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BIFURCATION AT MULTIPLE EIGENVALUES FOR SYSTEMS WITH LIPSCHITZ MAPPINGS

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ABSTRACT. Some results on the existence of bifurcation at multiple eigenvalues for abstract systems concerning Lipschitz continuous mappings in Banach spaces are proved. The obtained results improve some well-known bifurcation results by Crandall and Rabinowitz, McLeod and Sattinger, Tan etc, in the case involving Lipschitz continuous mappings. An application to a system of partial differential equations will be given.

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1. INTRODUCTION

Bifurcation problems play a very important role in different areas of applied mathematics and have been intensively studied in the literature. Several methods have been used, for instance, variational, topological, analytical and numerical methods etc, (cf [1, 2, 3, 4, 9, 10, 11]). In general, the bifurcation problem consists in determining bifurcation points of equations depending on a parameter in Banach spaces of the form

$$F(\lambda, v) = 0, \qquad (\lambda, v) \in \Lambda \times \overline{D} \tag{1.1}$$

where Λ is a subset of a normed space, D is a neighbourhood of the origin in a Banach space X with the closure $\overline{D}$ and F is a nonlinear mapping from $\Lambda \times \overline{D}$ into another Banach spaces Y with $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. A point $(\lambda, 0)$ is called a trivial solution.

Definition 1.1. A point $(\overline{\lambda}, v) \in \Lambda \times D$ is said to be a bifurcation point of the equation (1.1) if and only if

$$(\overline{\lambda}, 0) \in cl\left\{(\lambda, v) \in \Lambda \times \overline{D}, F(\lambda, v) = 0 \text{ and } v \neq 0\right\},$$

where $clA = \overline{A}$, the closure of the set A.

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In the case the mapping F is differentiable, using the Implicit Function Theorem one can easily verify that $(\overline{\lambda},0)$ is a bifurcation point of (1.1) only if $\overline{\lambda}$ belongs to the spectral set of $F_x(\lambda,0)$, i.e, $F_x(\overline{\lambda},0)u_0=0$ for some $u_0\in X, u_0\neq 0$.

The purpose of this paper is to study the existence of bifurcation points of the system (1.1), with F of the form

$$F(\lambda,u)=-T(u)+L(\lambda,u)+H(\lambda,u)+K(\lambda,u), \qquad (\lambda,u)\in\Lambda\times\overline{D},$$

$T=(T_1,T_2)$, $L=(L_1,L_2)$, $H=(H_1,H_2)$ and $K=(K_1,K_2)$, where Λ is an open subset of a normed space Z with the norm defined by $|.|_\Lambda$, D is a neighbourhood of the origin in a Banach space X. For any fixed $\lambda\in\Lambda$ and for any $i=1,2$, T_i, $L_i(\lambda,.)$ are linear continuous mappings from X into another Banach space Y_i, $H_i(\lambda,.)$, $K_i(\lambda,.)$ are nonlinear Lipschitz continuous mappings of "higher order term" from $\overline{D}$ into Y_i with $H_i(\lambda,0)=K_i(\lambda,0)=0$ and $H_i(\lambda,.)$ satisfies an a_i-homogeneous condition to be described later with $a_i>1$.

Let $\overline{\lambda}\in\Lambda$ be a characteristic value of the pair (T,L) (i.e $T(v)-L(\overline{\lambda},v)=0$ for some $v\in X, v\neq 0$) such that the mapping $T-L\left(\overline{\lambda},.\right)$ is Fredholm with nullity p and index zero. We shall prove, under some sufficient conditions, that $(\overline{\lambda},0)$ is a bifurcation point of the system (1.1) with F as above, using the Lyapunov-Schmidt procedure, the Banach Contraction Principle and the topological degree theory. Furthermore, we also describe parameter families of nontrivial solutions of the system (1.1) in a neighbourhood of $(\overline{\lambda},0)$ in analytical form. Our result in Section 2 generalize some well-known results obtained by Crandall and Rabinowitz [7], McLeod and Sattinger [12], Buchner, Marsden and Schecter [5], Tan [13]. They always need the differentiability conditions on those mappings. Lastly, in Section 3, we apply the obtained result to investigate the bifurcation points of the system of equations of the form

$$\begin{cases} -\Delta u=\eta v+u\,|uv| \text{ in } G\\ -\Delta v=\mu u+v\,|u|^{\sigma} \text{ in } G\\ u=v=0 \text{ in } \partial G,\end{cases}$$

where $(\eta,\mu)\in\mathbb{R}^2, \sigma>2, G=[0,1]\times[0,1]\times[0,1], (u,v)\in X=H_0^1(G)\times H_0^1(G)$.

2. THE MAIN RESULT

2.1. **Notations.** Throughout this paper, X, Y_1 and Y_2 are always supposed to be real Banach spaces with duals X^*, Y_1^* and Y_2^*, respectively, Y denoted the product space $Y_1\times Y_2$ with dual Y^*. Λ is an open subset of a normed space Z. The norm and the pairings between elements of X, X^*; and Y, Y^* are denoted by the same symbols $\|.\|$ and $\langle\cdot,\cdot\rangle$, respectively. The norm of the normed space containing Λ restricted to Λ is denoted by $|.|_\Lambda$. In this section we consider the existence of bifurcation points of the system (1.1) with F mentioned above. This means that we investigate the existence of bifurcation points of the system

$$(2.1)\qquad \begin{cases} T_1(v) = L_1(\lambda, v) + H_1(\lambda, v) + K_1(\lambda, v), & (\lambda, v) \in \Lambda \times \overline{D} \\ T_2(v) = L_2(\lambda, v) + H_2(\lambda, v) + K_2(\lambda, v), & (\lambda, v) \in \Lambda \times \overline{D}, \end{cases}$$

where the mappings T_i, L_i, H_i, and $K_i, i = 1, 2$ are as in the introduction.

Now let $\overline{\lambda} \in \Lambda$ be a characteristic value of the pair (T, L) with multiplicity $p, p \geq 1$, such that the mapping $T - L\left(\overline{\lambda}, .\right)$ is Fredholm with nullity p and index zero, where $T = (T_1, T_2)$ and $L = (L_1, L_2)$. It follows that the null space $\ker(T - L(\overline{\lambda}, .))$ is p-dimensional. We assume

$$(2.2)\qquad \ker(T - L(\overline{\lambda}, .)) = [v_1, \dots, v_p],$$

where the right side is the subspace of X spanned by $v_1, \dots, v_p$.

By $(T - L(\overline{\lambda}, .))^*$ we denote the adjoint mapping of the mapping $T - L(\overline{\lambda}, .)$, defined on Y^*, and assume

$$(2.3)\qquad \ker(T - L(\overline{\lambda}, .))^* = [\psi_1, \dots, \psi_p].$$

Given $j \in \{1, \dots, p\}$, we define $\psi_j^1 : Y_1 \to \mathbb{R}$ and $\psi_j^2 : Y_2 \to \mathbb{R}$ by $\left\langle y, \psi_j^1 \right\rangle = \psi_j(y, 0)$ and $\left\langle y, \psi_j^2 \right\rangle = \psi_j(0, y)$.
By the Hahn-Banach Theorem one can find p functionals $\gamma_1, \dots, \gamma_p$ on X such that

$$\langle v_m, \gamma_n \rangle = \delta_{mn},\ m, n = 1, \dots, p.$$

and $2p$ elements, $z_1^i, \dots, z_p^i$ in $Y_i, i = 1, 2$, such that

$$\left\langle z_m^i, \psi_n^i \right\rangle = \delta_{mn},\ m, n = 1, \dots, p,$$

with δ_{mn} denoting the Kronecker δ. We set

$$\begin{cases} X^0 = [v_1, \dots, v_p], \\ X^1 = \{x \in X, \langle x, \gamma_k \rangle = 0, k = 1, \dots, p\}. \end{cases}$$

For $j = 1, \dots, p$, we set $z_j = (z_j^1, z_j^2)$ and

$$\begin{cases} Y^0 = [z_1, \dots, z_p], \\ Y^1 = \{y \in Y, \langle y, \psi_k \rangle = 0,\ k = 1, \dots, p\}. \end{cases}$$

It can be seen that $Y = Y^0 \bigoplus Y^1$ and the restriction of the mapping $T - L(\overline{\lambda}, .)$ on X^1 is a one-to-one linear continuous mapping onto Y^1. The projections $P_X : X \to X^0$, $Q_X : X \to X^1$, $P_Y : Y \to Y^0$ and $Q_Y : Y \to Y^1$, are defined by

$$P_X(x) = \sum_{k=1}^{p} \langle x, \gamma_k \rangle v_k, \quad Q_X(x) = x - P_X(x), \qquad x \in X,$$

$$P_Y(y) = \sum_{k=1}^{p} \langle y, \psi_k \rangle z_k, \quad Q_Y(y) = y - P_Y(y), \qquad y \in Y.$$

2.2. **Hypotheses.** Concerning the main results in this paper we impose the following hypotheses on the mappings T_i, L, H_i and K_i:

(H1) : There is a real number b such that $\alpha L(\overline{\lambda}, v) = L\left(\alpha^b \overline{\lambda}, v\right)$ holds for all $\alpha \in [0,1]$ and $v \in \overline{D}$.

(H2) : There exist two real numbers $a_i > 2$ and two real increasing function $\rho_i : \mathbb{R} \to \mathbb{R}$ with $\lim\limits_{\delta \to 0} \rho_i(\delta) = 0$ such that,

i) The mappings $H_i, i = 1, 2$ are k_i-Lipschitz.

ii) $H_i(\lambda, tv) = t^{a_i} H_i(\lambda, v)$ holds for all $t \in [0,1]$, $(\lambda, v) \in \Lambda \times \overline{D}$.

iii) $\alpha^{-a_i} P_Y K_i(\dfrac{\overline{\lambda}}{(1+\alpha^{a_i-1})^b}, \alpha v)$ tend to zero as $\alpha \to 0$, uniformly to $v \in \overline{D}$, where b is from Hypothesis (H1).

iv) $\|K_i(\lambda, v) - K_i(\lambda', v')\| \leq \rho_i(|\lambda - \lambda'|_\Lambda + \|v - v'\|)(|\lambda - \lambda'|_\Lambda + \|v - v'\|)$, holds for all $(\lambda, v), (\lambda', v') \in \Lambda \times D$.

Further, we put

$$a = \min(a_1, a_2) \tag{2.4}$$

and for $i = 1, 2$, we define the mapping $\mathcal{A}_i : \mathbb{R}^p \to \mathbb{R}^p, \mathcal{A}_i = (\mathcal{A}_i^1, \ldots, \mathcal{A}_i^p)$ by

(2.5)

$$\mathcal{A}_i^k(x) = \left\langle T_i(\sum_{j=1}^{p} x_j v_j) - c_i H_i(\overline{\lambda}, \sum_{j=1}^{p} x_j v_j), \psi_k^i \right\rangle,\ x = (x_1, \ldots, x_p),\ k = 1, \ldots, p,$$

with

$$c_i = \begin{cases} 1 \text{ if } a_i = a, \\ 0 \text{ if not,} \end{cases}$$

and make the following hypothesis:

(H3) : There is a point $\overline{x} \in \mathbb{R}^p$ and an open neighborhood U^* of $\overline{x}$ not containing the origin in $\mathbb{R}^p$, such that the topological degree $\deg(\mathcal{A}_1 + \mathcal{A}_2, U^*, 0)$ of the mapping $\mathcal{A}_1 + \mathcal{A}_2$ with respect to U^* and the origin is defined and different from zero.

2.3. **The main result.** We have the following result:

Theorem 2.1. *Under hypotheses (H1)-(H3), $(\overline{\lambda}, 0)$ is a bifurcation point of the system* (2.1)*. More precisely, to any given $\delta > 0$ there exists a neighborhood I of zero in $\mathbb{R}$ such that for each $\alpha \in I, \alpha \neq 0$, we can find $x(\alpha) = (x_1(\alpha), \ldots, x_p(\alpha)) \in U^*$ and a nontrivial solution $(\lambda(\alpha), v(\alpha))$ of the system (2.1) with*

$$\begin{aligned} \lambda(\alpha) &= \frac{\overline{\lambda}}{(1+|\alpha|^{a-1})^b}, \\ v(\alpha) &= \sum_{j=1}^{p} |\alpha|\, x_j(\alpha) v_j + o(|\alpha|), \end{aligned}$$

satisfying

$$\left|\lambda(\alpha) - \overline{\lambda}\right|_\Lambda < \delta \text{ and } 0 < \|v(\alpha)\| < \delta.$$

The proof of this theorem will be done in two steps. In the first, we will give three lemmas which will reduce the resolution of the system (2.1) to a finite dimension. In the second step we will solve this finite dimensional problem.

The system (2.1) can be written as (2.6).

$$T(v) = L(\lambda, v) + H(\lambda, v) + K(\lambda, v) \qquad (\lambda, v) \in \Lambda \times \overline{D}. \tag{2.6}$$

Let us fix $\overline{\lambda}$ satisfying (H1)-(H3). Let I_1 be a neighborhood of zero in $\mathbb{R}$, $I_1 \subset [-1, 1]$, such that $\dfrac{\overline{\lambda}}{(1 + |\alpha|)^b} \in \Lambda$ holds for all $\alpha \in I_1$, where b is from hypothesis (H1) and let $U_1 = U(0, r_1)$ be an open ball with the center at the origin in $\mathbb{R}^p$ and the radius $r_1 > 0$, such that $\sum_{i=1}^{p} x_i v_i \in P_X(D)$ for all $(x_1, \ldots, x_p) \in U_1$. Without loss of generality we may assume that $|\alpha|^{a-1} \in I_1, |\alpha|\, x \in U_1$ hold for all $\alpha \in I_1, x \in U_1$. Setting $D_1 = Q_X(D)$ and by choosing D_1' smaller if necessary, we may assume that $D_1 = D_1(0, r_2)$, the open ball with the center at the origin in X^1 and radius $r_2 > 0$.

We define the mapping $G : I_1 \times U_1 \times \overline{D_1} \to X^1$ by

$$G(\alpha, \varepsilon, \omega) = -SQ_Y\{|\alpha|\, T(\sum_{i=1}^{p} \varepsilon_i v_i + \omega) - (1 + |\alpha|) M(\frac{\overline{\lambda}}{(1 + |\alpha|)^b}, \sum_{i=1}^{p} \varepsilon_i v_i + \omega)\},$$

where $M = (H_1 + K_1, H_2 + K_2)$ and S is the inverse of the restriction of the mapping $T - L(\overline{\lambda}, .)$ on Y^1 onto X^1.

2.4. **Lemmas.**

Lemma 2.2. *Let hypotheses (H1), (H2) be satisfied and let I_1, U_1, D_1 be as above. Then there exist neighborhoods I_2 of zero in $\mathbb{R}, I_2 \subset I_1, D_2$ of the origin in X^1 such that for any $(|\alpha|^{a-1}, |\alpha|\, x) \in I_2 \times U_1$ one can find a point $\omega = \omega(|\alpha|^{a-1}, |\alpha|\, x) \in D_2$ satisfying*

$$G(|\alpha|^{a-1}, |\alpha|\, x, \omega) = \omega.$$

Proof. For fixed $t \in [0, 1]$, we set $I(t) = tI_1$ and $D(t) = tD_1$. Then, any $\alpha \in I(t), \omega \in D(t)$ can be written as $\alpha = t\alpha', \omega = t\omega'$ with $\alpha' \in I_1, \omega' \in D_1$. Now, let $(|\alpha|^{a-1}, |\alpha|\, x) \in I(t) \times U_1$. We will prove that there exists

$$t_0 \in (0, 1] \tag{2.7}$$

such that the map $G(|\alpha|^{a-1}, |\alpha|\, x, .)$ is a strict contraction from $D(t_0)$ into $D(t_0)$.

Let us write $\alpha = t\alpha', \omega^j = t\omega'^j$ for any $\omega^j \in D(t), j = 1, 2$, and set

$$\gamma = \|SQ_Y\| \tag{2.8}$$

and

$$\Delta = \left\|G(|\alpha|^{a-1}, |\alpha|\, x, \omega^1) - G(|\alpha|^{a-1}, |\alpha|\, x, \omega^2)\right\|.$$

Then, by the definition of the map G, we obtain

$$\begin{aligned}\Delta &= \|SQ_Y\{|\alpha|^{a-1}T(W^1)-(1+|\alpha|^{a-1})M(\frac{\overline{\lambda}}{(1+|\alpha|^{a-1})^b},W^1)\}-\\ &\quad SQ_Y\{|\alpha|^{a-1}T(W^2)-(1+|\alpha|^{a-1})M(\frac{\overline{\lambda}}{(1+|\alpha|^{a-1})^b},W^2)\}\|\end{aligned}$$

with $W^j=\sum_{i=1}^{p}|\alpha|\,x_iv_i+\omega^j$ for $j=1,2$.
By (2.8), the hypotheses (H2i), (H2ii) and (H2iv) we deduce

$$\begin{aligned}\Delta &\le \gamma\{|\alpha|^{a-1}\|T\|\left\|\omega^1-\omega^2\right\|+2t^a(k_1+k_2)\left\|\omega'^1-\omega'^2\right\|+\\ &\quad 2(\rho_1+\rho_2)(\left\|\omega^1-\omega^2\right\|)\left\|\omega^1-\omega^2\right\|\}.\end{aligned}$$

Since ρ_1 and ρ_2 are increasing functions and $\|\omega^1-\omega^2\|\le 2r_2t$, then

$$\Delta\le\gamma\{|\alpha|^{a-1}\|T\|+2t^{a-1}(k_1+k_2)+2(\rho_1+\rho_2)(2r_2t)\}\left\|\omega^1-\omega^2\right\|.$$

Therefore, setting

$$G_1(t)=\gamma\{|\alpha|^{a-1}\|T\|+2t^{a-1}(k_1+k_2)+2(\rho_1+\rho_2)(2r_2t)\}$$

and since $|\alpha|^{a-1}\in I(t)=tI_1$, we can see that

$$\lim_{t\to 0}G_1(t)=0. \tag{2.9}$$

Further for $\omega=t\omega',\omega'\in D$, we have similarly as above

$$\begin{aligned}&\left\|G(|\alpha|^{a-1},|\alpha|\,x,\omega)\right\|\\ \le\ &\gamma\{|t\alpha'|^{a-1}\|T\|\,(|t\alpha'x|+\|t\omega'\|)+2t^a(k_1+k_2)(|\alpha'x|+\|\omega'\|)+\\ &2t(\rho_1+\rho_2)(t(|\alpha'x|+\|\omega'\|))(|\alpha'x|+\|\omega'\|)\}\\ \le\ &t\gamma\{|t\alpha'|^{a-1}\|T\|+2t^{a-1}(k_1+k_2)+\\ &2(\rho_1+\rho_2)(t(r_2+r_1))\}(r_2+r_1).\end{aligned}$$

Setting

$$G_2(t)=t\gamma\{|t\alpha'|^{a-1}\|T\|+2t^{a-1}(k_1+k_2)+2(\rho_1+\rho_2)(t(r_2+r_1))\},$$

we can see that

$$\lim_{t\to 0}\frac{G_2(t)}{t}=0. \tag{2.10}$$

Consequently by (2.9) and (2.10), we deduce the existence of a real $t_0\in(0,1]$, such that, $0<G_1(t_0)<1$ and $G_2(t_0)\le t_0r_2$. Putting $I_2=t_0I_1, D_2=t_0D_1$, then the mapping $G(|\alpha|^{a-1},|\alpha|\,x,.)$ is a strict contraction mapping and it maps $\overline{D_2}$ into itself. Applying the Banach Contraction Principle, we conclude that $G(|\alpha|^{a-1},|\alpha|\,x,.)$ possesses a fixed point $\omega(|\alpha|^{a-1},|\alpha|\,x)$ in $\overline{D_2}$, i.e,

$$G(|\alpha|^{a-1},|\alpha|\,x,\omega(|\alpha|^{a-1},|\alpha|\,x))=\omega(|\alpha|^{a-1},|\alpha|\,x).$$

□

Lemma 2.3. *Under the hypotheses of lemma 2.2, there exists a constant $k > 0$ such that for any $|\alpha|^{a-1} \in I_2, |\alpha|\, x^1 \in U_1, |\alpha|\, x^2 \in U_1$, we have*

$$\left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\| \leq k \left\|x^1 - x^2\right\|.$$

In particular for all $\alpha \in I_2, \omega(|\alpha|^{a-1}, |\alpha|\,.)$ is a continuous mapping with respect to $x \in U_1$.

Proof. Let t_0 and γ be given by (2.7) and (2.8), respectively, and let $\omega(|\alpha|^{a-1}, |\alpha|\, x^j) = t_0\omega'(|\alpha|^{a-1}, |\alpha|\, x^j), j = 1, 2$. Then by lemma 2.2, the definition of the map G and the hypotheses (H1), (H2), we obtain:

$$\begin{aligned}
&\left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\| \\
&= \left\|G(|\alpha|^{a-1}, |\alpha|\, x^1, \omega(|\alpha|^{a-1}, |\alpha|\, x^1)) - G(|\alpha|^{a-1}, |\alpha|\, x^2, \omega(|\alpha|^{a-1}, |\alpha|\, x^2))\right\| \\
&\leq \gamma\{|\alpha|^{a-1} \|T\| \,(|\alpha|\, \|x^1 - x^2\| + \left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\|)+ \\
&2t_0^{a-1}(k_1 + k_2)(|\alpha|\, \|x^1 - x^2\| + \left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\|)+ \\
&2(\rho_1 + \rho_2)(|\alpha|\, \|x^1 - x^2\| + \left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\|) \\
&(|\alpha|\, \|x^1 - x^2\| + \left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\|)\} \\
&\leq \gamma\{t_0 \|T\| + 2t_0^{a-1}(k_1 + k_2) + 2(\rho_1 + \rho_2)(2(r_1 + r_2)t_0)\}(\|x^1 - x^2\| + \\
&\left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\|)
\end{aligned}$$

Setting

$$\beta(t_0) = \gamma\{t_0 \|T\| + 2t_0^{a-1}(k_1 + k_2) + 2(\rho_1 + \rho_2)(2(r_1 + r_2)t_0)\}$$

and choosing t_0 rather small, one can suppose that $0 < \beta(t_0) < 1$. Hence we obtain

$$\left\|\omega(|\alpha|^{a-1}, |\alpha|\, x^1) - \omega(|\alpha|^{a-1}, |\alpha|\, x^2)\right\| \leq \frac{\beta(t_0)}{1 - \beta(t_0)} \left\|x^1 - x^2\right\|,$$

and then, we take $k = \dfrac{\beta(t_0)}{1 - \beta(t_0)}$. □

2.5. **Proof of the main result.**

2.5.1. *Reduction of the resolution in finite dimension.* Let I_1, U_1, D_1 be from Section 2.3 and let $\delta > 0$ be given. Using lemma 2.2, we conclude that there is a neighborhood I_2 of zero in $\mathbb{R}, I_2 \subset I_1$, such that for $\alpha \in I_2, \alpha \neq 0, |\alpha|\, x \in U_1$ we can find a fixed point ω of the mapping $G(|\alpha|^{a-1}, |\alpha|\, x, .)$. Moreover

$$\begin{aligned}
\omega \;\; = \;\; & -SQ_Y\{|\alpha|^{a-1} T(\sum_{i=1}^{p} |\alpha|\, x_i v_i + \omega) - \\
& (1 + |\alpha|^{a-1})M(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, \sum_{i=1}^{p} |\alpha|\, x_i v_i + \omega)\}.
\end{aligned}$$

Applying $S^{-1} = T - L(\overline{\lambda}, .)$ we obtain realizing $S^{-1}\omega = Q_Y S^{-1}\omega$ that

$$Q_Y\{T(\omega) - L(\overline{\lambda}, \omega) + |\alpha|^{a-1} T(\sum_{i=1}^{p} |\alpha| x_i v_i + \omega) - (1 + |\alpha|^{a-1}) M(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, \sum_{i=1}^{p} |\alpha| x_i v_i + \omega)\} = 0.$$

Together with (2.2), we have

$$T(\sum_{i=1}^{p} |\alpha| x_i v_i) - L(\overline{\lambda}, \sum_{i=1}^{p} |\alpha| x_i v_i) = 0.$$

Then

$$Q_Y\{(1 + |\alpha|^{a-1}) T(\sum_{i=1}^{p} |\alpha| x_i v_i + \omega) - L(\overline{\lambda}, \sum_{i=1}^{p} |\alpha| x_i v_i + \omega) - (1 + |\alpha|^{a-1}) M(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, \sum_{i=1}^{p} |\alpha| x_i v_i + \omega)\} = 0. \tag{2.11}$$

By multiplying (2.11) with $1/(1 + |\alpha|^{a-1})$ and by using hypothesis (H1), we deduce

$$Q_Y\{T(\sum_{i=1}^{p} |\alpha| x_i v_i + \omega) - L(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, \sum_{i=1}^{p} |\alpha| x_i v_i + \omega) - M(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, \sum_{i=1}^{p} |\alpha| x_i v_i + \omega)\} = 0, \tag{2.12}$$

for all $\alpha \in I_2, \alpha \neq 0, |\alpha| x \in U_1$, which reduce the resolution of the system (2.1) in finite dimension. By shrinking I_2 and D_2 if necessary, we may assume that :

$$\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b} \in \Lambda, \left| \frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b} - \overline{\lambda} \right|_{\Lambda} < \delta \text{ and } \alpha U^* \subset U^* \text{ for all } \alpha \in I_2.$$

2.5.2. *Resolution in finite dimension.* For each $(t, \alpha, x) \in [0, 1] \times I_2 \times U^*, t \neq 0, \alpha \neq 0$, $k = 1, \ldots, p$, we put $\omega(t) := \omega(|t\alpha|^{a-1}, |t\alpha| x)$, $\lambda(\alpha) = \dfrac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}$ and

$$\begin{aligned}
g_{1_k}(t, \alpha, x) &= \left\langle T(\sum_{i=1}^{p} x_i v_i + \frac{\omega(t)}{|t\alpha|}), \psi_k \right\rangle \\
g_{2_k}(t, \alpha, x) &= -\left\langle (1 + t|\alpha|^{a-1}) |t\alpha|^{-a} H(\lambda(\alpha), \sum_{i=1}^{p} |t\alpha| x_i v_i + \omega(t)), \psi_k \right\rangle \\
g_{3_k}(t, \alpha, x) &= -\left\langle (1 + t|\alpha|^{a-1}) |t\alpha|^{-a} K(\lambda(\alpha), \sum_{i=1}^{p} |t\alpha| x_i v_i + \omega(t)), \psi_k \right\rangle,
\end{aligned}$$

and for each $\alpha \in I$, we define the function $\mathcal{A}_\alpha : [0,1]\times\overline{U^*} \to \mathbb{R}^p, \mathcal{A}_\alpha = (\mathcal{A}_{\alpha_1}, \ldots, \mathcal{A}_{\alpha_p})$, by

$$\mathcal{A}_{\alpha_k}(t,x) = \begin{cases} \sum_{m=1}^{3} g_{m_k}(t,\alpha,x) \text{ if } t \neq 0 \text{ and } \alpha \neq 0 \\ (\mathcal{A}_1^k + \mathcal{A}_2^k)(x) \text{ if } t = 0 \text{ or } \alpha = 0, \end{cases}$$

where $k = 1, \ldots, p$, $\mathcal{A}_i^k(x), i = 1, 2$, are from (2.5).
By lemma 2 in [13], we have

$$\left\|\omega(|\alpha|^{a-1}, |\alpha|\, x)\right\| = o\,(|\alpha|) \tag{2.13}$$

as $\alpha \to 0$ uniformly in $x \in U_1$. Then

$$\lim_{t\alpha\to 0} \frac{\omega(|t\alpha|^{a-1}, |t\alpha|\, x)}{t\alpha} = 0 \tag{2.14}$$

uniformly in $x \in U_1$. By using the hypotheses (H2ii) and (H2iii) it follows that for $k = 1, \ldots, p$, we have

$$\lim_{t\alpha\to 0} \mathcal{A}_{\alpha_k}(t,x) = (\mathcal{A}_1^k + \mathcal{A}_2^k)(x). \tag{2.15}$$

By (2.15) and lemma 2.3, we conclude that $\mathcal{A}_\alpha$ is a continuous mapping from $[0,1]\times\overline{U^*}$ into $\mathbb{R}^p$. Now, we claim that there is a neighborhood I of zero, $I \subset I_2$, such that

$$\mathcal{A}_\alpha(t,x) \neq 0 \ \ \forall (t,\alpha,x) \in [0,1] \times I \times \partial U^*. \tag{2.16}$$

Indeed, by contradiction, we take the sequence $\{\overline{I_n}\}$ of neighborhoods of zero, $\overline{I_{n+1}} \subset \overline{I_n} \subset I_2, \cap \overline{I}_n = \{0\}$ and assume that for any $n = 3, 4, \ldots$ there are $(t_n, \alpha_n, x_n) \in [0,1] \times \overline{I}_n \times \partial U^*$ with $\mathcal{A}_{\alpha_n}(t_n, x_n) = 0$. By extracting subsequences if necessary, we may suppose $(t_n, \alpha_n, x_n) \to (t^*, 0, x^*), (t^*, 0, x^*) \in [0,1] \times \overline{I} \times \partial U^*$. It implies from the continuity of $\mathcal{A}_\alpha$ that $\mathcal{A}_{\alpha_n}(t_n, x_n)$ tends to $(\mathcal{A}_1 + \mathcal{A}_2)(x^*)$ and $(\mathcal{A}_1 + \mathcal{A}_2)(x^*) = 0$, which contradicts (H3). Thus we have the proof of (2.16). It then follows that for any fixed $\alpha \in I$ the mapping $\mathcal{A}_\alpha(1,.)$ is homotopic to $\mathcal{A}_\alpha(0,.) = \mathcal{A}_1 + \mathcal{A}_2$ on U^*. Therefore by the basic theorem on the topological degree of continuous mapping in a finite dimensional space and by the hypothesis (H3), we deduce

$$\deg(\mathcal{A}_\alpha(1,.), U^*, 0) = \deg(\mathcal{A}_1 + \mathcal{A}_2, U^*, 0) \neq 0.$$

Therefore, we conclude that for each $\alpha \in I, \alpha \neq 0$, there is a point $x(\alpha) = (x_1(\alpha), \ldots, x_p(\alpha)) \in U^*$ such that

$$\mathcal{A}_\alpha(1, x(\alpha)) = 0.$$

By the definition of $\mathcal{A}_\alpha(1,.)$ we obtain

$$\left\langle T(\frac{v(\alpha)}{|\alpha|}) - (1 + |\alpha|^{a-1})\, |\alpha|^{-a}\, M(\lambda(\alpha), v(\alpha)), \psi_k \right\rangle = 0, \tag{2.17}$$

for all $k = 1, \dots, p$, where

$$\begin{aligned} v(\alpha) &= \sum_{i=1}^{p} |\alpha|\, x_i(\alpha) v_i + \omega(|\alpha|^{a-1}, |\alpha|\, x(\alpha)) \\ &= \sum_{i=1}^{p} |\alpha|\, x_i(\alpha) v_i + o(|\alpha|) \text{ as } |\alpha| \to 0. \end{aligned}$$

Multiplying both sides of (2.17) with $|\alpha|^a$, we obtain

$$\left\langle |\alpha|^{a-1} T(v(\alpha)) - (1 + |\alpha|^{a-1}) M(\lambda(\alpha), v(\alpha)), \psi_k \right\rangle = 0, \tag{2.18}$$

for all $k = 1, \dots, p$.

Together with the fact that

$$\left\langle T(v(\alpha)) - L(\overline{\lambda}, v(\alpha)), \psi_k \right\rangle = \left\langle v(\alpha), \left(T - L(\overline{\lambda}, .)\right)^* \psi_k \right\rangle$$

and by (2.3), we have

$$\left\langle T(v(\alpha)) - L(\overline{\lambda}, v(\alpha)), \psi_k \right\rangle = 0, \tag{2.19}$$

for all $k = 1, \dots, p$.

Adding (2.18) and (2.19) we obtain

$$\left\langle (1 + |\alpha|^{a-1}) T(v(\alpha)) - L(\overline{\lambda}, v(\alpha)) - (1 + |\alpha|^{a-1}) M(\lambda(\alpha), v(\alpha)), \psi_k \right\rangle = 0 \tag{2.20}$$

for all $k = 1, \dots, p$.

Dividing it by $(1 + |\alpha|^{a-1})$ we get

$$\left\langle T(v(\alpha)) - L(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, v(\alpha)) - (M(\lambda(\alpha), v(\alpha)), \psi_k \right\rangle = 0 \tag{2.21}$$

for all $k = 1, \dots, p$, therefore

$$P_Y(T(v(\alpha)) - L(\frac{\overline{\lambda}}{(1 + |\alpha|^{a-1})^b}, v(\alpha)) - M(\lambda(\alpha), v(\alpha))) = 0. \tag{2.22}$$

This together with (2.12) finishes the proof of Theorem 2.1.

3. **APPLICATION**

In this section, we are interested in studying the bifurcation points and bifurcating solutions of the system of equations :

$$\begin{cases} -\Delta u = \eta v + u\, |uv| \text{ in } G \\ -\Delta v = \mu u + v\, |u|^\sigma \text{ in } G \\ u = v = 0 \text{ on } \partial G, \end{cases} \tag{3.1}$$

with $(\eta, \mu) \in \mathbb{R}^2, \sigma > 2, G = [0,1] \times [0,1] \times [0,1], (u, v) \in X = H_0^1(G) \times H_0^1(G)$.

We define the norm $\|.\|_X$ in X by

$$\|(u, v)\|_X = \left(\int_G |\nabla u|^2\, dG + \int_G |\nabla v|^2\, dG \right)^{1/2}, \quad (u, v) \in X$$

and the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \int_G \nabla u_1 \nabla u_2 dG + \int_G \nabla v_1 \nabla v_2 dG, \quad (u_1, v_1), (u_2, v_2) \in X.$$

Definition 3.1. We say that $(\eta, \mu, u, v) \in \mathbb{R}^2 \times H_0^1(G) \times H_0^1(G)$ is a solution of the system (3.1) if

$$\left\{ \begin{array}{l} u = \eta \Delta^{-1} v + \Delta^{-1}(u\,|uv|) \text{ in } G \\ v = \mu \Delta^{-1} u + \Delta^{-1}(v\,|u|^{\sigma}) \text{ in } G \\ u = v = 0 \text{ on } \partial G, \end{array} \right. \tag{3.2}$$

where Δ^{-1} is the inverse of $-\Delta$.

Remark 3.2. Evidently, for any $(\eta, \mu) \in \mathbb{R}^2$, $(\eta, \mu, 0, 0)$ is a solution of the above system.

Next, we define the mappings $L : \mathbb{R}^2 \times X \to X$, and $H : \mathbb{R}^2 \times X \to X$ by

$$\begin{array}{l} L(\lambda, u, v) = (L_1(\lambda, u, v), L_2(\lambda, u, v)) = (\eta \Delta^{-1} v, \mu \Delta^{-1} u) \\ H(\lambda, u, v) = (H_1(\lambda, u, v), H_2(\lambda, u, v)) = (\Delta^{-1}(u\,|uv|), \Delta^{-1}(v\,|u|^{\sigma})) \end{array}$$

for any $\lambda := (\eta, \mu) \in \Lambda := \mathbb{R}^2$ and $(u, v) \in X$. Then (3.1) can be written as

$$\left\{ \begin{array}{l} (u, v) = L(\eta, \mu, u, v) + H(\eta, \mu, u, v) \text{ in G}, \\ (u, v) = 0 \text{ on } \partial G. \end{array} \right. \tag{3.3}$$

The space $H_0^1(G)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_0$ given by

$$\langle u, v \rangle_0 = \int_G \nabla u \nabla v dG,$$

and the system of functions

$$w_{m,n,l}(x, y, z) = \frac{2\sqrt{2}}{\pi (m^2 + n^2 + l^2)^{1/2}} \sin m\pi x \sin n\pi y \sin l\pi z, m, n, l = 1, 2, \ldots$$

forms an orthonormal system in $H_0^1(G)$. Further if we denote by $S(L)$ the set of all characteristic values for the pair (id, L), we can see

$$S(L) = \{(\eta, \mu) \in \mathbb{R}^2, \eta\mu = \pi^2 \left(m^2 + n^2 + l^2\right) \text{ for some } m, n, l = 1, 2, \ldots\}.$$

In the sequel, we assume that $(\overline{\eta}, \overline{\mu}) \in S(L)$ be fixed. Let $(m_k, n_k, l_k, p) \in {\mathbb{N}^*}^4$ such that the set $\{(m_k, n_k, l_k), k = 1, \ldots, p\}$ be the set of natural numbers satisfying $\overline{\eta}\,\overline{\mu} = \pi^2 (m_k^2 + n_k^2 + l_k^2)$. We suppose that

$$m_1 = \max_{1 \le k \le p} \{m_k, n_k, l_k\}. \tag{3.4}$$

We can easily verify that the couples $(w_k, \overline{w}_k), k = 1, \ldots, p$, with

$$\begin{array}{rcl} w_k & = & w_{m_k, n_k, l_k} \\ \overline{w}_k & = & \dfrac{1}{\overline{\eta}} w_k \end{array}$$

are eigenfunctions associated with $(\overline{\eta},\overline{\mu})$ and $\dim\ker(id-L(\overline{\eta},\overline{\mu},.,.))=p$. Further, by a simple calculation, we obtain

$$(id-L(\overline{\eta},\overline{\mu},.))^{*}=id-L^{*}(\overline{\eta},\overline{\mu},.),$$

with

$$L^{*}(\overline{\eta},\overline{\mu},u,v)=\left(\overline{\mu}\Delta^{-1}u,\overline{\eta}\Delta^{-1}v\right)$$

and

$$\ker(id-L(\overline{\eta},\overline{\mu},.))^{*}=[\psi_1,\ldots,\psi_p]$$

where $\psi_k=\left(\psi_1^k,\psi_2^k\right)=\left(w_k,\dfrac{1}{\overline{\eta}}w_k\right)$. Let us denote

$$d=\frac{8\left(\overline{\eta}.\overline{\mu}\right)^3}{27\overline{\mu}},\overline{x}_1=\sqrt{2d}\quad\text{and}\quad\overline{x}=(\overline{x}_1,0,\ldots,0)\in\mathbb{R}^p.$$

We have:

Theorem 3.3. $(\overline{\eta},\overline{\mu},0)$ *is a bifurcation point of the system* (3.1). *More precisely, there exists an open neighborhood* U^* *of* $\overline{x}$ *not containing the origin in* $\mathbb{R}^p$, *to any given* $\delta>0$ *there exists a neighborhood* I *of zero in* $\mathbb{R}$ *such that for each* $\alpha\in I,\alpha\neq0$, *we can find* $x(\alpha)=(x_1(\alpha),\ldots,x_p(\alpha))\in U^*$ *and a nontrivial solution* $(\eta(\alpha),\mu(\alpha),u(\alpha),v(\alpha))$ *of the system (3.1) with*

$$\begin{aligned}(\eta(\alpha),\mu(\alpha)) &= (\frac{\overline{\eta}}{1+\alpha^2},\frac{\overline{\mu}}{1+\alpha^2}),\\ (u(\alpha),v(\alpha)) &= (\sum_{j=1}^{p}|\alpha|\,x_j(\alpha)w_j,\sum_{j=1}^{p}|\alpha|\,x_j(\alpha)\overline{w}_j)+o(|\alpha|),\end{aligned}$$

satisfying

$$|(\eta(\alpha),\mu(\alpha))-(\overline{\eta},\overline{\mu})|<\delta\ \textit{and}\ 0<\|(u(\alpha),v(\alpha))\|<\delta.$$

Proof. Here we can take $D=X=H_0^1(G)\times H_0^1(G),Y_1=Y_2=H_0^1(G),\Lambda=\mathbb{R}^2$, $T_1=T_2=id$ and $K_1=K_2=0$. We verify (H1)-(H3) and apply Theorem 2.1.
Hypothesis (H1) :
It suffices to take $b=1$.
Hypothesis (H2) :
We take $a_1=3$ and $a_2=\sigma+1$. Then it suffice to prove that H_1 and H_2 are Lipschitz. Let $(u_1,v_1),(u_2,v_2)$ be two elements of the unit ball of X and let $(\eta,\mu)\in\mathbb{R}^2$, then

$$\|H_1(\eta,\mu,u_1,v_1)-H_1(\eta,\mu,u_2,v_2)\|_{H_0^1(G)}=\left\|\Delta^{-1}\left(u_1\left|u_1v_1\right|-u_2\left|u_2v_2\right|\right)\right\|_{H_0^1(G)}.$$

Hence there exists a constant $k>0$, such that

$$\|H_1(\eta,\mu,u_1,v_1)-H_1(\eta,\mu,u_2,v_2)\|_{H_0^1(G)}\leq k\left\|u_1\left|u_1v_1\right|-u_2\left|u_2v_2\right|\right\|_{L^2(G)}.$$

It follows that

$$\begin{aligned}\left\|H_1(\eta,\mu,u_1,v_1)-H_1(\eta,\mu,u_2,v_2)\right\|_{H_0^1(G)} &\leq k\left\|u_1\left|u_1v_1\right|-u_1\left|u_2v_2\right|\right\|_{L^2(G)}\\ &\quad+k\left\|u_1\left|u_2v_2\right|-u_2\left|u_2v_2\right|\right\|_{L^2(G)}\\ &\leq k\left\|u_1v_1-u_2v_2\right\|_{L^2(G)}\\ &\quad+k\left\|u_1-u_2\right\|_{L^2(G)}\\ &\leq k\left\|v_1-v_2\right\|_{L^2(G)}\\ &\quad+2k\left\|u_1-u_2\right\|_{L^2(G)}.\end{aligned}$$

Then, by using Poincare's inequality, there exists a constant $k_1>0$, such that

$$\left\|H_1(\eta,\mu,u_1,v_1)-H_1(\eta,\mu,u_2,v_2)\right\|_{H_0^1(G)}\leq k_1\left\|(u_1,v_1)-(u_2,v_2)\right\|_X.$$

Hence H_1 is Lipschitz. By the same manner and by using that the map $x\mapsto|x|^\sigma$ is Lipschitz in $\mathbb{R}$, we deduce that H_2 is also Lipschitz and the hypothesis (H2) follows.
Hypothesis (H3) :
Following (2.5) and realizing $a_1<1+\sigma=a_2$, we define the mappings $\mathcal{A}_i:\mathbb{R}^p\to\mathbb{R}^p$, $\mathcal{A}_i=(\mathcal{A}_i^1,\ldots,\mathcal{A}_i^p)$, $i=1,2$, by

$$\begin{aligned}\mathcal{A}_1^k(x) &= \left\langle\sum_{j=1}^p x_jw_j-H_1(\overline{\eta},\overline{\mu},\sum_{j=1}^p x_jw_j,\sum_{j=1}^p x_j\overline{w}_j),\psi_1^k\right\rangle,\\ \mathcal{A}_2^k(x) &= \left\langle\sum_{j=1}^p x_j\overline{w}_j,\psi_2^k\right\rangle,\end{aligned}$$

with $x=(x_1,\ldots,x_p)$ and $k=1,\ldots,p$.
It follows that :

$$\begin{aligned}\left(\mathcal{A}_1^k+\mathcal{A}_2^k\right)(x) &= \left\langle\sum_{j=1}^p x_jw_j-H_1(\overline{\eta},\overline{\mu},\sum_{j=1}^p x_jw_j,\sum_{j=1}^p x_j\overline{w}_j),w_k\right\rangle\\ &\quad+\left\langle\sum_{j=1}^p x_j\overline{w}_j,\overline{w_k}\right\rangle.\end{aligned}$$

By the orthonormality of the system (w_k) it comes that :

$$\left(\mathcal{A}_1^k+\mathcal{A}_2^k\right)(x)=2x_k-\left\langle H_1(\overline{\eta},\overline{\mu},\sum_{j=1}^p x_jw_j,\sum_{j=1}^p x_j\overline{w}_j),w_k\right\rangle.$$

For $j=1,\ldots,p$, we have $\overline{w}_j=\dfrac{1}{\overline{\eta}}w_j$. Then

$$\begin{aligned}\left(\mathcal{A}_1^k+\mathcal{A}_2^k\right)(x) &= 2x_k-\frac{1}{\overline{\eta}}\left\langle\Delta^{-1}\left((\sum_{j=1}^p x_jw_j)^3\right),w_k\right\rangle\\ &= 2x_k-\frac{1}{\overline{\eta}}\int_G\nabla\left(\Delta^{-1}\left((\sum_{j=1}^p x_jw_j)^3\right)\right).\nabla w_k dG.\end{aligned}$$

By Green's formula it follows that

$$\left(\mathcal{A}_1^k + \mathcal{A}_2^k\right)(x) = 2x_k - \frac{1}{\overline{\eta}} \int_G \left(\sum_{j=1}^{p} x_j w_j \right)^3 \overline{w}_k dG. \tag{3.5}$$

It comes that :

$$\left(\mathcal{A}_k^k + \mathcal{A}_2^k\right)(\overline{x}) = 2\overline{x}_k - \frac{1}{\overline{\eta}}(\overline{x}_1)^3 \int_G w_1^3 w_k dG.$$

Using the formula $sin^4\theta = \frac{1}{8}cos4\theta - \frac{1}{2}cos2\theta + \frac{3}{8}$, we obtain

$$\int_G w_1^4 dG = \frac{27}{8\left(\pi^2\left(m_1^2+n_1^2+l_1^2\right)\right)^2} = \frac{27}{8\left(\overline{\eta}.\overline{\mu}\right)^2}, \tag{3.6}$$

then

$$\left(\mathcal{A}_1^1 + \mathcal{A}_2^1\right)(\overline{x}) = \overline{x}_1\left(2 - \frac{27\overline{\mu}}{8\left(\overline{\eta}.\overline{\mu}\right)^3}\overline{x}_1^2\right) = 0. \tag{3.7}$$

If $k \neq 1$ then $(m_k, n_k, l_k) \neq (m_1, n_1, l_1)$ and by using the formula $\sin^3\theta \sin\omega = \frac{1}{8}\left(\cos(3\theta+\omega) - \cos(3\theta-\omega) + 3\cos(\theta-\omega) - 3\cos(\theta+\omega)\right)$, we obtain

$$\int_G w_1^3 w_k dG = 0.$$

Therefore

$$(\mathcal{A}_1^k + \mathcal{A}_2^k)(\overline{x}) = 0. \tag{3.8}$$

In the other hand by (3.5), for $q = 1, \dots, p$, we have

$$\frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_q}(x) = \begin{cases} 2 - \frac{3}{\overline{\eta}} \int_G \left(\sum_{j=1}^{p} x_j w_j\right)^2 w_k^2 dG & \text{if } q = k, \\ -\frac{3}{\overline{\eta}} \int_G \left(\sum_{j=1}^{p} x_j w_j\right)^2 w_q w_k dG & \text{if not.} \end{cases}$$

It follows that

$$\frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_q}(\overline{x}) = \begin{cases} 2 - \frac{3\overline{x}_1^2}{\overline{\eta}} \int_G w_1^2 w_k^2 dG & \text{if } q = k, \\ -\frac{3\overline{x}_1^2}{\overline{\eta}} \int_G w_1^2 w_q w_k dG & \text{if not.} \end{cases}$$

By (3.6) we have

$$\frac{\partial(\mathcal{A}_1^1 + \mathcal{A}_2^1)}{\partial x_1}(\overline{x}) = -4. \tag{3.9}$$

If $q = k \neq 1$ then $(m_k, n_k, l_k) \neq (m_1, n_1, l_1)$, using the formula $\sin^2\theta \sin^2\omega = \frac{1}{8}\left(\cos 2(\theta+\omega) + \cos 2(\theta-\omega) - 2\cos 2\theta - 2\cos 2\omega + 2\right)$ and by a simple calculation we obtain :

$$\frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_q}(\overline{x}) = -2 \quad \text{or} \quad \frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_q}(\overline{x}) = \frac{2}{9}. \tag{3.10}$$

If $q \neq k$ then $(m_k, n_k, l_k) \neq (m_q, n_q, l_q)$, using the formula

$$\begin{aligned}\sin^2\theta \sin\omega \sin\phi &= \frac{1}{8}(\cos(2\theta+\omega+\phi) + 2\cos(\omega-\phi) + \cos(2\theta-\omega-\phi) \\ &\quad -2\cos(\omega+\phi) - \cos(2\theta+\omega-\phi) - \cos(2\theta-\omega+\phi))\end{aligned}$$

and the supposition (3.4), we can easily seen that

$$\int_G w_1^2 w_q w_k dG = 0,$$

then

$$\frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_q}(\overline{x}) = 0. \tag{3.11}$$

Using (3.10), (3.9) and (3.11) it follows that

$$\left| \det\left(\frac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_j}(\overline{x})\right)_{k,j=1,\dots,p} \right| \geq 4\left(\frac{2}{9}\right)^{p-1}. \tag{3.12}$$

Finally by (3.8), (3.7) and (3.12) it comes that :

$$\begin{cases} (\mathcal{A}_1^k + \mathcal{A}_2^k)(\overline{x}) = 0, \\ \det\left(\dfrac{\partial(\mathcal{A}_1^k + \mathcal{A}_2^k)}{\partial x_j}(\overline{x})\right)_{k,j=1,\dots,p} \neq 0, \end{cases}$$

which gives the hypothesis (H3) by the definition of the topological degree (cf [8]). Then by theorem 2.1 we deduce theorem 3.3. □

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REFERENCES

[1] S. Amraoui and M. Iguernane, Bifurcation of Solutions for an Elliptic Degenerate Problem. Acta Mathematica Vietnamica, Vol 28, 3:279-295, 2003.

[2] S. P. Banks. Three-dimensional stratification, knots and bifurcations of two-dimensional dynamical systems, International Journal of Bifurcation and Chaos, Vol. 12, 1:,1-21, 2002.

[3] H. Berestycki and M. Esteban. Existence and bifurcation of solutions for an elliptic degenerate problem, 3-25, 1997.

[4] H. Berestycki. J P. Dias. M. J. Esteban. and M. Figueira. Eigenvalue problems for some nonlinear Wheeler-De Witt operators. J. Math pure Appl. 72:493-515, 1993.

[5] M. Buchner, J. Marsden and S. Schecter, Application of the blowing-up construction and algebraic geometry to bifurcation problems, J. Differential Equations 48:404-433, 1983.

[6] N. Chafee, The bifurcation of one or more closed orbits from an equilibrium point of an autonomous differential equation, J. Differential Equations 4:661-679, 1968.

[7] M. Crandall and P. Rabinowitz, Bifurcation at simple eigenvalues, J. Funct. Anal. 8:321-340, 1971.

[8] K. Deimling. Nonlinear Functional Analysis. Springer-verlag, 1985.

[9] J P. Dias. and M. Figueira. The Cauchy problem for a nonlinear Wheeler-De Witt equation. Ann. Inst. H. Poincaré. Anal. Nonlinéaire 10:99-108, 1993.

[10] P. Drábek, A. Elkhalil and A. Touzani. A result on the bifurcation from the principal eigenvalue of A_p-Laplacian, Abstract and Applied Analysis, Vol 2, 3-4:185-195, 1997.

[11] P. Drábek and Y. X. Huang. Bifurcation problems for the p-Laplacian in $\mathbb{R}^N$, Trans. Amer. Math. Soc. 349:171-188, 1997.

[12] J. B. McLeod and D. H. Sattinger, Loss of stability and bifurcation at double eigenvalue, J. Funct. Anal. 14:62-84, 1973.

[13] N. X. Tan, Bifurcation problems for equations involving Lipschitz continuous mappings, J. of Math. Anal. and Appl. 154:22-42, 1991.