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ON HERMITE INTERPOLATING L_2-APPROXIMANTS

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ABSTRACT. We consider L_2-appoximation of a real-valued square integrable function by polynomials that satisfy certain Hermite interpolation conditions. The solution of the modified minimization problem is found by constructing an orthogonal basis of the underlying approximating subspace. A convergence problem related to the best approximants is considered in a restricted set-up. Some computational aspects based on discretization of the underlying measure are also discussed in detail.

Key words. Hermite interpolation, Least squares approximation, Interpolating orthogonal polynomials.

AMS (MOS) Subject Classification. 05E35, 41A29, 42C05, 65D05, 65F25, 93E24.

1. INTRODUCTION

In his elegant piece of work [12, 14], Walter Gautschi considered a least squares approximation problem subject to interpolatory constraints. Here, we discuss some theoretical and computational aspects related to the problem. For its description and further discussions, we shall use most of the notations and terminology opted by Gautschi in [12]. Suppose that $d\lambda$ is a positive measure on the real line $\mathbb{R}$ for which $\int_{\mathbb{R}} t^k d\lambda$ exists for all non-negative integers k and let $\mathbb{L}^2_{d\lambda}$ denote the class of real-valued functions f for which

$$\|f\|^2_{d\lambda} := \int_{\mathbb{R}} |f(t)|^2 \, d\lambda < \infty.$$

Suppose that $\mathbb{P}_n$ denotes the set of all polynomials of degree $\leq n$. For an $f \in \mathbb{L}^2_{d\lambda}$, set

$$\Phi_n = \{\phi \in \mathbb{P}_n : \phi(s_j) = f(s_j), \, j = 0, 1, \ldots, m\}, \tag{1.1}$$

where $m+1$ distinct points $s_j \in \mathrm{Dom}(f)$ with $m \leq n$. The problem under consideration [12; (2.1), (2.6)] is to find a polynomial $p^* \in \mathbb{P}_n$ that solves the constrained

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minimization problem:

$$\text{(P)} \qquad \min_{p\in\Phi_n} \|f-p\|_{d\lambda}\,.$$

In a way, a similar problem was considered by J.L. Walsh in a different set-up [18; Sec. 11.3 and 11.5]. An explicit solution p^* of (P) is as follows [12]:

$$p^*(t) = p_m(t,f) + s_m(t)\sum_{j=0}^{n-m} d_j\widehat{\pi}_j(t), \tag{1.2}$$

where

- $p_m(t,f)$ is the unique polynomial in $\mathbb{P}_m$ interpolating f at points s_i, $i=0,1,\ldots,m$.
- $s_m(t) := \prod_{i=0}^{m}(t-s_i)$
- $\hat{\pi}_j(t) := \pi_j(t; s_m^2(t)d\lambda(t))$ are the monic orthogonal polynomials of degree j relative to the measure $d\hat{\lambda}(t) = s_m^2(t)d\lambda(t)$ for $j=0,1,\ldots,n-m$.
- d_j , $j=0,1,\ldots,n-m$ are the Fourier coefficients of the function

  $$\Delta(t) = \frac{f(t)-p_m(t,f)}{s_m(t)}$$

  relative to the orthogonal system $\hat{\pi}_j, j=0,1,2,\ldots$.

The solution p^* (cf (1.2)) essentially requires the construction of orthogonal polynomials which Gautschi has carried out via the famous 3-term recurrence relation:

$$\hat{\pi}_{j+1}(x) = (x-\hat{\alpha}_j)\hat{\pi}_j(x) - \hat{\beta}_j\hat{\pi}_{j-1}(x). \tag{1.3}$$

While doing so, he assumes [12; Sec. 2] the availability of orthogonal polynomials relative to the measure $d\lambda$. Then the recursion coefficients $\hat{\alpha}_j$ and $\hat{\beta}_j, j=0,1,2,\ldots$ are generated [12; (7.12)] by repeated applications of an algorithm due to Galant [8] involving the theory of quasi-definite measures and kernel polynomials [4; Chapter 1]. Gautschi's procedure also restricts the location of the preassigned points $s_j, i=0,1,\ldots,m$ outside the support interval of the underlying measure $d\lambda$. More precisely, following his explanation in [12; Sec. 7.2], we note that certain numbers $q_k := \frac{\pi_{k+1}(t)}{\pi_k(t)}$ are required in the computation of $\hat{\alpha}_k$ and $\hat{\beta}_k$, where π_k's are the orthogonal monic polynomials relative to the measure $d\lambda$ [12; Sec. 7.1, 7.2]. Therefore, if any of the s_j's happens to be a zero of π_k, the proposed procedure loses its theoretical justification. In this paper, we present an alternative procedure for the construction of orthogonal polynomials that eliminates the restriction on the location of s_j's. Our procedure for finding the orthogonal polynomials will be different from that of Gautschi in the sense that we shall neither count on the availability of any kind of orthogonal polynomials nor modify the measure $d\lambda$. To achieve our objective, we shall explicate the notion of *interpolating orthogonal polynomials*. See Remark 1 and also [1].

The paper is organized as follows. We reformulate the problem (P) by introducing general Hermite interpolation conditions defined in the constraint set Φ_n (cf (1.1)) and then discuss some convergence aspects of the proposed problem in the next two sections. A procedure of computing the best approximants which depends on discretized orhogonal polynomials is discussed at length in Section 4. Here, we shall take advantage of Gautschi's work with some modifications. The last section deals with three examples that illustrate the proposed computational strategies.

2. MINIMIZATION PROBLEM

We begin by introducing some notations. For preassigned $k+1$ positive integers n_i, $0 \le i \le k$, we set

$$I_k := \{0, 1, 2, \dots, k\}\,; \qquad N_i := \{0, 1, \dots, n_i - 1\}\,, \quad i \in I_k. \tag{2.1}$$

As usual, the inner product of $f, g \in \mathbb{L}^2_{d\lambda}$ is given by

$$\langle f, g\rangle_{d\lambda} := \int_{\mathbb{R}} f(x)g(x)d\lambda. \tag{2.2}$$

A formal statement of our approximation problem which we shall refer as to (P1) is as follows:

Problem. *Let* $f \in \mathbb{L}^2_{d\lambda}$. *Let* k and n_i, $1 \le i \le k$, *be fixed positive integers. Set*

$$l := \sum_{i \in I_k} n_i. \tag{2.3}$$

Then for a given set $\{s_i : i = 0, 1, \dots, k\} \subset Dom(f)$ *and a finite sequence of real numbers*

$$\gamma = \{\gamma_{ij}\}_{i \in I_k, j \in N_i}\,, \tag{2.4}$$

find the polynomial ϕ_n^* , $n \ge l-1$, *in*

$$\Phi_{n,\gamma} = \left\{\phi \in \mathbb{P}_n : \phi^{(j)}(s_i) = \gamma_{ij},\, \forall i \in I_k,\ \forall j \in N_i\right\} \tag{2.5}$$

that solves the minimization problem[1]*:*

$$\min_{\phi\ \in \Phi_{n,\gamma}} \|f - \phi\|_{d\lambda}\,. \tag{P1}$$

Solution of (P1). Let $H_{l-1}(x, \gamma)$ denote the unique polynomial of degree $\le l-1$ satisfying the l (cf. (2.3)) interpolatory constraints

$$H_{l-1}^{(j)}(s_i, \gamma) = \gamma_{ij}, \forall i \in I_k,\ \forall j \in N_i$$

and let

$$f_{H,\gamma}(x) := f(x) - H_{l-1}(x, \gamma). \tag{2.6}$$

[1]Replacing each n_i by 1 and β_{i0} by $f(s_i)$ in (P1), we observe that (P1) reduces to the minimization problem (P) discussed in [12].

Note that each $\phi \in \Phi_{n,\gamma}$ can be expressed as $\phi(x) = H_{l-1}(x,\gamma) + p(x)$ for some $p \in \mathbb{P}_n$, which satisfies the conditions $p^{(j)}(s_i) = 0, i \in I_k, j \in N_i$. Therefore, we can write $\Phi_{n,\gamma} = H_{l-1}(x,\gamma) + \mathbb{P}_n^*$, where

$$\mathbb{P}_n^* = \left\{ p \in \mathbb{P}_n : p^{(j)}(s_i) = 0, i \in I_k, j \in N_i \right\}. \tag{2.7}$$

This observation leads us to reformulate the minimization problem (P1) as follows:

Find a polynomial p_n^* in $\mathbb{P}_n^*$ that solves the problem

$$\min_{p \in \mathbb{P}_n^*} \|f_{H,\gamma} - p\|_{d\lambda}. \tag{RP}$$

It may be noted that $\mathbb{P}_n^*$ is an $(n-l+1)$-dimensional subspace of $\mathbb{P}_n$ for which the polynomials $x^k W(x), k = 0, 1, \ldots, n-l$ form a basis where

$$W(x) := \prod_{i \in I_k} (x - s_i)^{n_i}. \tag{2.8}$$

Once we construct an orthogonal basis of $\mathbb{P}_n^*$, say $\left\{\pi_j^*\right\}_{j=0}^{n-l}$, the polynomial

$$\wp_n(f_{H,\gamma}) = \sum_{i=0}^{n-l} c_i \pi_i^* \tag{2.9}$$

with

$$c_i := \frac{\langle f_{H,\gamma}, \pi_i^* \rangle_{d\lambda}}{\langle \pi_i^*, \pi_i^* \rangle_{d\lambda}} \tag{2.10}$$

solves the minimization problem (RP). Thus, the optimal solution ϕ_n^* of the problem (P1) is given by

$$\phi_n^* = \begin{cases} H_{l-1}(.,\gamma), & n = l-1, \\ \wp_n(f_{H,\gamma}) + H_{l-1}(.,\gamma), & n \geq l. \end{cases} \tag{2.11}$$

Remark 1. Each polynomial π_j^* considered in (2.9) satisfies the interpolating properties: $\left(\pi_j^*\right)^{(m)}(s_i) = 0$, $m \in N_i$ and $i \in I_k$. Because of this observation, we may call each π_j^*, $j = 0, 1, 2, \ldots$, an *interpolating orthogonal polynomial* relative to the data

$$\{(s_i, n_i) : n_i \text{ is multiplicity of the node } s_i,\ i \in I_k\}$$

and relative to the measure $d\lambda$.

3. CONVERGENCE

This section briefly describes the convergence of the sequence $\{\wp_n(f_{H,\gamma})\}_{n=l}^{\infty}$ (cf (2.9)) subject to certain conditions on $d\lambda$ and γ which are given below in Theorem 3.1. Let $\mathbb{C}^m(I)$ denote the class of real-valued functions that are m-times continuously differentiable on an interval I. Then taking into account all the notations explained in the earlier sections, we have the following convergence result:

Theorem 3.1. *Suppose that the measure $d\lambda$ has a bounded support, say $I_{\text{supp}}(d\lambda)$, and that I is a closed interval that contains $I_{\text{supp}}(d\lambda)$ and the preassigned nodes $s_i, i = 0, 1, \ldots, k$. For a given $f \in \mathbb{C}^{n^*}(I)$ where $n^* = (\max_{i\in I_k} n_i) - 1$, set*

$$\gamma = \left\{ f^{(j)}(s_i) \right\}_{i\in I_k, j\in N_i}.$$

Then

$$\lim_{n\to\infty} \|\wp_n(f_{H,\gamma}) - f_{H,\gamma}\|_{d\lambda} = 0.$$

Proof of this theorem is a routine exercise based on a convergence result [2, Theorem 6.1] and is omitted here.

Remark 2. It may be interesting to establish a similar covergence result when $I_{\text{supp}}(d\lambda)$ happens to be unbounded.

4. COMPUTATIONAL PROCEDURE

As noted above in (2.9), the optimal solution of the problem (P1) depends on the interpolating orthogonal polynomials $\pi_j^*(t; d\lambda(t))$. We can generate these polynomials by applying the techniques similar to those used in the construction of classical monic orthogonal polynomials $\pi_j(t; d\lambda(t))$, but with appropriate modifications. Like $\pi_j(t; d\lambda(t))$, the polynomials $\pi_j^*(t; d\lambda(t))$ also satisfy the 3-term recurrence relation of the form

$$\pi_{j+1}^*(x) = (x - \alpha_j^*)\pi_j^*(x) - \beta_j^*\pi_{j-1}^*(x), \tag{4.1}$$

where the recursion coefficients α_j^* and β_j^* are real constants with $\beta_j^* > 0$. However, a difference between the two constructions arises with the choice of first orthogonal polynomial. We initiate with $\pi_0^*(x) := W(x)$ (cf (2.8)) as the first interpolating orthogonal polynomial in relation (4.1) whereas the corresponding polynomial in any classical case is the 0-degree polynomial $\pi_0(.; d\lambda) \equiv 1$. Because of this difference, we also find that the polynomial π_{n-1}^* is orthogonal to the subclass $\mathbb{P}_n^*$ (cf (2.7)) rather than to the entire class $\mathbb{P}_n$.

In order to generate the coefficients α_j^* and β_j^* required in (4.1), we may consider two classical methods which have their roots in the work of Chebyshev [3] and Stieltjes [17]. We shall consider in detail the Stieltjes approach[2] that involves the inner product induced by the underlying measure $d\lambda$.

[2]The second method depends on Chebyshev algorithm. This algorithm in its general form was developed by Sack and Devon [16] by using the modified moments and therefore, is referred to as the modified Chebyshev algorithm in the literature [11].

4.1. **Stieltjes Approach.** Here, the computation of α_j^* and β_j^* is based on the knowledge of π_{j-1}^* and π_j^* and in return π_{j+1}^* is constructed from relation (4.1). More precisely, we proceed as follows to compute the optimal solution ϕ_n^*:

Step 1. (*Initialization*)

Set

$$\begin{aligned}\pi_0^*(x) &= W(x),\\ \alpha_0^* &= \langle x\pi_0^*, \pi_0^*\rangle_{d\lambda},\\ \pi_1^*(x) &= (x-\alpha_0^*)\pi_0^*(x).\end{aligned}$$

Step 2. (*Computation of recursion coefficients*)

By alternating between (4.2) and (4.1), evaluate for each $j = 1, 2, \ldots, n-l$,

$$\left.\begin{aligned}\alpha_j^* &= \langle x\pi_j^*, \pi_j^*\rangle_{d\lambda},\\ \beta_j^* &= \tfrac{\langle \pi_j^*, \pi_j^*\rangle_{d\lambda}}{\langle \pi_{j-1}^*, \pi_{j-1}^*\rangle_{d\lambda}}.\end{aligned}\right\} \tag{4.2}$$

In this way, the procedure bootstraps up to any desired order of the recursion coefficients as it does in case of classical orthogonal polynomials.

Step 3. (*Computation of* $H_{l-1}(x,\gamma)$)

Use the Newton's interpolation formula to compute the polynomial $H_{l-1}(x,\gamma)$. For this, evaluate the divided differences based on the given set of finite data [15; Sec. 6.3]

Step 4. (*Fourier coefficients*)

Compute the inner products $\langle f_{H,\gamma}, \pi_i^*\rangle_{d\lambda}$ and $\langle \pi_i^*, \pi_i^*\rangle_{d\lambda}$ to obtain the Fourier coefficients

$$c_i := \frac{\langle f_{H,\gamma}, \pi_i^*\rangle_{d\lambda}}{\langle \pi_i^*, \pi_i^*\rangle_{d\lambda}}, \qquad j = 0, 1, 2, \ldots, n-l. \tag{4.3}$$

Step 5. (*Optimal solution of* (*P1*))

Set (cf (2.11))

$$\phi_n^* = H_{l-1}(x,\gamma) + \sum_{i=0}^{n-l} \frac{\langle f_{H,\gamma}, \pi_i^*\rangle_{d\lambda}}{\langle \pi_i^*, \pi_i^*\rangle_{d\lambda}} \pi_i^*.$$

4.2. **Conte-de Boor Procedure.** Both the inner products $\langle \pi_i^*, \pi_i^*\rangle_{d\lambda}$ and the Fourier coefficients c_i's (cf (4.3)) decrease rapidly with an increase in i. As a result, the cancellation error occurs with the computation of inner products $\langle f_{H,\gamma}, \pi_i^*\rangle_{d\lambda}$. In order to assuage the error impact, we follow an alternative method due to Conte and de Boor for the computation of c_i's [5]. Their strategy modifies Steps (4) and (5) of the Stieltjes procedure for the computation of ϕ_n^* and is implemented as follows [13; Ch. 2, Sec. 1.2]:

Step 4´. *(Fourier coefficients)*

Set

$$\begin{aligned}
&s_{-1} = 0,\\
&\text{For } i = 0, 1, 2, \ldots, n-l,\\
&\quad c_i = \frac{1}{\langle \pi_i^*, \pi_i^* \rangle_{d\lambda}} \left\langle f_{H,\gamma} - s_{i-1}, \pi_j^* \right\rangle_{d\lambda},\\
&\quad s_i = s_i + c_i \pi_i^*.
\end{aligned}$$

Step 5´. *(Optimal Solution)*

Set

$$\phi_n^* = H_{l-1}(x, \gamma) + s_{n-l}.$$

4.3. **Discretization.** It may be noted that Stieltjes-like procedures suffer from the effects of ill-conditioning usually caused by a rapid propagation of round-off errors in the computation of inner products. This effect may be eliminated by introducing a discrete N-point measure, say $d\lambda_N$[3]. Here, the inner products are converted into finite sums without involving the moments. This type of procedure, in general, has been found to be quite stable [7].

For definiteness, we assume that $d\lambda = \omega(t)dt$ with $\mathrm{supp}(d\lambda) = (-1, 1)$[4] where ω is a non-negative weight function. The discretization is obtained by approximating the integrals with a suitable quadrature rule:

$$\int_{-1}^{1} h(t) d\lambda \approx \sum_{i=1}^{N} w_j^N h(t_i^N) \omega(t_i^N) =: \int_{-1}^{1} h(t) d\lambda_N \tag{4.4}$$

with nodes $t_i^N \in (-1, 1)$ and weights $w_i^N > 0$. This rule enables us to compute recursive coefficients α_j^* and β_j^* relative to the discrete measure $d\lambda_N$, and ultimately the corresponding discrete orthogonal polynomials $\pi_{j,N}^*$.

Remark 3. As $N \to \infty$, it can be shown that for any fixed j,

$$\left.\begin{array}{r}
\alpha_j^N := \alpha_j^*(d\lambda_N) \to \hat{\alpha}_j(d\lambda)\\
\beta_j^N := \beta_j^*(d\lambda_N) \to \beta_j^*(d\lambda)\\
\pi_{j,N}^* := \pi_j^*(d\lambda_N) \to \pi_j^*(d\lambda)
\end{array}\right\}$$

provided that the property

$$\lim_{N\to\infty} \int_{-1}^{1} h(t) d\lambda_N = \int_{-1}^{1} h(t) d\lambda \tag{4.5}$$

holds whenever $h \in \mathbb{P}_n^*$ in (4.2).

[3]It may be noted that the ill-conditioning effects in the modified Chebyshev algorithm remains evident in its discretized version [10]. The details on descretization of this algorithm may be found in [11, 19]

[4]This restriction does not affect the generality; see Section 4.5.

4.4. **Choice of Nodes and Weights.** In order to implement the discretized procedure (4.4), a suitable choice of nodes and weights (t_i^N, w_i^N) is required that may preserve the convergence condition (4.5). Among these, the set of uniformly distributed nodes with identical weights given by

$$t_i^N := -1 + \frac{2i}{N+1}, \qquad w_i^N := \frac{2}{N+1}; \qquad i = 1, 2, \ldots, N \tag{4.6}$$

is a reasonable choice which meets the convergence requirement. In fact, the integral on the right side of (4.5) with the suggested nodes and weights turns out to be an N-th Riemann sum of the polynomial p over the interval $[-1, 1]$. Nevertheless, formula (4.2) together with (4.4) suffers from convergence problems if the weight function $\omega(x)$ associated with $d\lambda$ is singular. On the other hand, the N-point Fejér rule [6] that involves the following nodes and weights

$$t_i^N := \cos\theta_i^N, \qquad w_i^N := \frac{2}{N}\left\{1 - 2\sum_{j=1}^{[N/2]} \frac{\cos(2j\theta_i^N)}{4j^2-1}\right\}; \qquad i = 1, 2, \ldots, N \tag{4.7}$$

where

$$\theta_i^N = \frac{(2i-1)}{2N} \tag{4.8}$$

is convergent even in the presence of monotone and integrable singularities occurring at the endpoints of the interval [9]. This rule, however, provides slow convergence if $\omega(x)$ has square root singularities at the end points. A more suitable choice in this situation comes from the Gauss-Chebyshev quadrature formula where the nodes and weighs are selected as follows:

$$t_i^N := \cos\theta_i^N, \qquad w_i^N := \frac{\pi}{N}\sin\theta_i^N; \qquad i = 1, 2, \ldots, N \tag{4.9}$$

where θ_i^N is given in (4.8). This choice, in addition, provides an exact equality:

$$\int_{-1}^{1} p(t)d\lambda_N = \int_{-1}^{1} p(t)d\lambda$$

for $p \in \mathbb{P}^*_{2n-1}$ and $d\lambda = (1-t)^{-1/2}dt$.

4.5. **Beyond** $[-1, 1]$**.** The procedures and techniques so far discussed are restricted to the measures with the support interval $[-1, 1]$. A similar approach is equally applicable to $d\lambda = \omega(x)dx$ where $\omega(x)$ is a positive weight function and the support interval of $d\lambda$ is of the form $[a, b]$ with $-\infty \le a < b \le +\infty$. In this situation, a suitable monotone function $\mu : [-1, 1] \to [a, b]$ modifies the discretization process (4.4) [10] to

$$\int_a^b h(x)d\lambda(x) = \int_{-1}^{1} h(\mu(t))\omega(\mu(t))\mu'(t)d(t) \approx \sum_{i=1}^{N} w_j^N h(\mu(t_i^N))\omega(\mu(t_i^N))\mu'(t_i^N).$$

5. COMPUTED EXAMPLES

We have implemented the computational procedures of Section 4 to different type of functions and determined their approximating polynomials subject to various sets of Hermite data. All the numerical experiments were subject to the weight function $w(x) \equiv 1$ on the interval $[-1,1]$. The discrete interpolating orthogonal polynomials required in the evaluation of ϕ_n^* were constructed separately for different choices of nodes and weights. The resulting mean-squared error $\|f-\phi_n^*\|_{d\lambda_N}$ between the function f and its approximating polynomials ϕ_n^* is tabulated for each choice of nodes and weights (t_i^N, w_i^N), (cf (4.6), (4.7), (4.9)), corresponding to various values of n, the number of interpolating orthogonal polynomials used in the construction of ϕ_n^*. The nodes and weights opted for the application of discretized Stieltjes procedure are referred to as follows in each table:

UNW:= Uniformly distributed nodes and weights (cf (4.6)),
FNW:= Fejer nodes and weights (cf (4.7)),
CNW:= Gauss-Chebyshev nodes and weights (cf (4.9)).

All computations are carried out in the MATLAB programming environment on an hp pavilion ze4200.

5.1. **Example A.** We seek an approximation of the function

$$f(x) = e^{-10x^2}$$

subject to the data $(s_1, s_2, s_{3,} s_{4,} s_5) = (-1, \frac{-1}{2\sqrt{5}}, 0, \frac{1}{2\sqrt{5}}, 1)$, $(n_1, n_2, n_{3,} n_{4,} n_5) = (2,1,1,1,2)$ and

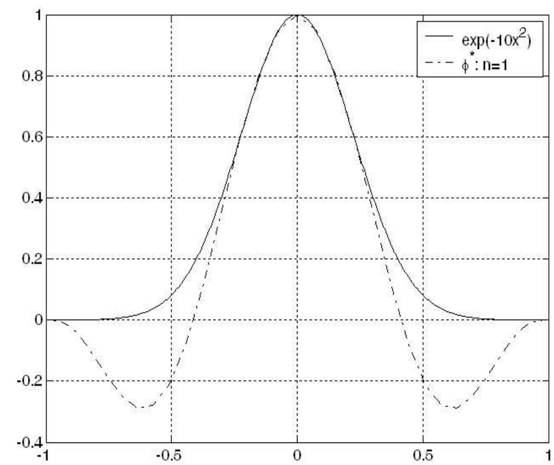
$$\gamma_{i,j} = \begin{cases} f^{(j)}(s_i), & i = 1,5; \quad j = 0,1 \\ f(s_i), & i = 2,3,4 \quad j = 0 \end{cases}$$

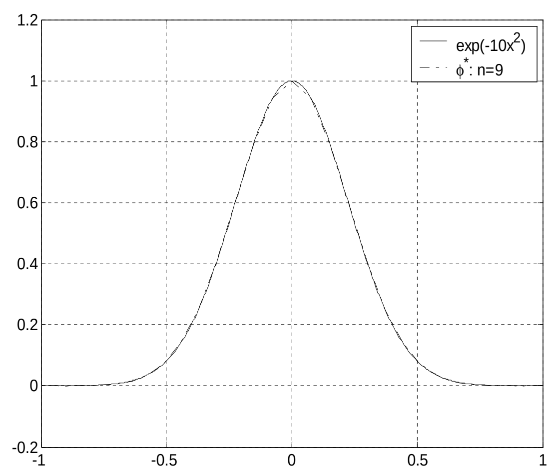
(cf (2.4), (2.5)). This function assumes its inflection points at $(\pm\frac{1}{2\sqrt{5}}, e^{-10(\frac{1}{2\sqrt{5}})^2})$ and the absolute maxima at $x = 0$. The mean-squared error of approximation is given in Table I.

$\|f-\phi_n^*\|_{d\lambda_N}$ when $N = 41$			
n	*UNW*	*WFN*	*CNW*
1	0.24057112147300	0.24057112783865	0.24057112784897
5	0.01114722202381	0.01114741730327	0.01114741750058
9	5.588242398119300e-004	5.521624739343474e-004	5.521652580405463e-004
19	1.549125709001037e-007	1.536038418078543e-007	1.535644939105453e-007
36	3.410220438849282e-013	3.639601972675804e-015	6.390586598069277e-016

Table I

We note that the choice of UNW provides convergence at a slower rate as compared to those of FNW and CNW. The following graphs refer to the choice of CNW with $N = 41$.





5.2. **Example B.** Here, we consider the function $f(x) = x^2 \sin \frac{1}{x}$ and the data $s_7 = -s_1 = 1$, $s_6 = -s_2 = 0.23446893787575$, $s_4 = 0$, $s_5 = -s_3 = 0.130260521042080$ and $n_i = 1, i = 1, 2, \ldots, 7$ with

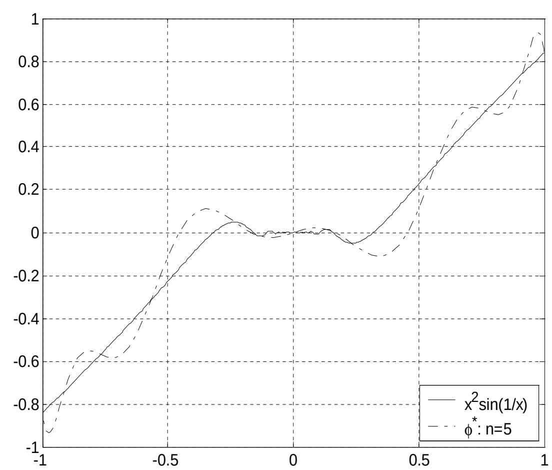
$$\gamma_{i,0} = f(s_i), i = 1, 2, \ldots, 7.$$

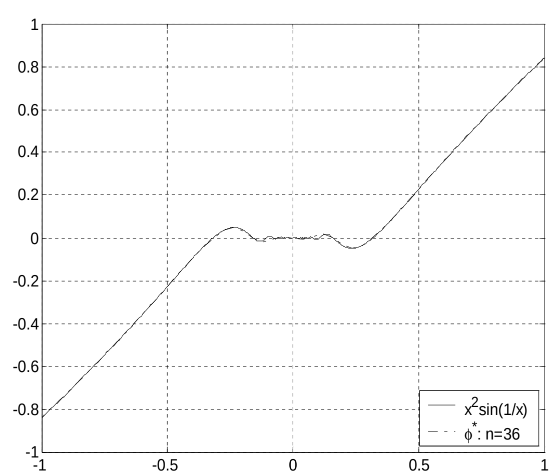
This function has infinitely many oscillations around $x = 0$. The resulting mean-squared error is given in Table II.

$\|f-\phi_n^*\|_{d\lambda_N}$ when $N=79$			
n	*UNW*	*FNW*	*CNW*
5	0.02322971712267	0.12528197576481	0.12528197636183
19	0.00829653795711	0.01288729119908	0.01288715813612
36	0.00409126602766	0.00328967357544	0.00328971097343
48	0.00240175814755	0.00251437424382	0.00251429288503
72	0.00399551734086	0.00100237845516	0.00100241148350
75	0.00399721372720	1.604498607362561e-004	1.604736714876855e-004
77	0.00519592471328	2.619705474787597e-014	3.886504986029689e-014

Table II

With the choice of *UNW*, the solution sequence $\{\phi_n^*\}$ behaves poorly in terms of convergence whereas a high number of interpolating orthogonal polynomials are required to meet a convergence criterion in case of *FNW* and *CNW*. The following graphs refer to *CNW* with $N=79$.





5.3. **Example C.** This example deals with the approximation of the step function

$$f(x) = \begin{cases} 0, & \text{if } -1 \le x < 0, \\ 1 & \text{if } 0 \le x \le 1, \end{cases}$$

subject to the data: $s_5 = -s_1 = 1$, $s_4 = -s_2 = 0.5$, $s_3 = 0$ and $n_i = 1$, $i = 1, 2, \ldots, 5$ with

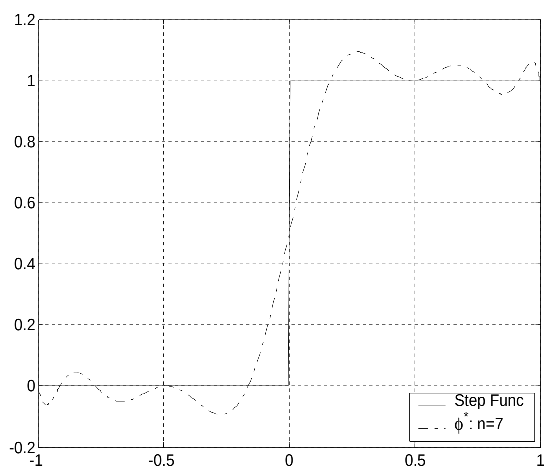
$$\gamma_{i,0} = \begin{cases} f(s_i), & i = 1, 2, 4, 5 \\ 0.5, & i = 3. \end{cases}$$

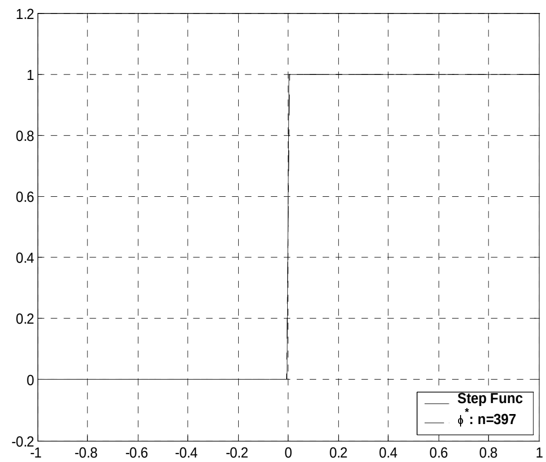
The resulting mean-squared error is given in Table III.

$\|f - \phi_n^*\|_{d\lambda_N}$ when $N = 401$			
n	*UNW*	*FNW*	*CNW*
7	0.16460460538042	0.16474350553809	0.15868781981448
27	0.09898355043318	0.09962814650159	0.08925900468607
47	0.07767983822823	0.07902646054334	0.06547201372501
147	0.04400858644398	0.05077292693036	0.02488568084039
247	0.03228630704425	0.04531343951629	0.00973223282446
347	0.02929886106656	0.04428865257400	0.00170105604207
397	0.03233530575547	0.04425597312364	8.74317684152439e-014

Table III

It may be noted that $f(x)$ in this example is discontinuous unlike the other two examples. However, the error $\|f - \phi_n^*\|_{d\lambda_N}$ tends to zero as $n \to \infty$ with the choice of *CNW* though at a slower rate. This outcome is based on an N-point discretization of the measure for a very large value of N. The following graphs refer to the choice *CNW* with $N = 401$.





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