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A BACKWARDS STOCHASTIC DIFFERENTIAL EQUATION MODEL IN LIFE INSURANCE

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ABSTRACT. In this article we obtain the ratio of risk investment and the optimal accumulated level of single premium endowment insurance in the case of dynamic investment strategies of life insurance company by BSDEs. It gives an illustration of traditional reserve valuation, and prudential rules.

Keywords. BSDE, liability reserve, assessed interest rate

AMS(MOS) Subject Classification. 34F05,60H10

1. INTRODUCTION

One of the main tasks of an actuary in the life insurance company is to decide the discounting rate for income and outgo in pricing and valuation. *In practice, usually the discounting rate assumption for the evaluation of liability reserve is more conservative, which always based on the interest rate of the riskless asset.* The aim of this article is to find a theoretical support for this rule in practice.

The traditional actuarial model is a simplified version of discounted cash flow, and it doesn't consider the dynamic adjustment to the investment portfolio. Until 1980, the stochastic control model has been applied in the fundamental research of insurance systems and pension funds (Haberman and Sung 2002). At the same time, the researchers also realized that the usual insurance system and the pension fund can all be understood as PRE-FUNDED SYSTEM (Taylor 2002), and have some of the same aspects in model description and the application of the results. The theory of backwards stochastic differential equation, abbreviated BSDE, is a technique developed in the last two decades and quite applicable to the financial market research. The study of its application in insurance system and pension fund is, however, at the very early stage of development. Under such a background, it is very important to find new ideas or deal with more complicated model via BSDE in its study. *In*

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the evaluation of liability reserve, the goal of the insurance fund is to pay the death benefit at the end of each policy year (the insurance fund accrue the death benefit and expiration benefit at the end of the n-th policy year), and the insurance fund is invested in the riskless asset and the risk asset. Our aim is to find the value of the insurance fund at the beginning of the plan. This problem is a backward problem and is suitable dealt with BSDE In this article we obtain the ratio of risk investment and the optimal accumulated level of endowment life insurance fund with a single premium in the case of dynamic investment strategies. It gives an illustration of traditional liability reserve valuation, and prudential rules.

2. MODELS AND ASSUMPTIONS

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space supporting all our random/stochastic quantities and $\{B_t\}_{t\geq 0}$ is a Brownian motion defined on a probability space$(\Omega, \mathcal{F}, P)$. Indeed $\{B_t\}_{t\geq 0}$ is a $\mathcal{F}_t$-martingale, where $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

We use the organizational stochastic model to consider the n-year endowment life insurance: *All the participants enter the plan at age x.* l_x is the number of homogenous policies, the initial fund is zero at time zero, we do not consider expenses and surrender. Benefit occurs at the end of the death year. h is the year of policies, $h = 1, 2, \cdots, n$. *D_{x+h-1} means the number of deaths and L_{x+h} means the number of survival in each policy year h. The number of deaths D_{x+h-1} and the number L of last survival stochastic vector is* $D = (D_x, D_{x+1}, \cdots, D_{x+n-1}, L_{x+n-1})$; D obeys the multi-normal distribution. We also assume that single premium policies group is self-financing. u_t is the insurance fund at time t. The fund is a right-continuous stochastic process and is subject to jump at the end of each death year due to payments on deaths. The goal of the insurance fund at the end of n-th year is expiration benefit of L_{x+n} and death benefit of D_{x+n-1}, it is suitable to be dealt with by BSDE.

Next we give some assumptions that will be in force below:

(I) Life insurance company can choose investment proportion in two assets: risk-free asset and risky asset. Let v_t denote the amount of money invested in risky asset. It represents the dynamic investment strategy and can change continuously.

(II) The *interest rate* of riskless asset is $r(t)$; here, we consider only the deterministic $r(t)$.

(III) The instantaneous yield of risky asset is governed by

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t),$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are continuous functions.

(IV) Assume that the survival random vector D is independent of $\{B_t\}_{t\geq 0}$.

(V) $\dfrac{\mu(t)-r(t)}{\sigma(t)} \in L^2$ on the interval $[0,n)$.

These assumptions are in common use.

Let

$$\begin{aligned}\mathcal{R}_i &= \sigma\{D_x, D_{x+1}, \cdots, D_{x+i-1}\}, \qquad i=1,2,\cdots,n-1,\\ \mathcal{R}_n &= \sigma\{D_x, D_{x+1}, \cdots, D_{x+n-1}, L_{x+n}\},\end{aligned}$$

and

$$\mathcal{M}_t = \begin{cases} \mathcal{F}_t, & 0 \le t < 1,\\ \sigma(\mathcal{F}_t \cup \mathcal{R}_1), & 1 \le t < 2,\\ \quad\vdots & \\ \sigma(\mathcal{F}_t \cup \mathcal{R}_{n-1}), & n-1 \le t < n,\\ \sigma(\mathcal{F}_t \cup \mathcal{R}_n), & t=n.\end{cases}$$

We now present our first result which deals with preservation of the martingale property while enlarging the filtration. While this is a delicate mathematical technicality, we nevertheless relegate the proof to the appendix so as not to break the continuity of the underlying theme of insurance.

Lemma 2.1. $\{B_t\}_{t\ge 0}$ *is a martingale with respect to* $\mathcal{M}_t$.

3. INSURANCE FUND

Since the goal of the insurance fund is to pay the death benefit at the end of each policy year (the insurance fund accrue the death benefit and expiration benefit at the end of the n-th policy year), and the insurance fund is invested in the riskless asset and the risk asset, see the assumptions in previous section, then the insurance fund in the n-th year satisfies the following SDE:

$$\begin{cases} du_t = [r(t)u_t + (\mu(t)-r(t))\nu_t]dt + \sigma(t)\nu_t dB_t \\ u_n = L_{x+n} + D_{x+n-1} \qquad n-1 \le t \le n \end{cases} \tag{3.1}$$

Remark 3.1. Here the equation is on the interval $n-1 \le t \le n$. For the calculation of the insurance is a backward process, we should calculate it from the last policy year.

In the following we investigate the expression of u_t.

Let $-\sigma(t)\nu_t = z_t$ in equation (3.1), then

$$\begin{cases} du_t = [r(t)u_t - \dfrac{\mu(t)-r(t)}{\sigma(t)} z_t]dt - z_t dB_t \\ u_n = L_{x+n} + D_{x+n-1} \end{cases} \tag{3.2}$$

From [7] we know that there exits a unique pair of solution of (3.2).

For any $t \in [n-1, n]$, suppose y_t^s, $t \leq s$, is the solution of the following SDE

$$\begin{cases} dy_t^s = y_t^s[-r(s)ds - \dfrac{\mu(s)-r(s)}{\sigma(s)}dB_s], \\ y_t^t = 1. \end{cases} \tag{3.3}$$

First of all, we give the expression of the solution of (3.3). By Ito's formula

$$\begin{aligned} d\ln y_t^s &= \frac{1}{y_t^s}dy_t^s - \frac{1}{2}\frac{1}{(y_t^s)^2}(dy_t^s)^2 \\ &= -r(s)ds - \frac{\mu(s)-r(s)}{\sigma(s)}dB_s - \frac{1}{2}\frac{\mu(s)-r(s)}{\sigma(s)}^2 ds. \end{aligned}$$

Hence

$$y_t^s = \exp\left\{\int_t^s [-r(\tau) - \frac{1}{2}\frac{\mu(\tau)-r(\tau)}{\sigma(\tau)}^2]d\tau - \int_t^s [\frac{\mu(\tau)-r(\tau)}{\sigma(\tau)}]dB_\tau\right\}.$$

Obviously, for any $t_1 \leq t_2 \leq t_3$, we have $y_{t_1}^{t_2} y_{t_2}^{t_3} = y_{t_1}^{t_3}$. Moreover, by assumption (V),

$$\int_t^s \frac{\mu(\tau)-r(\tau)}{\sigma(\tau)}dB_\tau$$

is a martingale with respect to $\mathcal{F}_t$ and satisfies the Novikov condition, *(8) see* [4]. Then we have

$$E\Big[\exp\left\{\int_t^s \frac{1}{2}[\frac{\mu(\tau)-r(\tau)}{\sigma(\tau)}]^2 d\tau + \int_t^s \frac{\mu(\tau)-r(\tau)}{\sigma(\tau)}dB_\tau\right\}\Big] = 1.$$

So

$$Ey_t^s = \exp\{\int_t^s [-r(\tau)]d\tau\}.$$

Next we will solve the BSDE (3.2). Once again, by Ito's formula

$$\begin{aligned} d(u_s y_t^s) &= u_s dy_t^s + y_t^s du_s + du_s dy_t^s \\ &= u_s y_t^s[-r(s)ds - \frac{\mu(s)-r(s)}{\sigma(s)}dB_s] - y_t^s z_s dB_s \\ &\quad + y_t^s[r(s)u_s - \frac{\mu(s)-r(s)}{\sigma(s)}z_s]ds + y_t^s z_s \frac{\mu(s)-r(s)}{\sigma(s)}ds \\ &= (-u_s - z_s)y_t^s \frac{\mu(s)-r(s)}{\sigma(s)}dB_s, \end{aligned}$$

and hence

$$u_t = u_n y_t^n + \int_t^n (u_s + z_s)y_t^s \frac{\mu(s)-r(s)}{\sigma(s)}dB_s.$$

By assumptions (IV) and Lemma 2.1, we know that

$$\int_0^t (u_s + z_s)y_t^s \, \frac{\mu(s)-r(s)}{\sigma(s)}dB_s$$

is a martingale w.r.t $\mathcal{M}_t$, and (u_s, z_s) is $\mathcal{M}_t$-adapted. We now get

$$u_t = E[u_n y_t^n | \mathcal{M}_t] = E[(L_{x+n} + D_{x+n-1})y_t^n | \mathcal{M}_t].$$

Then

$$u_{n-1} = E[(L_{x+n} + D_{x+n-1})y^n_{n-1}|\mathcal{M}_{n-1}],$$

The insurance fund at the end of $n-1$-th policy year should pay the death benefit of D_{x+n-2}, so we set

$$u^-_{n-1} = E[(L_{x+n} + D_{x+n-1})y^n_{n-1}|\mathcal{M}_{n-1}] + D_{x+n-2}.$$

In the $(n-1)$-th period, the insurance fund should satisfy the SDE

$$\begin{cases} du_t = [r(t)u_t - \dfrac{\mu(t)-r(t)}{\sigma(t)}z_t]dt - z_t dB_t \\ u^-_{n-1} = E[(L_{x+n} + D_{x+n-1})y^n_{n-1}|\mathcal{M}_{n-1}] + D_{x+n-2}, n-2 \le t < n-1 \end{cases} \tag{3.4}$$

We can see that

$$u^-_{n-1} = \lim_{t\to(n-1)-} u_t$$

Remark 3.2. In the rest of the paper, $u^-_t = \lim_{s\to t-} u_s$, $v^-_t = \lim_{s\to t-} v_s$.

We can similarly obtain that

$$\begin{aligned} u_{n-2} &= E[(u_{n-1} + D_{x+n-2})y^{n-1}_{n-2}|\mathcal{M}_{n-2}] \\ &= E[\big(E[(L_{x+n} + D_{x+n-1})y^n_{n-1}|\mathcal{M}_{n-1}] + D_{x+n-2}\big)\, y^{n-1}_{n-2}|\mathcal{M}_{n-2}] \\ &= E[E[(L_{x+n} + D_{x+n-1})y^n_{n-2}|\mathcal{M}_{n-1}]|\mathcal{M}_{n-2}] + E[D_{x+n-2}y^{n-1}_{n-2}|\mathcal{M}_{n-2}] \\ &= E[(L_{x+n} + D_{x+n-1})y^n_{n-2}|\mathcal{M}_{n-2}] + E[D_{x+n-2}y^{n-1}_{n-2}|\mathcal{M}_{n-2}] \\ &= E[(L_{x+n} + D_{x+n-1})y^n_{n-2} + D_{x+n-2}y^{n-1}_{n-2}|\mathcal{M}_{n-2}]. \end{aligned}$$

More generally we can get

$$u_{n-k} = E[(L_{x+n} + D_{x+n-1})y^n_{n-k} + D_{x+n-2}y^{n-1}_{n-k} + \cdots + D_{x+n-k}y^{n-(k-1)}_{n-k}|\mathcal{M}_{n-k}].$$

$$\begin{aligned} u_{n-k+1} &= E[(u_{n-k} + D_{x+n-(k+1)})y^{n-k}_{n-(k+1)}|\mathcal{M}_{n-k+1}] \\ &= E[\big(E[(L_{x+n} + D_{x+n-1})y^n_{n-k} + D_{x+n-2}y^{n-1}_{n-k} + \cdots + D_{x+n-k}y^{n-(k-1)}_{n-k}|\mathcal{M}_{n-k}] \\ &\quad + D_{x+n-(k+1)}\big)y^{n-k}_{n-(k+1)}|\mathcal{M}_{n-k+1}] \\ &= E[(L_{x+n} + D_{x+n-1})y^n_{n-(k+1)} + D_{x+n-2}y^{n-1}_{n-(k+1)} + \cdots + D_{x+n-k}y^{n-(k-1)}_{n-(k+1)} \\ &\quad + D_{x+n-(k+1)}y^{n-k}_{n-(k+1)}|\mathcal{M}_{n-k+1}]. \end{aligned}$$

Then, by assumption (IV)

$$\begin{aligned} Eu_{n-k} &= \exp\{\int_{n-k}^{n} -r(\tau)d\tau\}E(L_{x+n} + D_{x+n-1}) + \exp\{\int_{n-k}^{n-1} -r(\tau)d\tau\}E(D_{x+n-1}) \\ &\quad + \cdots + \exp\{\int_{n-k}^{n-(k-1)} -r(\tau)d\tau\}E(D_{x+n-k}). \end{aligned}$$

In the case of $r(t) = \sum_{i=1}^{n} r_i I_{[i-1\leq t\leq i]}$, we have

$$\begin{aligned} Eu_{n-k} &= \exp\{-\sum_{i=n-k}^{n} r_i\}E(L_{x+n}+D_{x+n-1}) + \exp\{-\sum_{i=n-k}^{n-1} r_i\}E(D_{x+n-1}) \\ &\quad + \cdots + \exp\{-r_{n-k}\}E(D_{x+n-k}). \end{aligned}$$

In particular, if $r_1 = r_2 = \cdots = r_n = r$ then

$$Eu_{n-k} = e^{-rk}E(L_{x+n}+D_{x+n-1}) + e^{-r(k-1)}E(D_{x+n-1}) + \cdots + e^{-r}E(D_{x+n-k}). \tag{3.5}$$

This is in accordance with the traditional formula in single premium.

Finally, we establish a more precise expression for the insurance fund.

Theorem 3.3.

$$\begin{aligned} u_{n-k} &= \exp\{\int_{n-k}^{n} -r(\tau)d\tau\}E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}] \\ &\quad + \exp\{\int_{n-k}^{n-1} -r(\tau)d\tau\}E[D_{x+n-1}|\mathcal{R}_{n-k}] + \cdots \\ &\quad + \exp\{\int_{n-k}^{n-(k-1)} -r(\tau)d\tau\}E[D_{x+n-k}|\mathcal{R}_{n-k}]. \end{aligned}$$

Proof. We shall only establish

$$E[(L_{x+n}+D_{x+n-1})y_{n-k}^{n}|\mathcal{M}_{n-k}] = \exp\{\int_{n-k}^{n} -r(\tau)d\tau\}E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}]. \tag{3.6}$$

To prove it, we need to show that for any $C \in \mathcal{M}_{n-k}$,

$$E[(L_{x+n}+D_{x+n-1})y_{n-k}^{n}I_C] = \exp\{\int_{n-k}^{n}[-r(\tau)]d\tau\}E[I_C E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}]].$$

Let

$$\begin{aligned} \mathcal{L}_{n-k} &= \{A\cap B : A \in \mathcal{F}_{n-k}, B \in \mathcal{R}_{n-k}\} \\ \mathcal{H}_{n-k} &= \{C \in \mathcal{M}_{n-k} : E[(L_{x+n}+D_{x+n-1})y_{n-k}^{n}I_C] = \\ &\quad \exp\{\int_{n-k}^{n}[-r(\tau)]d\tau\}E[I_C E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}]]\}. \end{aligned}$$

For any $A\cap B \in \mathcal{L}_{n-k}$, we have

$$\begin{aligned} &E[(L_{x+n}+D_{x+n-1})y_{n-k}^{n}I_{A\cap B}] \\ &\quad = E[(L_{x+n}+D_{x+n-1})y_{n-k}^{n}I_A I_B] \\ &\quad = E(I_A)E[(L_{x+n}+D_{x+n-1})I_B]E(y_{n-k}^{n}) \\ &\quad = E(I_A)E[I_B E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}]]E(y_{n-k}^{n}) \\ &\quad = \exp\{\int_{n-k}^{n}[-r(\tau)]d\tau\}E[I_{A\cap B}E[L_{x+n}+D_{x+n-1}|\mathcal{R}_{n-k}]]. \end{aligned}$$

As in the proof of lemma 2.1, we note that $\mathcal{H}_{n-k}$ is a λ-system. Then by the monotone class theorem we get (3.6). $\square$

From Theorem 3.3, we can see that in the assumption (IV), the insurance fund in every period is not related to the risky asset and is determined by the risk-free market, D, and L_{x+n}.

4. RATIO OF RISKY INVESTMENT

Let $v_t = \alpha_t u_t$, where α_t is the ratio of risk investment $0 \leq \alpha_t \leq 1$ a.s. In the $(n-k)$-th period, use Ito's formula to $\frac{u_t}{v_t}$, $n-k-1 \leq t < n-k$, $\alpha^-_{n-k} = \frac{v^-_{n-k}}{u^-_{n-k}}$. Then

$$d\frac{1}{\alpha_t} = d\frac{u_t}{v_t} = \frac{1}{v_t}du_t + u_t d\frac{1}{v_t} + d\frac{1}{v_t}du_t,$$

and

$$\begin{aligned} d\frac{1}{v_t} &= -\frac{1}{{v_t}^2}dv_t + \frac{1}{{v_t}^3}d{v_t}^2 \\ &= \frac{1}{v_t}(\sigma^2(t) - \mu(t))dt - \frac{1}{v_t}\sigma(t)dB_t. \end{aligned}$$

Hence

$$\begin{aligned} d\frac{1}{\alpha_t} &= \frac{1}{\alpha_t}[\sigma^2(t) - \mu(t)]dt - \frac{1}{\alpha_t}\sigma(t)dB_t \\ &\quad + \frac{1}{\alpha_t}\{[r(t) - \alpha_t(\mu(t) - r(t)) - \sigma^2(t)]dt + \sigma(t)dB_t\} \\ &= (\frac{1}{\alpha_t} - 1)(\sigma^2(t) - \mu(t) + r(t))dt - (\frac{1}{\alpha_t} - 1)\sigma(t)dB_t. \end{aligned}$$

Let $k_t = \frac{1}{\alpha_t} - 1$. Now,

$$d\ln k_t = (\frac{1}{2}\sigma^2(t) - \mu(t) + r(t))dt - \sigma(t)dB_t,$$

and

$$k_t = k^-_{n-k}\exp\{\int_t^{n-k}(\mu(s) - \frac{1}{2}\sigma^2(s) - r(s))ds + \int_t^{n-k}\sigma(s)dB_s\}.$$

Therefore

$$\alpha_t = \frac{1}{1 + (\frac{1}{\alpha^-_{n-k}} - 1)\exp\{\int_t^{n-k}(\mu(s) - r(s) - \frac{1}{2}\sigma^2(s))ds + \int_t^{n-k}\sigma(s)dB_s\}}.$$

Obviously

$$d\ln u_t\alpha_t = d\ln v_t = (\mu(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dB_t.$$

Then

$$\frac{\alpha^-_{n-k}}{\alpha_{n-k-1}} = \frac{E[u^-_{n-k}|\mathcal{M}_{n-k-1}]}{u^-_{n-k}}\exp\left\{\int_{n-k-1}^{n-k}(\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k}\sigma(t)dB_t\right\}.$$

On the other hand

$$\frac{\alpha^-_{n-k}}{\alpha_{n-k-1}} = \alpha^-_{n-k} + (1-\alpha^-_{n-k}) \exp\left\{\int_{n-k-1}^{n-k} (\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k} \sigma(t)dB_t\right\}.$$

So

$$\alpha^-_{n-k} = \frac{\left(\frac{E[u^-_{n-k}|\mathcal{M}_{n-k-1}]}{u^-_{n-k}} - 1\right) \exp\left\{\int_{n-k-1}^{n-k}(\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k}\sigma(t)dB_t\right\}}{1 - \exp\left\{\int_{n-k-1}^{n-k}(\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k}\sigma(t)dB_t\right\}}.$$

By Theorem 3.3, we get

$$\alpha^-_{n-k} = \frac{\frac{E[u_{n-k}|\mathcal{R}_{n-k-1}] - u_{n-k}}{u_{n-k} + D_{x+n-k-1}} \exp\left\{\int_{n-k-1}^{n-k}(\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k}\sigma(t)dB_t\right\}}{1 - \exp\left\{\int_{n-k-1}^{n-k}(\mu(t) - \frac{1}{2}\sigma^2(t) - r(t))dt + \int_{n-k-1}^{n-k}\sigma(t)dB_t\right\}}.$$

Thus we get the proportion of the fund applied to risky investments.

5. DISCUSSION

The Relation (3.5) is consistent with the traditional liability reserve valuation formula, but it only assumes that the number of deaths in each of the past years is a random variable in the deduction in traditional formula, and the discounting interest rate is any fixed effective annual interest rate. Making use of BSDE and more practical assumption, we illuminate that the liability evaluation formula that is derived from the predigested assumption is optimal.

In the case of life insurance companies adopting dynamic investment strategies, we can see that the riskless investment yield rate as assessment interest rate is identical with the usual rules. And we also illustrate that, in self-financing, the insurance fund is not related to the risky investment rate, it is only related to the riskless investment rate, the number of death and the valid policies at the end of policy year.

6. APPENDIX

Proof of Lemma 2.1:

For any fixed t and $s \le t$, we assume, without loss of generality, that $k \le s \le k+1$ and $k = 0, 1, \cdots, n$. Let

$$\mathcal{L}_s = \{A \cap B : A \in \mathcal{F}_s, B \in \mathcal{R}_k\}, \quad \mathcal{H}_s = \{A \in \mathcal{M}_s : E[I_A B_t] = E[I_A B_s]\},$$

as we know that $\mathcal{F}_s$ and $\mathcal{R}_k$ are all σ-algebras, it is easy to see that $\mathcal{L}_s$ is a π-system; $\mathcal{F}_s \subset \mathcal{L}_s$ and $\mathcal{R}_k \subset \mathcal{L}_s$. So $\sigma(\mathcal{L}_s) = \mathcal{M}_s$.

Next we prove that $\mathcal{H}_s$ is a $\lambda-$system. In fact,

1) $\Omega \in \mathcal{H}_s$,

2) for any $A_1, A_2 \cdots \in \mathcal{H}_s$ with $A_i \uparrow A,\ as\ i \to \infty$, we have

$$\begin{aligned} |E[I_A B_t - I_A B_s]| &= |E[I_A B_t - I_{A_i} B_t + I_{A_i} B_t - I_A B_s]| \\ &= |E[I_A B_t - I_{A_i} B_t + I_{A_i} B_s - I_A B_s]| \\ &\leq E|I_{A \setminus A_i} B_t| + E|I_{A \setminus A_i} B_s| \\ &\leq P(A \setminus A_i) E^{\frac{1}{2}}[B_t^2] + P(A \setminus A_i) E^{\frac{1}{2}}[B_s^2] \\ &= (\sqrt{t} + \sqrt{s}) P(A \setminus A_i) \to 0\ (i \to \infty), \end{aligned}$$

then $E[I_A B_t] = E[I_A B_s]$, that is $A \in \mathcal{H}_s$.
3) $\forall A, B \in \mathcal{H}_s$ with $A \subset B$, and $B \setminus A \neq \phi$

$$E[I_{B \setminus A} B_t] = E[I_B B_t] - E[I_A B_s] = E[I_{B \setminus A} B_s]$$

that is $B \setminus A \in \mathcal{H}_s$, so $\mathcal{H}_s$ is a λ-system. $\forall A \cap B \in \mathcal{L}_s$, we have

$$E[I_{A \cap B} B_t] = E[I_A I_B B_t] = E[I_B] E[I_A B_t] = E[I_B] E[I_A B_s] = E[I_{A \cap B} B_s]$$

therefore $\mathcal{L}_s \subset \mathcal{H}_s$.

By the monotone class argument, we know that $\sigma(\mathcal{L}_s) \subset \mathcal{H}_s$, and $\mathcal{H}_s \subset \mathcal{M}_s$, $\sigma(\mathcal{L}_s) = \mathcal{M}_s$ so $\mathcal{M}_s = \mathcal{H}_s$, that is $\{B_t\}_{t \geq 0}$ is a martingale w.r.t $\mathcal{M}_t$.

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