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SETVALUED PERTURBED HYBRID INTEGRO-DIFFERENTIAL EQUATIONS AND STABILITY IN TERMS OF TWO MEASURES

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ABSTRACT. We study some stability criteria in terms of two measures for setvalued perturbed hybrid integro-differential equations with fixed moments of impulse. Stability properties of perturbed system are obtained via a comparison result which connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system.

Keywords and Phrases. Perturbed hybrid setvalued integro-differential equations, Stability in terms of two measures, Variation of Lyapunov second method.

AMS (MOS) Subject Classifications. 34K20, 34K25, 45J05.

1. INTRODUCTION

The subject of setvalued differential equations initiated as an independent subject, has been addressed by many authors, for instance, see [1-5] and the references therein. The interesting feature of the setvalued differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping. Moreover, setvalued differential equations, that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions [6]. Set differential equations can also be utilized to investigate fuzzy differential equations [2].

In the perturbation theory of nonlinear differential systems, a flexible mechanism known as variation of Lyapunov second method, was introduced in [7]. This technique, which essentially connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system using a comparison principle, was extended to integral equations in [8-9]. The concept of stability in terms of two measures [10] which unifies a number of stability concepts such as Lyapunov stability, partial

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stability, conditional stability, etc. has become an important area of investigation in the qualitative analysis [11-15].

Impulsive hybrid dynamical systems form a class of hybrid systems in which continuous time states are reset discontinuously when the discrete event states change. Recently, a number of research papers has dealt with dynamical systems with impulsive effect as a class of general hybrid systems [16-20]. In this paper, we develop the stability criteria in terms of two measures for setvalued perturbed hybrid integro-differential equations with fixed moments of impulsive effect through the variation of Lyapunov second method.

2. PRELIMINARIES AND COMPARISON RESULT

Let $K(R^n)$ denote the collection of nonempty, compact and convex subsets of R^n. We define the Hausdorff metric as

$$D[X,Y] = \max[\sup_{y \in Y} d(y,X), \sup_{x \in X} d(x,Y)], \tag{1}$$

where $d(y,X) = \inf[d(y,x) : x \in X]$ and X, Y are bounded subsets of R^n. Notice that $K(R^n)$ with the metric is a complete metric space. Moreover, $K(R^n)$ equipped with the natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [6,21]. The Hausdorff metric (1) satisfies the following properties:

$$D[X+Z, Y+Z] = D[X,Y] \text{ and } D[X,Y] = D[Y,X], \tag{2}$$

$$D[\mu X, \mu Y] = \mu D[X,Y], \tag{3}$$

$$D[X,Y] \le D[X,Z] + D[Z,Y], \tag{4}$$

$\forall\ X, Y, Z \in K(R^n)$ and $\mu \in R_+$.

Definition 2.1. The set $Z \in K(R^n)$ satisfying $X = Y + Z$ is known as the Hukuhara difference of the sets X and Y in $K(R^n)$ and is denoted as $X - Y$.

Definition 2.2. For any interval $I \in R$, the mapping $F : I \to K(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0+h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0-h)}{h}, \tag{5}$$

exist in the topology of $K(R^n)$ and each one is equal to $D_H F(t_0)$.

By embedding $K(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, it is found that if

$$F(t) = X_0 + \int_0^t \Phi(\eta)d\eta, \quad X_0 \in K(R^n), \tag{6}$$

where $\Phi : I \to K(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and

$$D_H F(t) = \Phi(t) \quad \text{a.e. on } I. \tag{7}$$

Consider the following perturbed set integro-differential equations with fixed moments of impulse

$$\begin{cases} D_H U(t) = F(t, U(t), L_1 U(t)), & t \neq t_k, \\ U(t_k^+) = U(t_k) + I_k(U(t_k)), & k = 1, 2, 3, \ldots, \\ U(t_0^+) = U_0, & t_0 \geq 0, \end{cases} \tag{8}$$

together with the unperturbed ones

$$\begin{cases} D_H V(t) = G(t, V(t), L_2 V(t)), & t \neq t_k, \\ V(t_k^+) = V(t_k) + J_k(V(t_k)), & k = 1, 2, 3, \ldots, \\ V(t_0^+) = U_0, & t_0 \geq 0, \end{cases} \tag{9}$$

where $F, G : R_+ \times K_c(R^n) \times K_c(R^n) \to K_c(R^n)$ are continuous on $(t_{k-1}, t_k] \times K_c(R^n) \times K_c(R^n)$, with G smooth enough or containing the linear terms of system (8), L_i denote the integral in sense of Hukuhara [22-23] and is defined by $L_i U(t) = \int_{t_0}^t K_i(t, \eta, U(\eta))d\eta$, $K_i : R_+ \times R_+ \times K_c(R^n) \to K_c(R^n)$ is continuous on $(t_{k-1}, t_k] \times (t_{k-1}, t_k] \times K_c(R^n)$, $i = 1, 2$, $I_k, J_k : K_c(R^n) \to K_c(R^n)$ and $\{t_k\}$ is a sequence of points such that $t_0 < t_1 < \cdots t_k < \cdots$ with $\lim_{k\to\infty} t_k = \infty$.

Letting ρ to be a positive real number, we define the following classes of functions:

$$\mathcal{K} = \{\nu : [0, \rho) \to R_+ \text{ is continuous, strictly increasing and } \nu(0) = 0\};$$

$$PC = \{\mu : R_+ \to R_+ \text{ is continuous on } (t_{k-1}, t_k] \text{ and } \mu \to \mu(t_k^+) \text{ exists as } t \to t_k^+\};$$

$$PC\mathcal{K} = \{\phi : R_+ \times [0, \rho) \to R_+, \quad \phi(., m) \in PC$$

for each $m \in [0, \rho)$, $\phi(t, .) \in \mathcal{K}$ for each $t \in R_+\}$;

$$\Gamma = \{h : R_+ \times K_c(R^n) \to R_+, \quad \inf_{U \in K_c(R^n)} h(t, U) = 0, \quad h(., U) \in PC,$$

for each $U \in K_c(R^n)$, and $h(t, .) \in C(K_c(R^n), R_+)$ for each $t \in R_+\}$;

$$S(h, \rho) = \{(t, U) \in R_+ \times K_c(R^n) : h(t, U) < \rho, h \in \Gamma\};$$

$$S(\rho) = \{U \in K_c(R^n) : (t, U) \in S(h, \rho) \text{ for each } t \in R_+\}.$$

Definition 2.3. Let $W : R_+ \times K_c(R^n) \to R_+$. Then W is said to belong to class W_0 if $W(t, U) \in PC$ for each $U \in S(\rho)$ and $W(t, U)$ is locally Lipschitzian in U.

Definition 2.4. Let $W \in W_0$ and $V(t,\eta,U)$ be any solution of (9). Then for any fixed $t > t_0$, $(\eta, U) \in (t_{k-1}, t_k) \times S(\rho)$, $t_0 \leq \eta < t$, we define

$$\begin{aligned} &D^+W(\eta, V(t,\eta,U)) \\ &\quad = \limsup_{h\to 0^+} \frac{1}{h}[W(\eta+h, V(t,\eta+h, U+hF(\eta,U,L_1U))) - W(\eta, V(t,\eta,U))], \end{aligned}$$

where $V(t,\eta,U)$ is any solution of (9) such that $V(\eta,\eta,U) = U$.

Remark. In order to show how the perturbation terms affect the stability properties of the perturbed system, we suppose that

$$F(t,U(t),L_1U(t)) = G(t,U(t),L_2U(t)) + R(t,U(t),LU(t)),$$

and the solution of (9) is differentiable with respect to initial value. Then we have

$$\begin{cases} \frac{\partial V}{\partial U_0}(t,t_0,U_0) = \Psi(t,t_0,U_0), \\ \frac{\partial V}{\partial t_0}(t,t_0,U_0) = -\Psi(t,t_0,U_0).G(t,t_0,L_2U_0),\ t \geq t_0, \end{cases}$$

where $\Psi(t,t_0,U_0)$ is the fundamental matrix solution of the corresponding variational equation. Setting $W(\eta,V) = \|V\|^2$ (for instance, $\|V\| = \sup_{v\in V}\|v\|$), we get

$$D^+W(\eta, V(t,\eta,U(t))) = 2V^T(t,\eta,U(t)).\Psi(t,t_0,U(t)).R(\eta,U(t),LU(t)),$$

which shows the desired effect.

Definition 2.5. Let h, $h_0 \in \Gamma$. We say that

(i) h_0 is finer than h if there exists a $\overline{\lambda} > 0$ and a function $\phi \in PCK$ such that

$$h_0(t,U) < \overline{\lambda} \text{ implies } h(t,U) \leq \phi(t, h_0(t,U));$$

(ii) h_0 is uniformly finer than h if (i) holds for $\phi \in \mathcal{K}$.

Definition 2.6. Let h, $h_0 \in \Gamma$ and $W \in W_0$. Then $W(t,U)$ is said to be

(i) h-positive definite if there exists a $\lambda > 0$ and a function $b \in \mathcal{K}$ such that

$$h(t,U) < \lambda \text{ implies } b(h(t,U)) \leq W(t,U);$$

(ii) weakly h_0-decrescent if there exists a $\lambda_1 > 0$ and a function $a \in PCK$ such that

$$h_0(t,U) < \lambda_1 \text{ implies } W(t,U) \leq a(t, h_0(t,U));$$

(iii) h_0-decrescent if (ii) holds with $a \in \mathcal{K}$.

Definition 2.7. Let h, $h_0 \in \Gamma$ and $U((t) = U(t,t_0,U_0)$ be any solution of (1), then the system (1) is said to be

(I) (h_0,h)-stable if for each $\epsilon > 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$h_0(t_0,U_0) < \delta \text{ implies } h(t,U(t))) < \epsilon,\ t \geq t_0;$$

(II) (h_0,h)-uniformly stable if (I) holds with δ independent of t_0;

(III) (h_0, h)-attractive if there exists a $\delta = \delta(t_0) > 0$ and for each $\epsilon > 0$, there exists $T = T(t_0, \epsilon) > 0$ such that

$$h_0(t_0, U_0) < \delta_0 \text{ implies } h(t, U(t)) < \epsilon,\ t \geq t_0 + T;$$

(IV) (h_0, h)-uniformly attractive if (III) holds with δ and T independent of t_0;

(V) (h_0, h)-asymptotically stable if it is (h_0, h)-stable and (h_0, h)-attractive;

(VI) (h_0, h)-uniformly asymptotically stable if it is (h_0, h)-uniformly stable and (h_0, h)-uniformly attractive.

Now, we prove a comparison result which is needed for the sequel.

Lemma 2.1. *Assume that*

($\mathbf{A_1}$) *The solution $V(t) = V(t, t_0, U_0)$ of (9) exists for all $t \geq t_0$, unique, continuous with respect to the initial values, locally Lipschitzian in U_0 and $V(t_0) = U_0$;*

($\mathbf{A_2}$) *$W \in C[R_+ \times K(R^n), K(R^n)]$ satisfies $|W(t, X) - W(t, Y)| \leq ND[X, Y]$, where N is the local Lipschitz constant, $X, Y \in K(R^n),\ t \in R_+$;*

($\mathbf{A_3}$) *For $(\eta, U) \in S(h, \rho),\ t_0 \leq \eta < t,\ W \in W_0$ satisfies the inequality*

$$\begin{cases} D^+W(\eta, V(t, \eta, U)) \leq g_1(\eta, W(\eta, V(t, \eta, U))), & t \neq t_k, \\ W(t_k^+, V(t, t_k^+, U(t_k^+))) \leq \psi_k(W(t_k, V(t, t_k, U(t_k))), & k = 1, 2, \ldots, \\ W(t_0^+, V(t, t_0^+, U_0)) \leq x_0, \end{cases}$$

where $g_1(t, x) \in PC$ for each $x \in R_+$ and $\psi_k : R_+ \to R_+$ are nondecreasing functions for all $= 1, 2, \ldots;$

($\mathbf{A_4}$) *The maximal solution $r(t) = r(t, t_0, x_0)$ of the following scalar impulsive differential equation exists on $[t_0, \infty)$*

$$\begin{cases} x' = g_1(t, x), & t \neq t_k, \\ x(t_k^+) = \psi_k(x(t_k)),\ k = 1, 2, \ldots, \\ x(t_0^+) = x_0 \geq 0. \end{cases} \tag{10}$$

Then $W(t, U(t, t_0, U_0)) \leq r(t, t_0, x_0)$.

Proof. Let $U(t) = U(t, t_0, U_0)$ be any solutions of (8) with $(t_0, U_0) \in S(h, \rho)$. We set $m(\eta) = W(\eta, V(t, \eta, U(\eta))$, $\eta \in [t_0, t]$ and $\lim_{\eta \to t-0} m(\eta) = m(t)$. For small $h > 0$, we consider

$$\begin{aligned} m(\eta + h) - m(\eta) &= W(\eta + h, V(t, \eta + h, U(\eta + h))) - W(\eta, V(t, \eta, U(\eta)) \\ &= W(\eta + h, V(t, \eta + h, U(\eta + h))) - W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta)))) \\ &\quad + W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta)))) - W(\eta, V(t, \eta, U(\eta))) \\ &\leq ND[V(t, \eta + h, U(\eta + h)), V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta)))] \\ &\quad + W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U(\eta), L_1U(\eta)))) - W(\eta, V(t, \eta, U(\eta))) \end{aligned}$$

where we have used the condition (A_2). Thus,

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)]$$
$$\leq D^+W(\eta, V(t,\eta,U(\eta)) + N^2 \limsup_{h\to 0^+} \frac{1}{h} D[U(\eta+h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta))].$$

Letting $U(\eta+h) = U(\eta) + Z(\eta)$, where $Z(\eta)$ is the Hukuhara difference of $U(\eta+h)$ and $U(\eta)$ for small $h > 0$ and is assumed to exist. Hence, employing the properties of $D[\cdot,\cdot]$, it follows that

$$D[U(\eta+h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta))]$$
$$= D[U(\eta) + Z(\eta), U(\eta) + hF(\eta, U(\eta), L_1U(\eta))] = D[Z(\eta), hF(\eta, U(\eta), L_1U(\eta))]$$
$$= D[U(\eta+h) - U(\eta), hF(\eta, U(\eta), L_1U(\eta))].$$

Consequently, we find that

$$\frac{1}{h} D[U(\eta+h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta))]$$
$$= D[\frac{U(\eta+h) - U(\eta)}{h}, F(\eta, U(\eta), L_1U(\eta))],$$

which, in view of the fact that $U(t)$ is a solution of (8), yields

$$\limsup_{h\to 0^+} \frac{1}{h} D[U(\eta+h), U(\eta) + hF(\eta, U(\eta), L_1U(\eta))]$$
$$= \limsup_{h\to 0^+} D[\frac{U(\eta+h) - U(\eta)}{h}, F(\eta, U(\eta), L_1U(\eta))]$$
$$= D[U'_H(\eta), F(\eta, U(\eta), L_1U(\eta))] = 0.$$

Hence, we have

$$D^+m(\eta) \leq g(\eta, m(\eta)), \quad t \neq t_k.$$

Also

$$m(t_k^+) = \psi_k(m(t_k)), \quad k = 1, 2, \ldots,$$

$$m(t_0) \leq x_0.$$

Now, from reference [11], it follows that $m(\eta) \leq r(\eta, t_0, x_0)$, $\eta \in [t_0, t]$, that is, $W(\eta, V(t,\eta,U(\eta)) \leq r(\eta, t_0, x_0)$, $\eta \in [t_0, t]$. Since $V(t, t, U(t)) = U(t)$, therefore we have

$$W(t, U(t, t_0, U_0)) = W(t, V(t, t, U(t))) \leq r(t, t_0, x_0).$$

This proves the assertion of the theorem. □

3. STABILITY CRITERIA FOR SETVALUED HYBRID INTEGRO-DIFFERENTIAL EQUATIONS

Theorem 3.1. *Assume that*

($\mathbf{B_1}$) *The solution $V(t) = V(t, t_0, U_0)$ of (2) exists for all $t \geq t_0$, unique, continuous with respect to the initial values, locally Lipschitzian in U_0 and $V(t_0) = U_0$.*

($\mathbf{B_2}$) *$K_i(t, s, 0) = 0$ so that $G(t, 0, 0) = G(t, 0) = 0$, $g_1(t, 0) = 0$ and $J_k(0) = 0$, $\psi_k(0) = 0$, $k = 1, 2, \ldots;$*

($\mathbf{B_3}$) *h_0, $h \in \Gamma$ such that $h_0(t, 0) = 0$ for $t \in R_+$ and h_0 is finer than h;*

($\mathbf{B_4}$) *$W \in W_0$ be such that $W(t, U)$ is h-positive definite and weakly h_0-decrescent for $(t, U) \in S(h, \rho)$, and satisfies the inequality*

$$\begin{cases} D^+W(\eta, V(t, \eta, U)) \leq g_1(\eta, W(\eta, V(t, \eta, U))), \ \eta \neq t_k, \\ (\eta, U) \in S(h, \rho), \eta \in [t_0, t), \\ W(t_k^+, V(t, t_k^+, U(t_k^+))) \leq \psi_k(W(t_k, V(t, t_k, U(t_k)))), \ k = 1, 2, \ldots; \end{cases}$$

($\mathbf{B_5}$) *There exists a $\rho_0 \in (0, \rho]$ such that*

$$h(t_k, U(t_k)) < \rho_0 \textit{ implies that } h(t_k^+, U(t_k^+)) < \rho, \ k = 1, 2, \ldots$$

Then the stability of the null solution of (9) *and the asymptotical stability of the null solution of* (10) *imply the* (h_0, h)*-asymptotical stability of* (8).

Proof. Let $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, t_0, U_0)$ and $x(t) = x(t, t_0, x_0)$ be any solutions of (8), (9) and (10) respectively. Since $W(t, U)$ is h-positive definite on $S(h, \rho)$, there exists $b \in \mathcal{K}$ such that

$$h(t, U) < \rho \text{ implies } b(h(t, U)) \leq W(t, U). \tag{11}$$

Also $W(t, U)$ is weakly h_0-decrescent and h_0 is finer than h, so there exists a $\lambda_0 > 0$ and $a \in PC\mathcal{K}$, $\phi \in PC\mathcal{K}$ such that

$$h(t, U) \leq \phi(t, h_0(t, U)) \text{ implies } W(t, U) \leq a(t, h_0(t, U)), \tag{12}$$

when $h_0(t, U) < \lambda_0$ and $\phi(t_0^+, \lambda_0) < \rho$. Since the null solution of (10) is stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$0 \leq x_0 < \delta_1 \text{ implies that } \ x(t, t_0, x_0) < b(\epsilon), \ t \geq t_0, \tag{13}$$

where $0 < \epsilon < \rho_0$ and $t_0 \in R_+$. Also, the trivial solution of (9) is stable, so there exists a $\delta_2 = \delta_2(t_0, \epsilon) > 0$ corresponding to δ_1 such that

$$\|U_0\| < \delta_2 \text{ implies that } \ \|V(t)\| < a^{-1}(t_0, \delta_1),$$

while, from (B_3), we have

$$h_0(t_0^+, U_0) < \delta_2 \text{ implies that } \ h_0(t_0^+, V(t)) < a^{-1}(t_0, \delta_1). \tag{14}$$

Select $\delta = \delta(t_0, \epsilon) > 0$ satisfying $\delta < \min\{\lambda_0, \delta_2\}$. Now if $h_0(t_0^+, U_0) < \delta$, then it follows from (11)–(14) that

$$b(h(t_0^+, U_0)) \leq W(t_0^+, U_0) \leq a(t_0^+, h_0(t_0^+, U_0)) < a(t_0^+, \delta_2) \leq \delta_1 \leq b(\epsilon),$$

which implies that $h(t_0^+, U_0)) < \epsilon$ when $h_0(t_0^+, U_0)) < \delta$.

We assert that

$$h(t, U(t)) < \epsilon \text{ whenever } h_0(t_0^+, U_0)) < \delta. \tag{15}$$

For the sake of contradiction, let us assume that (15) is false and there exists $t^* > t_0$ such that $h(t^*, U(t^*)) \geq \epsilon$. For $h \in \Gamma$, there are two cases: (i) $t_0 < t^* \leq t_1$ (ii) $t_k < t^* \leq t_{k+1}$ for some $k = 1, 2, \ldots$.

(i) Without loss of generality, let $t^* = \inf\{t : h(t, U(t)) \geq \epsilon\}$ and $h(t^*, U(t^*)) = \epsilon$. Using Lemma 2.1 and (11)–(12) together with the fact that $r(t, t_0, x_1) \leq r(t, t_0, x_2)$ if $x_1 \leq x_2$ (which follows from Lemma 2.1), we obtain

$$\begin{aligned} W(t^*, U(t^*)) &\leq r(t^*, t_0, W(t_0^+, V(t^*, t_0, U_0))) \leq r(t^*, t_0, a(t_0, h(t_0^+, V(t^*, t_0, U_0))) \\ &\leq r(t^*, t_0, \delta_1) < b(\epsilon). \end{aligned} \tag{16}$$

On the other hand, it follows from (11) that

$$W(t^*, U(t^*)) \geq b(h(t^*, U(t^*))) = b(\epsilon),$$

which contradicts (15).

(ii) In view of the impulse effect, we have

$$h(t^*, U(t^*)) \geq \epsilon \text{ and } h(t, U(t)) < \epsilon, \ t \in [t_0, t_k].$$

Since $0 < \epsilon < \rho_0$, it follows from assumption (B_5) that

$$h(t_k^+, U(t_k^+)) = h(t_k^+, U(t_k) + I_k(U(t_k))) < \rho.$$

Consequently, there exists a $t^{**} \in (t_k, t^*]$ such that

$$\epsilon \leq h(t^{**}, U(t^{**})) < \rho \text{ and } h(t, U(t)) < \rho, \ t \in [t_0, t_1) \tag{17}$$

Now, by virtue of Lemma 2.1 and (11)-(12), we obtain

$$\begin{aligned} W(t^{**}, U(t^{**})) &\leq r(t^{**}, t_0, W(t_0^+, V(t^{**}, t_0, U_0))) \leq r(t^{**}, t_0, a(t_0, h(t_0^+, V(t^{**}, t_0, U_0))) \\ &\leq r(t^{**}, t_0, \delta_1) < b(\epsilon), \end{aligned}$$

whereas (11) and (17) yields

$$W(t^{**}, U(t^{**})) \geq b(h(t^{**}, U(t^{**}))) \geq b(\epsilon),$$

which is again a contradiction. Thus our assertion is true and the (h_0, h)-stability of the system (8) is proved.

Next it is assumed that the null solution of (10) is asymptotically stable. In view of (h_0, h)-stability of the system (8), we set $\epsilon = \rho_0$ and $\delta = \delta_3 = \delta_3(t_0, \rho_0) > 0$ in (15) and obtain

$$h(t, U(t)) < \rho_0 < \rho \text{ whenever } h_0(t_0^+, U_0)) < \delta_3,\ t \geq t_0.$$

In order to prove the (h_0, h)-attractive of system (8), let the null solution of (10) be attractive, that is, for $t_0 \in R_+$, there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that

$$x_0 < \delta_0^* \text{ implies that } \lim_{t\to\infty} x(t, t_0, x_0) = 0.$$

Now, for this δ_0^*, there is a $\delta_1^* = \delta_1^*(t_0, \delta_0^*) > 0$ such that

$$h_0(t_0^+, U_0) < \delta_1^* \text{ implies that } h_0(t_0^+, V(t)) < a^{-1}(t_0, \delta_0^*).$$

Taking $\delta_0 = \delta_0(t_0)$ (independent of ϵ) such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and applying the earlier arguments, we find that

$$b(h(t, U(t))) \leq W(t, U(t)) \leq r(t, t_0, W(t_0^+, V(t, t_0, U_0))) \leq r(t, t_0, \delta_0^*) \to 0,$$

as $t \to \infty$ when $h_0(t_0^+, U_0)) < \delta_0$. This implies that $\lim_{t\to\infty} h(t, U(t)) = 0$ when $h_0(t_0^+, U_0)) < \delta_0$, that is, system (8) is (h_0, h)-attractive. Hence system (8) is (h_0, h)-asymptotically stable. □

Theorem 3.2. *Assume that all the assumptions of Theorem* 3.1 *hold except* (B_3) *and* (B_4) *which are modified as*

($\mathbf{B_3^*}$) h_0 *is uniformly finer than* h *instead of finer in* (B_3)*;*
($\mathbf{B_4^*}$) W *is* h_0*-decrescent instead of weakly* h_0*-decrescent in* (B_4)*.*

Then the uniform stability of the null solution of (9) *and the uniformly asymptotically stability of the null solution of* (10) *imply the* (h_0, h)*-uniformly asymptotically stability of* (8)*.*

Proof. From (B_3^*) and (B_4^*), it follows that there exists a $\lambda_0 > 0$ and a, $\phi \in \mathcal{K}$ such that

$$h(t, U) \leq \phi(h_0(t, U)) \text{ implies that } W(t, U) \leq a(h_0(t, U)), \tag{18}$$

when $h_0(t, U) < \lambda_0$ with $\phi(\lambda_0) < \rho$. The null solution of (10) is uniformly stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(\epsilon) > 0$ independent of t_0 such that

$$0 \leq x_0 < \delta_1 \text{ implies that } x(t, t_0, x_0) < b(\epsilon),\ t \geq t_0, \tag{19}$$

where $0 < \epsilon < \rho_0$ and $t_0 \in R_+$. From the hypothesis that the trivial solution of (9) is uniformly stable, for the above δ_1, there exists a $\delta_2 = \delta_2(\epsilon) > 0$ independent of t_0 such that

$$\|U_0\| < \delta_2 \text{ implies that } \|V(t)\| < a^{-1}(\delta_1),$$

while from (B_3^*), we have

$$h_0(t_0^+, U_0) < \delta_2 \text{ implies that } h_0(t_0^+, V(t)) < a^{-1}(\delta_1). \tag{20}$$

Now, applying the arguments similar to the ones used in the proof of Theorem 3.1, we conclude that

$$h_0(t_0^+, U_0) < \delta \text{ implies that } h(t_0^+, U(t)) < \epsilon,\ t \geq t_0,$$

where δ is independent of t_0 and satisfies $0 < \delta = \delta(\epsilon) < \min\{\lambda_0, \delta_2\}$. Thus, the system (8) is (h_0, h)-uniformly stable.

Next, from the hypothesis that the null solution of (10) is uniformly asymptotically stable, we can find a $\delta_0^* > 0$ independent of t_0 and any ϵ satisfying $0 < \epsilon < \rho_0$ such that there exists a $\tau = \tau(\epsilon)$ so that

$$0 < x_0 < \delta_0^* \text{ implies that } x(t, t_0, x_0) < b(\epsilon),\ t \geq t_0 + \tau(\epsilon),\ t_0 \in R_+. \tag{21}$$

In view of that fact that (9) is uniformly stable, there is a δ_1^* independent of t_0 corresponding to δ_0^* such that

$$h_0(t_0^+, U_0) < \delta_1^* \text{ implies that } h_0(t_0^+, V(t)) < a^{-1}(\delta_0^*).$$

Since uniformly asymptotically stability of (10) implies its asymptotically stability, so system (8) is (h_0, h)-uniformly stable. For $\epsilon = \rho_0$, there exists a $\delta^* = \delta^*(\rho_0)$ such that

$$h_0(t_0^+, U_0) < \delta^* \text{ implies that } h(t, U(t)) < \rho_0 < \rho,\ t \geq t_0.$$

Choosing δ_0 such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and using the arguments employed in Theorem 3.1, we find that $h(t, U(t))) \leq \epsilon$, $t \geq t_0 + \tau$, when $h_0(t_0^+, U_0)) < \delta_0$, where δ_0 and τ are independent of t_0. This implies that system (8) is (h_0, h)-uniformly attractive. Hence system (8) is (h_0, h)-uniformly asymptotically stable. □

4. CONCLUSIONS

(**A**) The stability criteria in term of two measures (h_0, h) enable us to unify a variety of stability notions found in the literature if we endow h_0, h with explicit form. In fact, our Definition 2.7, for instance, takes the form of

- **(a)** the well known stability of the trivial solution $U(t) = \theta$ of (8) if $h(t, U) = h_0(t, U) = D[U, \theta]$, $U \in K_c(R^n)$;
- **(b)** the stability of the prescribed motion $U_0(t)$ of (8) if $h(t, U) = h_0(t, U) = D[U, U_0(t)]$;
- **(c)** the stability of the invariant set $\Omega \subset K_c(R^n)$ if $h(t, U) = h_0(t, U) = D_0[U, \Omega]$, where $D_0[U, \Omega]$ is the distance function of U from the set Ω;
- **(d)** the stability of an asymptotically invariant set Ω if $h(t, U) = h_0(t, U) = D[U, \Omega] + \psi(t)$, where $\psi(t) > 0$ is a decreasing function such that $\psi(t) \to 0$ as $t \to \infty$:

(e) the stability of a conditionally invariant set Ω_1 with respect to Ω_2 ($\Omega_2 \subset \Omega_1 \subset K_c(R^n)$) if $h(t,U) = D_0[U,\Omega_1], h_0(t,U) = D_0[U,\Omega_2]$;

(f) the orbital stability if $h(t,U) = h_0(t,U) = D[U, B(t_0,W_0)]$, where $B(t_0,W_0) = U_0([t_0,\infty), t_0, W_0)$ is a closed set in $K_c(R^n)$ and $U_0(t,t_0,W_0)$ is a prescribed solution of (8);

(g) partial stability if $h(t,U_1) = D[U_1,\theta]$, $h_0(t,U) = D[U,\theta]$, where U_1 is compact convex subset of $U \in K_c(R^n)$.

For several other definitions and stability results in terms of two measures for ordinary differential systems, we refer the reader to [12].

(B) The (h_0,h)-equatability of (8) can be established on the same pattern if we require $\delta = \delta(t_0,\epsilon)$ in Definition 2.7 to be a continuous function in t_0 for each ϵ.

(C) Setting $L_1U \equiv 0 \equiv L_2U$, our results reduce to ones corresponding to setvalued perturbed hybrid differential equations with fixed moments of impulse. Moreover, if the solution $U(t)$ of (8) is a single valued mapping, and Hukuhara derivative and integral used here reduce to the ordinary vector derivative and integral, then we get the results obtained in [15].

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