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UNIQUENESS IMPLIES UNIQUENESS FOR NONLOCAL BOUNDARY VALUE PROBLEMS FOR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. It is assumed that solutions of the differential equation $y''' = f(x, y, y', y'')$, with certain boundary conditions comprised of function values at $m + n$ points, are unique, when they exist. It is shown that, for any integers p and q such that $1 \leq p \leq m$, $1 < q \leq n$, solutions for similar boundary value problems are unique, when they exist.

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1. INTRODUCTION

This paper is concerned with solutions of nonlocal boundary value problems for the third order ordinary differential equation,

$$y''' = f(x, y, y', y''). \tag{1.1}$$

In particular, we will discuss the uniqueness of solutions of certain boundary value problems for (1.1) implying the uniqueness of solutions of similar boundary value problems.

The "uniqueness implies uniqueness" result of this paper plays an important role in demonstrating the existence of the solutions in question. For other examples of work on the topic, please see papers by Gray [3], Henderson and Jackson [4], Jackson and Schrader [5], and Lasota and Luczyński [6, 7]. For other discussions of third order, nonlocal boundary value problems for ordinary differential equations, please see papers by Benbouziane, Boucherif, and Bouguima [1], Du, Lin, and Ge [2], and Liu, Zhong, and Jiang [8].

2. RESULT

Theorem 2.1 (Uniqueness Implies Uniqueness). *Suppose the following three conditions hold for the differential equation* (1.1).

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) *Solutions of initial value problems for* (1.1) *are unique and exist on all of* (a, b)*;*

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(C) *For some $m, n \in \mathbb{N}$ with $n > 1$, any $a < x_1 < x_2 < \cdots < x_{m+n} < b$, and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem for* (1.1) *with boundary conditions*

$$y(x_1) - \sum_{i=2}^{n-1} y(x_i) = y_1, \tag{2.1}$$

$$y(x_n) = y_2, \tag{2.2}$$

$$y(x_{m+n}) - \sum_{j=n+1}^{m+n-1} y(x_j) = y_3, \tag{2.3}$$

are unique, when they exist. We take the boundary condition (2.1) *to mean $y(x_1) = y_1$ in the case that $n = 2$, and the boundary condition* (2.3) *is taken to be $y(x_{n+1}) = y_3$ in the case that $m = 1$.*

Then, for any integers p and q such that $1 \leq p \leq m$, $1 < q \leq n$, solutions of the boundary value problem for (1.1) *with boundary conditions*

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{2.4}$$

$$y(x_q) = y_2, \tag{2.5}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3, \tag{2.6}$$

where the boundary condition (2.4) *is taken to mean $y(x_1) = y_1$ in the case that $q = 2$, and the boundary condition* (2.6) *is taken to be $y(x_{q+1}) = y_3$ in the case that $p = 1$, are unique, when they exist.*

Proof. There are three nontrivial cases: (i) $p = m$ and $1 < q < n$, (ii) $1 \leq p < m$ and $q = n$, and (iii) $1 \leq p < m$ and $1 < q < n$. We prove only the second and third cases in this paper, and do so using induction.

Proof of Case (ii): $1 \leq p < m$ and $q = n$. We begin by proving the case $m = 2$, $p = 1$, and $q = n \geq 2$ is some positive integer. That is, we are assuming Conditions (A) and (B), and that, for any $a < x_1 < x_2 < \cdots < x_{n+2} < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem for (1.1) satisfying boundary conditions

$$y(x_1) - \sum_{i=2}^{n-1} y(x_i) = y_1, \tag{2.7}$$

$$y(x_n) = y_2, \tag{2.8}$$

$$y(x_{n+2}) - y(x_{n+1}) = y_3, \tag{2.9}$$

are unique, when they exist.

Suppose that, for some $a < x_1 < x_2 < \cdots < x_{n+1} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, we have distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (1.1) satisfying boundary conditions

$$\begin{aligned} y(x_1) - \sum_{i=2}^{n-1} y(x_i) &= y_1, \\ y(x_n) &= y_2, \\ y(x_{n+1}) &= y_3. \end{aligned}$$

Set $w(x) = u(x) - v(x)$. We have

$$\begin{aligned} u(x_1) - \sum_{i=2}^{n-1} u(x_i) &= v(x_1) - \sum_{i=2}^{n-1} v(x_i), \\ u(x_n) &= v(x_n), \\ u(x_{n+1}) &= v(x_{n+1}), \end{aligned}$$

so that

$$w(x_1) - \sum_{i=2}^{n-1} w(x_i) = w(x_n) = w(x_{n+1}) = 0.$$

Well, $w(x)$ cannot have a zero on (x_n, x_{n+1}), else we would get an immediate contradiction to our assumption that $u(x)$ and $v(x)$ are distinct, by our uniqueness condition. Therefore, since $w(x_n) = 0$, $w(x_{n+1}) = 0$, and $w(x) = u(x) - v(x)$ is a continuous function, it must be the case that $w(x)$ has a local extremum on (x_n, x_{n+1}). Then we may choose $\alpha, \beta \in (x_n, x_{n+1})$ such that $w(\alpha) = w(\beta)$. We have

$$\begin{aligned} w(\alpha) &= w(\beta), \\ u(\alpha) - v(\alpha) &= u(\beta) - v(\beta), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta). \end{aligned}$$

Therefore, $u(x)$ and $v(x)$ are solutions of (1.1) satisfying

$$\begin{aligned} u(x_1) - \sum_{i=2}^{n-1} u(x_i) &= v(x_1) - \sum_{i=2}^{n-1} v(x_i), \\ u(x_n) &= v(x_n), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta). \end{aligned}$$

Then our uniqueness condition implies $u(x) \equiv v(x)$ on (a, b), which contradicts our assumption that $u(x)$ and $v(x)$ are distinct. Case (ii) holds if $m = 2$, $p = 1$, and $q = n \geq 2$ is some positive integer.

It remains to prove the case $p = m - 1$ and $q = n$. Suppose that Conditions (A), (B), and (C) hold, and let $p = m - 1$ and $q = n$. Additionally, for some $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, assume that $u(x)$ and $v(x)$ are distinct solutions of the boundary value problem for (1.1) satisfying

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{2.10}$$

$$y(x_q) = y_2, \tag{2.11}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{2.12}$$

Set $w(x) = u(x) - v(x)$. We have

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= v(x_1) - \sum_{i=2}^{q-1} v(x_i), \\ u(x_q) &= v(x_q), \\ u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) &= v(x_{p+q}) - \sum_{j=q+1}^{p+q-1} v(x_j), \end{aligned}$$

so that

$$w(x_1) - \sum_{i=2}^{q-1} w(x_i) = w(x_q) = w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = 0.$$

Claim: It cannot be the case that $w(x_j) = 0$ for each $j \in \{q+1, \ldots, p+q-1\}$. To establish the claim, assume to the contrary that $w(x_j) = 0$ for every j among $\{q+1, \ldots, p+q-1\}$. Then $w(x_{p+q}) = 0$, since $w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = 0$. Now, since $w(x_{p+q}) = w(x_{p+q-1}) = 0$ and $w(x) = u(x) - v(x)$ is continuous, there exist $\alpha, \beta \in (x_{p+q-1}, x_{p+q})$ such that $\alpha < \beta$ and $w(\alpha) = w(\beta)$, or equivalently, $w(\beta) - w(\alpha) = w(x_{p+q}) = 0$. We have

$$\begin{aligned} 0 &= w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) \\ &= w(\beta) - w(\alpha) - \sum_{j=q+1}^{p+q-1} w(x_j) \\ &= w(\beta) - \left(\sum_{j=q+1}^{p+q-1} w(x_j) + w(\alpha) \right), \end{aligned}$$

so that, for $a < x_1 < x_2 < \cdots < x_{p+q-1} < \alpha < \beta < b$, $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= v(x_1) - \sum_{i=2}^{q-1} v(x_i), \\ u(x_q) &= v(x_q), \\ u(\beta) - \left(\sum_{j=q+1}^{p+q-1} u(x_j) + u(\alpha) \right) &= v(\beta) - \left(\sum_{j=q+1}^{p+q-1} v(x_j) + v(\alpha) \right). \end{aligned}$$

Condition (C) implies that $u(x) \equiv v(x)$ on (a, b), but this contradicts our assumption that $u(x)$ and $v(x)$ are distinct. The claim is true.

By our claim, we have that $w(x_j) \neq 0$ for some $i \in \{q+1, \ldots, p+q-1\}$. Without loss of generality, assume $w(x_{q+1}) \neq 0$. Then since $w(x_q) = 0$, and $w(x) = u(x) - v(x)$ is continuous, we may choose $\alpha, \beta \in (x_q, x_{q+1})$ with $\alpha < \beta$ such that $w(x_{q+1}) = w(\alpha) + w(\beta)$. Then, we have

$$w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = w(x_{p+q}) - \left(w(\alpha) + w(\beta) + \sum_{j=q+2}^{p+q-1} w(x_j) \right) = 0,$$

so $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= v(x_1) - \sum_{i=2}^{q-1} v(x_i), \\ u(x_q) &= v(x_q), \end{aligned}$$

$$\begin{aligned} u(x_{p+q}) - \left(u(\alpha) + u(\beta) + \sum_{j=q+2}^{p+q-1} u(x_j) \right) &= \\ v(x_{p+q}) - \left(v(\alpha) + v(\beta) + \sum_{j=q+2}^{p+q-1} v(x_j) \right)&, \end{aligned}$$

for $a < x_1 < x_2 < \cdots < x_q < \alpha < \beta < x_{q+2} < \cdots < x_{p+q} < b$. We see that $u(x) \equiv v(x)$ by Condition (C), contradicting our assumption that $u(x) \neq v(x)$. This completes the proof of Case (ii).

Proof of Case (iii): $1 \leq p < m$ and $1 < q < n$. Assume that Conditions (A) and (B) hold, and that Condition (C) holds for $m = 2$ and $n = 3$. That is, for any $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, we are assuming uniqueness of solutions of the boundary value problem for (1.1) that satisfy the boundary conditions

$$\begin{aligned} y(x_1) - y(x_2) &= y_1, \\ y(x_3) &= y_2, \\ y(x_5) - y(x_4) &= y_3. \end{aligned}$$

Now, for some $a < x_1 < x_2 < x_3 < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, suppose that $u(x)$ and $v(x)$ are distinct solutions of the boundary value problem for (1.1) that both satisfy the boundary conditions

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3.$$

Set $w(x) = u(x) - v(x)$. Then we have $u(x_i) = v(x_i)$, for $i = 1, 2, 3$, so that $w(x_i) = 0$, for $i = 1, 2, 3$. Therefore, since $w(x_i) = 0$, for $i = 1, 2, 3$, and since $w(x) = u(x) - v(x)$

is continuous, we may choose $\alpha, \beta \in (x_1, x_2)$ such that $\alpha < \beta$ and $w(\alpha) = w(\beta)$, and we may choose $\eta, \theta \in (x_2, x_3)$ such that $\eta < \theta$ and $w(\eta) = w(\theta)$. Thus we have

$$\begin{aligned} w(\alpha) &= w(\beta), \\ u(\alpha) - v(\alpha) &= u(\beta) - v(\beta), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta), \end{aligned}$$

and

$$\begin{aligned} w(\eta) &= w(\theta), \\ u(\eta) - v(\eta) &= u(\theta) - v(\theta), \\ u(\eta) - u(\theta) &= v(\eta) - v(\theta), \\ u(\theta) - u(\eta) &= v(\theta) - v(\eta). \end{aligned}$$

Therefore, $u(x)$ and $v(x)$ are solutions of the boundary value problem for (1.1) that satisfy

$$u(\alpha) - u(\beta) = v(\alpha) - v(\beta),$$
$$u(x_2) = v(x_2),$$
$$u(\theta) - u(\eta) = v(\theta) - v(\eta),$$

with $a < \alpha < \beta < x_2 < \eta < \theta < b$. Our uniqueness condition says that $u(x) \equiv v(x)$ on (a, b), which is a contradiction of our assumption that $u(x)$ and $v(x)$ are distinct. We conclude that Case (iii) holds for $m = 2$ and $n = 3$.

Suppose that Conditions (A), (B), and (C) hold for positive integers $m > 2$ and $n > 3$. Set $p = m - 1$ and $q = n - 1$. Assume that, for some $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, there exist distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (1.1) that satisfy the boundary conditions of (2.4)-(2.6). Set $w(x) = u(x) - v(x)$. We then have that

$$u(x_1) - \sum_{i=2}^{q-1} u(x_i) = v(x_1) - \sum_{i=2}^{q-1} v(x_i),$$
$$u(x_q) = v(x_q),$$
$$u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = v(x_{p+q}) - \sum_{j=q+1}^{p+q-1} v(x_j),$$

or equivalently,

$$\begin{aligned} w(x_1) - \sum_{i=2}^{q-1} w(x_i) &= 0, \\ w(x_q) &= 0, \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= 0. \end{aligned}$$

From arguments in the proofs of Case (i) (not shown) and Case (ii), we know that our assumption that $u(x)$ and $v(x)$ are distinct implies that it cannot be the case that $w(x_i) = 0$ for all $i \in \{1, 2, \ldots, q-1\}$, nor can $w(x_j) = 0$ for all $j \in \{q+1, \ldots, p+q\}$. Assume without loss of generality that $w(x_{q-1}) \neq 0$ and $w(x_{q+1}) \neq 0$. Thus, since $w(x_{q-1}) \neq 0$ and $w(x_{q+1}) \neq 0$, $w(x_q) = 0$, and $w(x) = u(x) - v(x)$ is a continuous function, we may choose $\alpha, \beta \in (x_{q-1}, x_q)$ such that $\alpha < \beta$ and $w(\alpha)+w(\beta) = w(x_{q-1})$, and we may choose $\eta, \theta \in (x_q, x_{q+1})$ such that $\eta < \theta$ and $w(\eta) + w(\theta) = w(x_{q+1})$. Hence, we have

$$w(x_1) - \sum_{i=2}^{q-1} w(x_i) = w(x_1) - \left(\sum_{i=2}^{q-2} w(x_i) + w(\alpha) + w(\beta) \right) = 0,$$

and

$$w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = w(x_{p+q}) - \left(w(\eta) + w(\theta) + \sum_{j=q+2}^{p+q-1} w(x_j) \right) = 0,$$

so $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \left(\sum_{i=2}^{q-2} u(x_i) + u(\alpha) + u(\beta) \right) &= \\ v(x_1) - \left(\sum_{i=2}^{q-2} v(x_i) + v(\alpha) + v(\beta) \right)&, \\ u(x_q) = v(x_q)&, \\ u(x_{p+q}) - \left(u(\eta) + u(\theta) + \sum_{j=q+2}^{p+q-1} u(x_j) \right) &= \\ v(x_{p+q}) - \left(v(\eta) + v(\theta) + \sum_{j=q+2}^{p+q-1} v(x_j) \right)&, \end{aligned}$$

for $a < x_1 < x_2 < \cdots < x_{q-2} < \alpha < \beta < x_q < \eta < \theta < x_{q+2} < \cdots < x_{p+q} < b$. Then, Condition (C) implies that $u(x) \equiv v(x)$ on (a, b), which contradicts our assumption that $u(x)$ and $v(x)$ are distinct. The proof of Theorem 2.1 is complete. $\square$

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