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FIRST ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS INVOLVING DISCONTINUOUS MULTIFUNCTIONS

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ABSTRACT. The present paper studies the existence as well as existence of the extremal solutions for the first order impulsive differential inclusions under generalized monotonic conditions and without assuming any kind of continuity of the multi-functions on the right hand side.

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1. STATEMENT OF THE PROBLEM

The impulsive differential equations have received much attention during the last decade, but the study of the impulsive differential inclusions is relatively late in the literature. The dynamical systems, which involve the jumps or discontinuities are modeled on the impulsive differential equations and inclusions. The exhaustive account on the topic appear in Samoilenko and Perestyuk [12] and Benchohra and Boucherif [2] and the references therein. Moreover, the impulsive differential inclusions with discontinuous multifunctions are better substitute to describe these situations. The existence theorems so far discussed in the literature involve either the use of lower semi-continuity or upper semi-continuity of the multi-functions. Here in the present study, we do not require any kind of the continuity conditions of the multi-functions and the existence as well as existence of the extremal solutions is proved under generalized monotonicity conditions. We claim that the results of this paper are new to the literature on impulsive differential inclusions.

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Let $t_0, \ldots t_{p+1}$ be the points in J such that $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$ and let $J' = J \setminus \{t_1, \ldots, t_p\}$. Denote $J_j = (t_j, t_{j+1}) \subset J$ for $j = 1, 2, \ldots, p$. We use the usual notation $\mathcal{P}_p(\mathbb{R})$ to denote the class of all non-empty subsets of $\mathbb{R}$ with the property p. Thus $\mathcal{P}_{cl}(\mathbb{R})$, $\mathcal{P}_{bd}(\mathbb{R})$ and $\mathcal{P}_{cp}(\mathbb{R})$ denote, respectively, the classes of all closed, bounded and compact subsets of $\mathbb{R}$.

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Now consider the initial value problem for the first order impulsive differential inclusions (in short IDI)

$$\left.\begin{aligned} x'(t) \in F(t, x(t)) \ \text{ a.e. } t \in J \setminus \{t_1, \dots, t_p\}, \\ x(t_j^+) - x(t_j^-) = I_j(x(t_j)), \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where, $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$, $x(t_j^+)$ and $x(t_j^-)$ are, respectively, the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$ for $j = 1, 2, \dots, p$; and $I_j(x(t_j))$ are the impulsive moments at the points $t = t_j$, $j = 1, 2, \dots, p$.

By a *solution* of the IDI (1.1) we mean a function $x \in AC(J_j, \mathbb{R})$, $j = 1, 2, \dots, p$ that satisfies the differential inclusion and the conditions in (1.1), where $AC(J_j, \mathbb{R})$ is the space of absolutely continuous real-valued functions on $J_j = (t_j, t_{j+1})$, $j = 1, 2, \dots, p$.

The IDI (1.1) has already been discussed in the literature under compactness and upper semi-continuity conditions of the multifunction F. Here in the present study, we do not make use of any kind of continuity of the multifunction F and prove the existence results under arguments based on the recent algebraic fixed point theorems of Dhage [4].

2. AUXILIARY RESULTS

Let $C(J_j, \mathbb{R})$ denote the class of continuous real-valued functions on the interval $J_j = (t_j, t_{j+1})$. Denote

$$\begin{aligned} X = \big\{x \in C(J_j, \mathbb{R}) \mid x(t_j^-) \text{ and } x(t_j^+) \text{ exists for } j = 1, \dots, p; \\ \text{and } x(t_j^-) = x(t_j)\big\}. \end{aligned} \tag{2.1}$$

Define a norm $\|\cdot\|$ in X by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and define the order relation $\leq$ in X by the cone K given by

$$K = \{x \in X \mid x(t) \geq 0 \ \text{ for all } t \in J\},$$

which is obviously a normal cone in X. Thus, we have

$$x \leq y \iff x(t) \leq y(t) \text{ for all } \ t \in J.$$

Clearly, X becomes a ordered Banach space with respect to the above norm and order relation in X. We also use the following notations in the sequel.

Let $A, B \in \mathcal{P}_p(X)$. Denote

$$A \pm B = \{a \pm b : a \in A \quad \text{and} \quad b \in B\},$$

$$\lambda A = \{\lambda a : \lambda \in \mathbb{R} \quad \text{and} \quad \in A\}.$$

Also denote

$$\|A\| = \{\|a\| : a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| : a \in A\}.$$

Let the Banach space X be equipped with the order relation $\leq$ and we define the different order relations in $\mathcal{P}_p(X)$ as follows.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$." Again $A \overset{d}{\leq} B$ means for each $b \in B$ there exists a $a \in A$ such that $a \leq b$. Furthermore, we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set. See Dhage [5] and references therein.

Definition 2.1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp. $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Again, Q is called monotone increasing if it is left as well as right monotone increasing on X. Similarly, a monotone decreasing multi-valued mapping $Q : X \to \mathcal{P}_p(X)$ is defined. Finally, Q is called a strict monotone increasing on X if $Qx \leq Qy$ for all $x, y \in X$ for which $x \leq y$, $x \neq y$.

Remark 2.2. Note that every strict monotone increasing multi-valued mapping is right monotone increasing, but the converse may not be true.

It is known that the monotone techniques are very useful tools for proving the existence of the extremal solutions for differential equations and inclusions. The exhaustive treatment of monotone technique for the discontinuous differential equations may be found in Heikkilä and Lakshmikantham [9]. But the use of monotone techniques in the theory of differential inclusions involving the discontinuous multi-valued functions is relatively new to the literature. Some recent results in this direction appear in Dhage [4, 5, 6, 7]. In the methods of monotone technique for operator inclusions, the multi-valued operators in question are required to satisfy certain monotonicity condition with respect to certain order relation on the domains of definition. The following two fixed point theorems are fundamental in the monotone theory for discontinuous differential inclusions involving the right or strict monotone increasing multi-valued functions.

Theorem 2.3 (Dhage [4])**.** *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$, has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$, then Q has a fixed point.*

Theorem 2.4 (Dhage [6]). *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a strict monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$, has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then Q has a least fixed point x_* a greatest fixed point x^* in $[a,b]$. Moreover,*

$$x_* = \min\{y \in [a,b] \mid Qy \leq y\} \quad \textit{and} \quad x_* = \max\{y \in [a,b] \mid y \leq Qy\}.$$

We remark that the more general versions of Theorems 2.3 and 2.4 appear in Dhage [4] and [6], but the above forms are more useful in the applications to differential and integral inclusions.

A single-valued mapping $T : X \to X$ is called **Lipschitz** if there exists a constant $k > 0$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in X$. The constant k is called the Lipschitz constant of T on X. Furthermore, if $k < 1$, then T is called a contraction on X with contraction constant k. A multi-valued mapping $T : X \to \mathcal{P}_{cp}(X)$ is called totally bounded if for any bounded subset S of X, $\bigcup T(S)$ is a totally bounded subset of X.

We also need the following two hybrid fixed point theorems in the sequel.

Theorem 2.5 (Dhage [4]). *Let $[a,b]$ be an order interval in an ordered Banach space X. Let $A : [a,b] \to X$ and $B : [a,b] \to \mathcal{P}_{cp}(X)$ be two multi-valued operators satisfying*

(a) *A is a nondecreasing and single-valued contraction with the contraction constant $k < 1/2$,*
(b) *B is totally bounded and right monotone increasing, and*
(c) *$Ax + By \subset [a,b]$ for all $x, y \in [a,b]$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a,b]$.

Theorem 2.6 (Dhage [6]). *Let $[a,b]$ be an order interval in an ordered Banach space X. Let $A : [a,b] \to X$ and $B : [a,b] \to \mathcal{P}_{cp}(X)$ be two multi-valued operators satisfying*

(a) *A is a nondecreasing and single-valued contraction with the contraction constant $k < 1/2$,*
(b) *B is totally bounded and strict monotone increasing, and*
(c) *$Ax + By \subset [a,b]$ for all $x, y \in [a,b]$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least and a greatest solution in $[a,b]$.

In the following section we prove our main existence theorems for the IDI (1.1) under suitable conditions.

3. EXISTENCE RESULTS

We need the following definitions in the sequel.

Definition 3.1. A multi-valued map $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

Definition 3.2. A multifunction $F(t,x)$ is called right monotone increasing in x almost everywhere for $t \in J$ if $F(t,x) \overset{i}{\leq} F(t,y)$ a.e. $t \in J$, for all $x, y \in \mathbb{R}$ for which $x \leq y$. Similarly, a multifunction $F(t,x)$ is called strict monotone increasing in x almost everywhere for $t \in J$ if $F(t,x) \leq F(t,y)$ a.e. $t \in J$ for all $x, y \in \mathbb{R}$ for which $x \leq y$, $x \neq y$.

Definition 3.3. A multi-valued function $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called Chandrabhan if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$, and
(ii) $F(t,x)$ is right monotone increasing in x almost everywhere for $t \in J$.

Furthermore, a Chandrabhan multifunction F is called L^1-Chandrabhan, if

(iii) for each $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that
$$\|F(t,x)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x)\} \leq h_r(t) \text{ a.e. } t \in J$$
for all $x \in \mathbb{R}$ with $|x| \leq r$.

For any $x \in C(J, \mathbb{R})$, denote
$$S^1_F(x) = \{v \in L^1(J, \mathbb{R}) \mid v(t) \in F(t, x(t)) \text{ a.e. } t \in J'\}.$$
The integral of the multi-valued function F is defined as
$$\int_0^t F(s, x(s))\, ds = \left\{ \int_0^t v(s)\, ds : v \in S^1_F(x) \right\}.$$

Definition 3.4. A function $a \in C(J, \mathbb{R})$ is called a strict lower solution of the IDI (1.1) if for all $v \in S^1_F(a)$,
$$a'(t) \leq v(t) \text{ a.e. } t \in J \setminus \{t_1, \ldots, t_p\},$$
$$a(t_j^+) - a(t_j^-) \leq I_j(a(t_j)),$$
$$a(0) \leq x_0.$$
A strict upper solution b to IDI (1.1) is defined similarly.

We consider the following set of hypotheses in the sequel.

(A_1) There exists constants $c_j > 0$ such that $|I_j(x)| \leq c_j$, $j = 1, 2, \ldots, p$, for all $x \in \mathbb{R}$.
(A_2) The maps $x \mapsto I_j(x)$ is nondecreasing in $x \in \mathbb{R}$ for each $j =, \ldots, p$.
(B_1) $F(t,x)$ is closed and bounded for each $t \in J$.

(B_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that the multifunction $F_h(t, x) = F(t, x) + h(t)x$ is Chandrabhan on $J \times \mathbb{R}$.

(B_3) $S^1_{F_h}(x) \neq \emptyset$ for all $x \in C(J, \mathbb{R})$.

(B_4) The multi-valued map $x \mapsto S^1_{F_h}(x)$ is right monotone increasing in $C(J, \mathbb{R})$.

(B_5) FDI (1.1) has a strict lower solution a and a strict upper solution b with $a \leq b$.

(B_6) The function $\ell : J \to \mathbb{R}$ defined by

$$\ell(t) = \|F_h(t, a(t))\|_{\mathcal{P}} + \|F_h(t, b(t))\|_{\mathcal{P}}$$

is Lebesgue integrable.

Remark 3.5. Note that if (B_2), (B_5)-(B_6) holds, then we have

$$\|F_h(t, x(t))\|_{\mathcal{P}} \leq \ell(t) \quad \text{a.e.} \quad t \in J$$

for all $x \in [a, b]$.

Hypotheses (A_1) and (A_2) are common in the literature. Some nice sufficient conditions for guarantying (B_3) appear in Deimling [3], and Lasota and Opial [11]. A mild form of (B_5) has been used in Halidias and Papageorgiou [8]. Hypothesis (B_5) holds in particular if F is bounded in $J \times \mathbb{R}$. Hypotheses (B_2), (B_3) and (B_6) are relatively new to the literature, but these are assumed for (B_4) to make sense and the special forms of these hypotheses have been appeared in the works of several authors(see Dhage [4, 5] and references therein).

Now consider the following impulsive differential inclusion

$$\left.\begin{aligned} x'(t) + h(t)x(t) \in F_h(t, x(t)) \ \text{ a. e. } t \in J \setminus \{t_1, \ldots, t_p\}, \\ x(t_j^+) - x(t_j^-) = I_j(x(t_j)), \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.1}$$

where, $F_h : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ defined by

$$F_h(t, x(t)) = F(t, x(t)) + h(t)x(t)$$

and $x(t_j^+)$, $x(t_j^+)$, $I_j(x(t_j))$ have their usual meanings as in IDI (1.1).

Note that if a function a is a lower solution of the IDI (1.1), then it is also a lower solution of the IDI (3.1) on J. The same fact is also true for an upper solution. Thus, x is a solution of the IDI (1.1) if and only if it is a solution of the IDI (3.1). We need the following result in the sequel.

Lemma 3.6. *Let $\sigma \in L^1(J, \mathbb{R})$. Then for any $h \in L^1(J, \mathbb{R})$, the function $x : J \to \mathbb{R}$ is a solution to the differential equation*

$$x'(t) + h(t)x(t) = \sigma(t) \ \text{ a. e. } t \in J \setminus \{t_1, \ldots, t_p\}, \tag{3.2}$$

$$x(t_j^+) - x(t_j^-) = I_j(x(t_j)), \tag{3.3}$$

$$x(0) = x_0 \in \mathbb{R}, \tag{3.4}$$

if and only if it is a solution of the integral equation

$$x(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)\sigma(s)\, ds + \sum_{0<t_j<t} k(t,t_j) I_j(x(t_j)) \tag{3.5}$$

for $t \in J$, where the kernel function k is given by

$$k(t,s) = e^{-H(t)+H(s)}, \quad \text{and} \quad H(t) = \int_0^t h(s)\, ds.$$

Proof. First note that the kernel k is a continuous and nonnegative real-valued function on $J \times J$. Therefore, we have $H(t) > 0$ on J provided h is not an identically zero function. Otherwise $H(t) \equiv 0$ on J.

First suppose that x is a solution of the IDE (3.2)-(3.4) on J. Then, we have

$$\left.\begin{aligned} \left(e^{H(t)}x(t)\right)' &= e^{H(t)}\sigma(t) \ \text{ a. e. } t \in J \setminus \{t_1, \dots, t_p\}, \\ x(t_j^+) - x(t_j^-) &= I_j(x(t_j)), \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.6}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^+)}x(t_1^+) - e^{H(0)}x(0) &= \int_0^{t_1} \left(e^{H(s)}x(s)\right)' ds \\ e^{H(t_2^+)}x(t_2^+) - e^{H(t_1^+)}x(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}x(s)\right)' ds \\ &\vdots \\ e^{H(t)}x(t) - e^{H(t_p^+)}x(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}x(s)\right)' ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}x(t) - \sum_{0<t_j<t} e^{H(t_j)} I_j(x(t_j)) = x_0 + \int_0^t e^{H(s)} h(s)\, ds,$$

or

$$x(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)\sigma(s)\, ds + \sum_{0<t_j<t} k(t,t_j) I_j(x(t_j)).$$

Conversely, suppose that x is a solution of the integral equation (3.5). Obviously x satisfies the conditions (3.3) and (3.4). By the definition of the kernel function k, we obtain

$$e^{H(t)}x(t) = x_0 + \int_0^t e^{H(s)}\sigma(s)\, ds + \sum_{0<t_j<t} e^{H(t_j)} I_j(x(t_j)). \tag{3.7}$$

Since $\sigma \in L^1(J,\mathbb{R})$, one has $\int_0^t e^{H(s)}\sigma(s)\,ds \in AC(J,\mathbb{R})$. So, a direct differentiation of (3.7) yields,

$$\big(e^{H(t)}x(t)\big)' = e^{H(t)}\sigma(t),$$

or

$$x'(t) + h(t)x(t) = \sigma(t),$$

for $t \in J$ satisfying $x(0) = x_0$ and (3.3). The proof of the lemma is complete. □

Theorem 3.7. *Assume that $(A_1) - (A_2)$ and $(B_1) - (B_6)$ hold. Then the FDI (1.1) has a solution in $[a,b]$ defined on J.*

Proof. Define an order interval $[a,b]$ in X which is well defined in view of hypothesis (B_5). Now the IDI (1.1) is equivalent to the integral inclusion

$$x(t) \in x_0 e^{-H(t)} + \int_0^t k(t,s)F_h(s,x(s))\,ds + \sum_{0<t_j<t} k(t,t_j)I_j(x(t_j)). \tag{3.8}$$

Define a multi-valued operator $Q : [a,b] \to \mathcal{P}_p(X)$ by

$$\begin{aligned} Qx &= \Big\{ u \in X : u(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)v(s)\,ds \\ &\qquad + \sum_{0<t_j<t} k(t,t_j)I_j(x(t_j)),\ v \in S^1_{F_h}(x) \Big\} \\ &= (\mathcal{L} \circ S^1_{F_h})(x), \end{aligned} \tag{3.9}$$

where $\mathcal{L} : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}v(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)v(s)\,ds + \sum_{0<t_j<t} k(t,t_j)I_j(x(t_j)). \tag{3.10}$$

Clearly, the operator Q is well defined in view of hypothesis (B_3). We shall show that Q satisfies all the conditions of Theorem 2.3.

Step I : First, we show that Q has compact values on $[a,b]$. Observe that if $t \in J$, then operator Q is equivalent to the composition $\mathcal{L} \circ S^1_{F_h}$ of two operators on $L^1(J,\mathbb{R})$, where $\mathcal{L} : L^1(J,\mathbb{R}) \to X$ is the continuous operator defined by (3.10). To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S^1_F$ has compact values on $[a,b]$. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F_h}(x)$. Then, by the definition of $S^1_{F_h}$, $v_n(t) \in F_h(t,x(t))$ a.e. for $t \in J$. Since $F_h(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F_h(t,x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on

J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t,\, \tau \in J$; then

$$\begin{aligned}
|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_0^t k(t,s)v_n(s)\,ds - \int_0^\tau k(\tau,s)v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_0^t k(t,s)v_n(s)\,ds - \int_0^\tau k(t,s)v_n(s)\,ds\right| \\
&\quad + \left|\int_0^\tau k(t,s)v_n(s)\,ds - \int_0^\tau k(\tau,s)v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_\tau^t v_n(s)\,ds\right| \\
&\quad + \left|\int_0^\tau |k(t,s) - k(\tau,s)|v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_\tau^t v_n(s)\,ds\right| \\
&\quad + \int_0^T |k(t,s) - k(\tau,s)||v_n(s)|\,ds \\
&\quad + \sum_{0<t_j<t} c_j|k(t,t_j) - k(\tau,t_j)|
\end{aligned}$$

Since $v_n \in L^1(J,\mathbb{R})$, the right hand side of above inequality tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L}\circ S^1_{F_h})(x)$ as $j \to \infty$, and so $(\mathcal{L}\circ S^1_{F_h})(x)$ is compact. Therefore, Q is a multi-valued operator with compact values on $[a,b]$.

Step II : Secondly, we show that Q is right monotone increasing and maps $[a,b]$ into itself. Let $x, y \in [a,b]$ be such that $x \le y$. Since $S^1_{F_h}(x) \overset{i}{\le} S^1_{F_h}(y)$, we have that $Q(x) \overset{i}{\le} Q(y)$. From (B_4), it follows that $a \le Qa$ and $Qb \le b$. Now Q is right monotone increasing, so we have

$$a \le Qa \overset{i}{\le} Qx \overset{i}{\le} Qb \le b$$

for all $x \in [a,b]$. Hence Q defines a right monotone increasing multi-valued operator $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$.

Step III : Finally, let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a sequence in $Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$. We will show that $\{y_n\}$

has a cluster point. This is achieved by showing that $\{x_n\}$ is a uniformly bounded and equi-continuous sequence.

Case I : First, we show that $\{y_n\}$ is uniformly bounded sequence. By the definition of $\{y_n\}$, there is a $v_n \in S^1_{F_h}(x_n)$ such that

$$y_n(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)v_n(s)\,ds + \sum_{0<t_j<t} k(t,t_j)I_j(x(t_j)).$$

Therefore, by Remark 3.1, we have $\ell \in L^1(J,\mathbb{R})$ such that

$$\begin{aligned}
|y_n(t)| &\le \left|x_0 e^{-H(t)}\right| + \left|\int_0^t k(t,s)v_n(s)\,ds\right| \\
&\quad + \Big|\sum_{0<t_j<t} k(t,t_j)I_j(x(t_j))\Big| \\
&\le |x_0| + \int_0^T \ell(s)\,ds + \sum_{j=1}^p e^{H(t_j)}c_j \\
&\le |x_0| + \|\ell\|_{L^1} + d
\end{aligned}$$

for all $t \in J$, where $d = \sum_{j=1}^p e^{H(t_j)}c_j$. Taking the supremum over t,

$$\|y_n\| = |x_0| + \|\ell\|_{L^1} + d$$

which shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a,b])$.

Next we show that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Let $t,\, \tau \in J$; then

$$\begin{aligned}
|y_n(t) - y_n(\tau)| &\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_0^t k(t,s)v_n(s)\,ds - \int_0^\tau k(\tau,s)v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_0^t k(t,s)v_n(s)\,ds - \int_0^\tau k(t,s)v_n(s)\,ds\right| \\
&\quad + \left|\int_0^\tau k(t,s)v_n(s)\,ds - \int_0^\tau k(\tau,s)v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_\tau^t v_n(s)\,ds\right| \\
&\quad + \left|\int_0^\tau |k(t,s) - k(\tau,s)|v_n(s)\,ds\right| \\
&\quad + \sum_{0<t_j<t} |k(t,t_j) - k(\tau,t_j)||I_j(x(t_j))| \\
&\le |x_0|\left|e^{-H(t)} - e^{-H(\tau)}\right| + \left|\int_\tau^t v_n(s)\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T |k(t,s)-k(\tau,s)||v_n(s)|\,ds\\
&+\sum_{0<t_j<t} c_j|k(t,t_j)-k(\tau,t_j)|\\
\leq{}& |x_0|\,\big|q(t)-q(\tau)\big| + |p(t)-p(\tau)|\\
&+\int_0^T |k(t,s)-k(\tau,s)|\ell(s)\,ds\\
&+\sum_{0<t_j<t} c_j|k(t,t_j)-k(\tau,t_j)|
\end{aligned}
$$

where, $q(t)=e^{-H(t)}$ and $p(t)=\displaystyle\int_0^t \ell(s)\,ds$.

From the above inequalities, it follows that

$$|y_n(t)-y_n(\tau)| \to 0 \quad \text{as} \ \ t\to\tau.$$

This shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$. Now $\{y_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point in view of the Arzelà-Ascoli theorem. Now the desired conclusion follows by an application of Theorem 2.3. □

Below we relax the boundedness assumption on the impulsive moments I_j on $\mathbb{R}$ for each $j=1,\dots,p$; instead, we assume a Lipschitz condition on I_j and prove the existence of solution for (1.1). We need the following hypothesis in the sequel.

(A_3) There exist constants $c_j>0$ such that

$$|I_j(x)-I_j(y)|\leq c_j|x-y|,\ j=1,2,\dots,p;$$

for all $x,y\in\mathbb{R}$.

Theorem 3.8. *Assume that the hypotheses $(A_2)-(A_3)$ and $(B_1)-(B_5)$ hold. Furthermore, if $\sum_{0<t_j<t} c_j < 1/2$, then the IDI (1.1) has a solution in the sector $[a,b]$ defined on J.*

Proof. Consider the order interval $[a,b]$ in X which is well defined in view of hypothesis (B_4). Define two operators $A:[a,b]\to X$ and $B:[a,b]\to\mathcal{P}_{cp}(X)$ by

$$Ax(t)=\sum_{0<t_j<t} k(t,t_j)I(x(t_j)) \tag{3.11}$$

and

$$Bx(t)=x_0e^{-H(t)}+\int_0^t k(t,s)F(s,x(s))\,ds. \tag{3.12}$$

We will show that A is a contraction on $[a,b]$. Let $x, y \in [a,b]$. By (A_3),

$$\begin{aligned}
|Ax(t) - Ay(t)| &\leq \left| \sum_{0<t_j<t} k(t,t_j) I(x(t_j)) - \sum_{0<t_j<t} k(t,t_j) I(y(t_j)) \right| \\
&\leq \sum_{0<t_j<t} |k(t,t_j)|\, |I(x(t_j)) - I(y(t_j))| \\
&\leq \sum_{0<t_j<t} c_j |x(t_j)) - y(t_j))| \\
&\leq \Big(\sum_{0<t_j<t} c_j \Big) \|x - y\|
\end{aligned}$$

for all $t \in J$. This further yields

$$\|Ax - Ay\| \leq \alpha \|x - y\|$$

for all $x, y \in [a,b]$, where $\alpha = \displaystyle\sum_{0<t_j<t} c_j < 1/2$. Hence, A is a contraction operator on the order interval $[a,b]$ in X.

It can be shown as in the proof of Theorem 3.7 that B is a totally bounded operator on $[a,b]$. Again, It is easy to verify that A is nondecreasing and B is right monotone increasing on $[a,b]$, and $Ax + By \in [a,b]$ for $x, y \in [a,b]$. Now the desired conclusion follows by an application of Theorem 2.5. $\square$

Next, we prove a result concerning the extremal solutions of the IDI (1.1) on J. We need the following definition in the sequel.

Definition 3.9. A multi-valued function $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called strict L^1-Chandrabhan if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $F(t,x)$ is strict monotone increasing in x almost everywhere for $t \in J$, and
(iii) for each $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t,x)\} \leq h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Remark 3.10. Note that if the multi-valued function $F(t,x)$ is strict L^1-Chandrabhan and the hypothesis (B_6) holds, then it is measurable in t and integrally bounded on $J \times \mathbb{R}$, and so, by a selection theorem, S_F^1 has non-empty and closed values on $[a,b]$, that is,

$$S_F^1(x) = \{v \in L^1(J,\mathbb{R}) \mid v(t) \in F(t,x(t)) \text{ a.e. } t \in J\} \neq \emptyset$$

for all $x \in [a,b] \subset C(J,\mathbb{R})$(see Deimling [3] and the references therein). Note also that if F is a strict L^1-Chandrabhan on $J \times \mathbb{R}$, then the multifunction F_h defined by

$F_h(t, x(t)) = F(t, x(t)) + h(t)x(t)$ is also a strict L^1-Chandrabhan on $J \times \mathbb{R}$, but the converse may not be true.

We consider the following hypothesis in the sequel.

(B_7) The multi-valued function F_h is a strict L^1-Chandrabhan on $J \times \mathbb{R}$.

Theorem 3.11. *Assume that the hypotheses* $(A_1) - (A_2)$ *and* $(B_1), (B_5), (B_7)$ *hold. Then the IDI* (1.1) *has a minimal and a maximal solution in* $[a, b]$ *defined on* J.

Proof. The proof is similar to Theorem 3.1. Here, $S^1_{F_h}(x) \neq \emptyset$ and the multi-valued map $x \mapsto S^1_{F_h}(x)$ is strictly monotone increasing on $[a, b]$. Consequently the multi-valued operator Q defined by (3.2) is strictly monotone increasing on $[a, b]$. Hence, the desired result follows by an application of Theorem 2.4. □

Theorem 3.12. *Assume that the hypotheses* (A_2), (A_3) *and* $(B_1), (B_5), (B_7)$ *hold. In addition, if* $\sum_{0<t_j<t} c_j < 1/2$, *then the IDI (1.1) has a minimal and a maximal solution in* $[a, b]$ *defined on* J.

Proof. The proof is similar to Theorem 3.11. Consider the order interval $[a, b]$ in X, which is well defined in view of hypothesis (B_4). Define two operators $A : [a, b] \to X$ and $B : [a, b] \to \mathcal{P}_{cp}(X)$ by (3.11) and (3.12) respectively. It can be shown as in the proofs of Theorems 3.7 and 3.8 that the operator A is contraction and B is totally bounded on $[a, b]$. Here, the operator A is nondecreasing on $[a, b]$. Also, $S^1_{F_h}(x) \neq \emptyset$ and the multi-valued map $x \mapsto S^1_{F_h}(x)$ is strictly monotone increasing on $[a, b]$, so the multi-valued operator B is strictly monotone increasing on $[a, b]$. The desired conclusion follows by an application of Theorem2.6. □

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REFERENCES

[1] J. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, New York, 1984.

[2] M. Benchohra and A. Boucherif, *An existence result for first order intial value problems for impulsive differential inclusions*, Arch. Math. (Brno) **36** (2000), 159–169.

[3] K. Deimling, *Multi-valued Differential Equations*, De Gruyter, Berlin 1998.

[4] B. C. Dhage, *Some algebraic fixed point theorems for multi-valued operators with applications*, Disc. Math. Differential inclusions, Control & Optimization **26** (2006), to appear.

[5] B. C. Dhage, *Monotone method for discontinuous differential inclusions*, Math. Sci. Res. Journal **8** (3) (2004), 104–113.

[6] B. C. Dhage, *Fixed point theorems for discontinuous multi-valued operators on ordered spaces with applications*, Computers & Mathematics with Applications **51** (3-4) (2006), 589–604.

[7] B. C. Dhage, *Monotone technique for second order discontinuous differential inclusions*, Applied Mathematics E-Notes **6** (2006), 66–73.

[8] N. Halidias and N. Papageorgiou, *Second order multi-valued boundary value problems*, Arch. Math. (Brno) **34** (1998), 267–284.

[9] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Pure and Applied Mathematics, Marcel Dekker, New York, 1994.

[10] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1989.

[11] A. Lasota and Z. Opial, *An application of the Kakutani- Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phy. **13** (1965), 781–786.

[12] A. N. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.