [16] one obtains

$$[f(x), z]_t = \int_0^t f'(x_s)d[x,z]_s, \tag{3.18}$$

for any continuous semimartingales x and z, with $z_0 = 0$. Thus the problem of existence of strong solutions to Stratonovich stochastic inclusion (3.4) can be reduced to the existence of strong solutions to the stochastic equation

$$x_t = x_0 + \int_0^t f(x_s)dz_s + \frac{1}{2}\int_0^t f'(x_s)d[x,z]_s + \int_0^t g(x_s)da_s \tag{3.19}$$

or equivalently to

$$x_t = x_0 + \int_0^t f(x_s)dz_s + \frac{1}{2}\int_0^t f'(x_s)f(x_s)d[z,z]_s + \int_0^t g(x_s)da_s. \tag{3.20}$$

For $k \geq 1$ let

$$f_k(u) := \begin{cases} f(u) & \text{for} \quad |u| \leq k \\ f(k) & \text{for} \quad u > k \\ f(-k) & \text{for} \quad u < k \end{cases} \tag{3.21}$$

We define g_k similarly. By the definition these functions are globally Lipschitz and f_k has a version of its derivative f'_k which is cádlág and globally bounded. Consider the equations

$$x_t = x_0 + \int_0^t f_k(x_s)dz_s + \frac{1}{2}\int_0^t f'_k(x_s)f_k(x_s)d[z,z]_s + \int_0^t g_k(x_s)da_s. \tag{3.22}$$

for every $k \geq 1$.

Functions f_k and g_k satisfy all assumptions of Theorem 4.14 in [16]. Hence, for every $k \geq 1$ there exists a unique minimal strong solution x^k of (3.22). Let us define stopping times $S^k := \inf\{t > 0 : |x^k_t| > k\}$. Then we have

$$x^k_{t\wedge S^k} = x_0 + \int_0^{t\wedge S^k} f_k(x^k_s)dz_s + \tfrac{1}{2}\int_0^{t\wedge S^k} f'_k(x^k_s)f_k(x^k_s)d[z,z]_s + \int_0^{t\wedge S^k} g_k(x^k_s)da_s. \tag{3.23}$$

Let us notice that $f(u) = f_k(u) = f_{k+1}(u)$ for $|u| \le k$, and $k = 1, 2, \ldots$. Similar property holds for functions $g,\ g_k,\ g_{k+1}$ and multiplications $f'f,\ f'_kf_k,\ f'_{k+1}f_{k+1}$. Since $x^k_{t\wedge S^k}$ belongs to the interval $[-k,k]$ we get

$$\begin{aligned} x^k_{t\wedge S^k} &= x_0 + \int_0^{t\wedge S^k} f(x^k_s)dz_s + \tfrac{1}{2}\int_0^{t\wedge S^k} f'(x^k_s)f(x^k_s)d[z,z]_s \\ &\quad + \int_0^{t\wedge S^k} g(x^k_s)da_s \\ &= x_0 + \int_0^{t\wedge S^k} f_{k+1}(x^k_s)dz_s + \tfrac{1}{2}\int_0^{t\wedge S^k} f'_{k+1}(x^k_s)f_{k+1}(x^k_s)d[z,z]_s \\ &\quad + \int_0^{t\wedge S^k} g_{k+1}(x^k_s)da_s. \end{aligned} \tag{3.24}$$

By the uniqueness of the minimal solution to (3.22) we deduce from (3.24) that $x^k = x^{k+1}$ on $[0, S^k]$.

Moreover, $S^k < S^{k+1}$ on $\{S^k < \infty\}$ and therefore, we can define a predictable stopping time $S := \lim_k S^k$ and the process x on the interval $[0,S]$ such that $x = x^k$ on $[0, S^k]$. Since the process x satisfies (3.22) on $[0, S^k]$ for $k = 1, 2, \ldots$, so it must satisfy also equation (3.19) on $[0, S)$. The stopping time S is the explosion time referred to, in the statement of the Theorem. This completes the proof. □

The case $P\{S = \infty\} = 1$ refers to nonexploding solution. The examples below show that both situations of the existence of solutions with or without explosions can appear.

Example 3.7. Let us take a set-valued function $F \equiv \{0\}$, a deterministic increasing process $a_t \equiv t,\ t \ge 0$, and a set-valued function $G(u) = [u^2 - \mathbf{1}_{Q\cap[0,1]}(u), 1]$, for $u \in [0,1]$. $\mathbf{1}_A$ denotes a characteristic function of the set A, while Q is a set of rationals.

Clearly, both F and G are upper separated, measurable and bounded. Moreover, G is not upper nor lower semicontinuous at any point. In this case Stratonovich inclusion (3.1) reduces to a deterministic Aumann -type integral inclusion:

$$x(t) \in 1 + \int_0^t G(x(s))ds,$$

for which the function $x(t) = (1-t)^{-1}$ is a solution up to the deterministic explosion time as $t \to 1^-$.

Example 3.8. Let us take $Z = W$ being a one dimensional Wiener process, $a_t \equiv t,\ t \geq 0$, and set-valued functions $F \equiv \{1\}$ and $G : [0, +\infty) \to [0, +\infty)$ defined by

$$G(u) := \begin{cases} [0, +\infty) & \text{for} \quad u = 0,\ u = 3^{-\frac{1}{3}} \\ [\frac{3}{2}u^2, \frac{1}{2u}] & \text{for} \quad u \in (0, 3^{-\frac{1}{3}}) \\ [\frac{1}{2u}, \frac{3}{2}u^2] & \text{for} \quad u > 3^{-\frac{1}{3}} \end{cases} \tag{3.25}$$

It is easy to see that G is upper separated but not Lipschitz continuous. It has not a linear growth. Let $x_0 = 1$. In this case Stratonovich inclusion (3.1) has the form

$$x_t \in 1 + W_t + \int_0^t G(x_s)ds.$$

This inclusion has solutions both with explosions and without them.

Indeed, taking selection $g(u) = \frac{3}{2}u^2$, we can consider stochastic equation:

$$x_t = 1 + W_t + \frac{3}{2}\int_0^t x_s^2 ds.$$

This equation has a strong solution on $[0, S)$. By Th.4.13 in [13] we obtain $P\{S = +\infty\} < 1$ in this case.

On the other hand taking a selection $g(u) = \frac{1}{2u}$ we arrive to the following stochastic equation:

$$x_t = 1 + W_t + \frac{1}{2}\int_0^t \frac{1}{x_s}ds.$$

This equation has a weak solution being the Bessel process with index 2. Using Feller test for explosions (Th. 5.5.29 and Example 5.5.29 [5]) one can show no explosion in this case.

In the above Examples we have shown that Stratonovich inclusion with upper separated set-valued functions have both exploding and nonexploding strong or weak solutions. The next result ensure the existence of nonexploding strong solutions.

Theorem 3.9. *Let z be a continuous and $\mathbf{F}_t$-adapted semimartingale and let a be a continuous, $\mathbf{F}_t$-adapted process of finite variation. Let $F, G : \mathbf{R^1} \to Conv\mathbf{R^1}$ be set-valued mappings such that*

(i) F is upper separated, satisfying linear growth condition and such that its lower bound $W_F(u) > \delta$, for some $\delta > 0$,

(ii) G is Lipschitz continuous.

Then there exists a strong and nonexploding solution to Stratonovich stochastic inclusion (3.1).

Proof. Similarly as in the proof of Theorem 3.6 there exists a convex and continuous selection f of F. It belongs to the class of antiderivatives of the space D of cádlág functions and satisfies the inequality $f^2 \geq W_F^2(u) > \delta^2 > 0$.

Moreover, there exists a Lipschitz selection g of G by Theorem 9.4.3 in [1]. Let us consider Stratonovich equation

$$x_t = x_0 + \int_0^t f(x_s) \circ dz_s + \int_0^t g(x_s) da_s. \tag{3.26}$$

By Corollary 5.17 of [16] there exists a unique strong solution to (3.26) which is clearly also a strong solution to inclusion (3.1). □

Remark 3.10. The valuable discussion on the existence of solutions to a single-valued Stratonovich equation of the type

$$x_t = x_0 + \int_0^t f(x_s) \circ dz_s$$

can be found among others in [8, 14]. The existence of solutions for such an equation is considered there for the case of continuously differentiable or cádlág differentiable function f. Therefore, it is quite natural to examine the set-valued case for the class of set-valued functions admitting differentiable selections. The class of Hukuhara differentiable multifunctions seems to be valuable for the problem. This approach is discussed in [10].

Let us mention that Theorems 3.6 and 3.9 essentially differ from the results obtained in [10], because the upper separated set-valued function need not be Hukuhara differentiable as we will see below.

Example 3.11. Define a set-valued function $F : R \to 2^{\mathbf{R^1}}$ by the formula

$$F(x) = \begin{cases} [\,|x| + 1, x^2 + 3\,] & for \quad x \in Q \\ [\,|x| + 2, x^2 + 5\,] & for \quad x \in \mathbf{R^1} \setminus Q. \end{cases}$$

where Q denotes the set of rationals. It is clear that the set-valued function F is not Lipschitz continuous, lower semicontinuous nor upper semicontinuous in any point. It does not satisfy any of monotone type conditions either. For such an F we obtain

$$\delta(x) := \text{ diam } F(x) = \begin{cases} x^2 - |x| + 2 & for \quad x \in Q \\ x^2 - |x| + 3 & for \quad x \in R \setminus Q \end{cases}$$

Since the function $x \to \delta(x)$ is not nondecreasing on any open subset of R, then by Proposition 4.1 of [2] F cannot be Hukuhara differentiable in any point. However, F is upper separated, $W_F(x) > 1 > 0$ and therefore, Theorem 3.9 can be applied to the inclusion (3.1) with set-valued F, G of such a type as above.

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