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INTEGRO-DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

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ABSTRACT. We apply Lakshmikantham's generalized quasilinearization method to an initial value problem involving a nonlinear integro-differential equation with initial time difference and obtain monotone sequences of lower and upper solutions converging uniformly and quadratically to the unique solution of the problem.

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1. INTRODUCTION

The method of quasilinearization developed by Bellman and Kalaba [1] and generalized by Lakshmikantham [2-3] later on, has been studied and extended in several diverse disciplines. In fact, it is generating a rich history and an extensive bibliography can be found in [4-10].

In the study of initial value problems involving nonlinear differential equations, we generally perturb or change the dependent (spatial) variable keeping the initial time unchanged. However, this approach is not realistic in the sense that it is impossible not to make the errors in the starting time [11] as the solution of unperturbed dynamical system may start at a different time in comparison with the perturbed dynamical system. Recently, the concept of changing initial time along with the dependent variable has been initiated and some results on the comparison theorems, global existence, stability criteria, the method of upper and lower solutions, monotone iterative technique, etc. can be found in the references [12-15].

In view of the extensive occurrence of the integro-differential equations in the mathematical modelling of physical problems, for example, see [16-19], the theory and applications of integro-differential equations have emerged as a new area of investigation [4, 20-23].

The objective of the present study is to develop the generalized quasilinearization

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method for an initial value problem involving Volterra integro-differential equations with initial time difference. In fact, we consider the following initial value problem

$$u'(t) = Af(t,u(t)) + B\int_{t_0}^{t} K(t,s,u(s))ds,\ t \in J = [t_0, t_0+T],\ t_0 \in R_+,\ T > 0,$$

$$u(t_0) = u_0, \tag{1}$$

where A, B are nonnegative real constants and obtain sequences of upper and lower solutions for the integro-differential equation that start at different initial times and bound the solution of the given nonlinear problem. It has also been shown that the lower and upper sequences of approximate solutions converge monotonically and quadratically to the unique solution of the problem (1).

2. PRELIMINARIES

Now, we state some important theorems which lay the foundation to prove the main result (for the proof of Theorem 1, one can apply the methodology of reference [14] while Theorem 2 is a known result [4]).

Theorem 1. Assume that

(**a**) Let $f \in C[R_+ \times R, R], K \in C[R_+ \times R_+ \times R, R], \alpha \in C^1[[t_0, t_0+T], R]$ and $\beta \in C^1[[\tau_0, \tau_0+T], R],\ \tau_0 > 0$ be such that

$$\alpha'(t) \le Af(t,\alpha(t)) + B\int_{t_0}^{t} K(t,s,\alpha(s))ds,\ t \in [t_0, t_0+T],$$

and

$$\beta'(t) \ge Af(t,\beta(t)) + B\int_{t_0}^{t} K(t,s,\beta(s))ds,\ t \in [\tau_0, \tau_0+T],$$

with

$$\beta(\tau_0) \ge \alpha(t_0).$$

(**b**) $f(t,u(t))$ is nondecreasing in t for each u and $K(t,s,u(s))$ is nondecreasing in t for each fixed $(s,u(s))$ and $\alpha(t) \le \beta(t+\eta)$ for $t \in [t_0, t_0+T],\ \eta = \tau_0 - t_0$ with $t_0 \le \tau_0$.

Then there exists a solution $u(t)$ of (1) such that $\alpha(t) \le u(t) \le \beta(t+\eta)$ for $t \in [t_0, t_0+T]$.

Theorem 2. Suppose that the following hold

(**a**) Let $f \in C[J \times R, R]$ and $K \in C[J \times J \times R, R]$ be such that $K(t,s,u(s))$ is monotone nondecreasing in u for each fixed $(t,s) \in J \times J$.

(**b**) Let $u, v \in C^1[J, R]$ be such that

$$u'(t) \le Af(t, u(t)) + B \int_{t_0}^{t} K(t, s, u(s))ds,\ t \in J,$$

$$v'(t) \ge Af(t, v(t)) + B \int_{t_0}^{t} K(t, s, v(s))ds,\ t \in J.$$

(**c**) $f(t,u) - f(t,v) \le \epsilon_1(u-v),\ K(t,s,u) - K(t,s,v) \le \epsilon_2(u-v)$, where $t \in J,\ (t,s) \in J \times J,\ u \ge v,\ \epsilon_1 \ge 0,\ \epsilon_2 \ge 0$.

Then $u(t) \le v(t)$ on J provided $u(t_0) \le v(t_0)$.

3. MAIN RESULT

Theorem 3. Assume that

($\mathbf{A_1}$) Let $f \in C[R_+ \times R, R], K \in C[R_+ \times R_+ \times R, R], \alpha \in C^1[[t_0, t_0+T], R]$ and $\beta \in C^1[[\tau_0, \tau_0 + T], R],\ \tau_0 > 0$ be such that

$$\alpha'(t) \le Af(t, \alpha(t)) + B \int_{t_0}^{t} K(t, s, \alpha(s))ds,\ t \in [t_0, t_0 + T],$$

and

$$\beta'(t) \ge Af(t, \beta(t)) + B \int_{t_0}^{t} K(t, s, \beta(s))ds,\ t \in [\tau_0, \tau_0 + T],$$

with

$$\beta(\tau_0) \ge \alpha(t_0).$$

($\mathbf{A_2}$) $f(t, u(t))$ is nondecreasing in t for each u and $K(t, s, u(s))$ is nondecreasing in t for each fixed $(s, u(s))$.

($\mathbf{A_3}$) $f, \phi \in C^{0,2}[[t_0, t_0+T], R]$ be such that $f_{uu}(t,u) + \phi_{uu}(t,u) \ge 0$ with $\phi_{uu}(t,u) \ge 0$ on Ω, where $\Omega = [(t,u) : t_0 \le t \le t_0 + T,\ \overline{\alpha}_0(t) \le \overline{u}(t) \le \overline{\beta}_0(t)]$ and $\overline{\beta}_0(t) = \beta(t + \eta_1),\ \eta_1 = \tau_0 - t_0,\ \overline{\alpha}_0(t) = \alpha(t)$ and $\overline{u}(t)$ is the unique solution of

$$\overline{u}'(t) = Af(t + \eta_2, \overline{u}(t)) + B \int_{t_0}^{t} K(t + \eta_2, s, \overline{u}(s))ds,$$

where $\eta_2 = \zeta_0 - t_0,\ t_0 < \zeta_0 < \tau_0$.

($\mathbf{A_4}$) $K(t, s, u)$ is monotone nondecreasing in u and $K_{uu}(t, s, u) \ge 0$ for each fixed $(t, s) \in J \times J$.

Then, there exist monotone sequences $\{\overline{\alpha}_n(t)\}$ and $\{\overline{\beta}_n(t + \eta\}$ of solutions which converge uniformly to the unique solution of (1) with $u(\zeta_0) = u_0$ on $[\zeta_0, \zeta_0 + T]$ and the convergence is quadratic.

Proof. From assumption (A_3), $\overline{\beta}_0(t) = \beta(t+\eta_1)$ so that $\overline{\beta}_0(t_0) = \beta(\tau_0) \geq \alpha(t_0) = \overline{\alpha}_0(t_0)$ and

$$\begin{aligned}\overline{\beta}_0'(t) &= \beta'(t+\eta_1)\\ &\geq Af(t+\eta_1, \beta(t+\eta_1)) + B\int_{t_0}^{t} K(t+\eta_1, s, \beta(s+\eta_1))ds\\ &= Af(t+\eta_1, \overline{\beta}_0(t)) + B\int_{t_0}^{t} K(t+\eta_1, s, \overline{\beta}_0(s))ds, \ t_0 \leq t \leq t_0+T.\end{aligned}$$

It also follows from (A_3) and (A_4) that $f(t,u)-f(t,v) \leq \epsilon_1(u-v), \ \epsilon_1 \geq 0, \ K(t,s,u)-K(t,s,v) \leq \epsilon_2(u-v), \ \epsilon_2 \geq 0$ whenever $\overline{\alpha}_0(t) \leq v \leq u \leq \overline{\beta}_0(t)$, for $t_0 \leq t, s \leq t_0+T$. Moreover, introducing $F(t,u) = f(t,u) + \phi(t,u)$, we find that

$$f(t,u) \geq F(t,v) + F_u(t,v)(u-v) - \phi(t,u), \tag{2}$$

$$K(t,s,u) \geq K(t,s,v) + K_u(t,s,v)(u-v). \tag{3}$$

Let $\overline{\alpha}_1$ and $\overline{\beta}_1$ be the solutions of the following initial value problems

$$\begin{aligned}\overline{\alpha}_1'(t) &= A[f(t+\eta_2, \overline{\alpha}_0) + \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_1 - \overline{\alpha}_0)]\\ &+ B\int_{t_0}^{t}[K(t+\eta_2, s, \overline{\alpha}_0(s)) + K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_1(s) - \overline{\alpha}_0(s))]ds,\end{aligned}$$

$$\overline{\alpha}_1(t_0) = u_0, \tag{4}$$

and

$$\begin{aligned}\overline{\beta}_1'(t) &= A[f(t+\eta_2, \overline{\beta}_0) + \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\beta}_1 - \overline{\beta}_0)]\\ &+ B\int_{t_0}^{t}[K(t+\eta_2, s, \overline{\beta}_0(s)) + K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\beta}_1(s) - \overline{\beta}_0(s))]ds,\end{aligned}$$

$$\overline{\beta}_1(t_0) = u_0, \tag{5}$$

where $\overline{\alpha}_0(t) \leq u_0 \leq \overline{\beta}_0(t)$. Now we prove that

$$\overline{\alpha}_0(t) \leq \overline{\alpha}_1(t) \leq \overline{\beta}_1(t) \leq \overline{\beta}_0(t), \tag{6}$$

in three steps:

(**i**) Set $p(t) = \overline{\alpha}_0(t) - \overline{\alpha}_1(t)$. Clearly $p(t_0) \leq 0$. Then

$$\begin{aligned}p'(t) &= \overline{\alpha}_0'(t) - \overline{\alpha}_1'(t)\\ &\leq Af(t, \overline{\alpha}_0) + B\int_{t_0}^{t} K(t,s,\overline{\alpha}_0(s))ds\\ &- A[f(t+\eta_2, \overline{\alpha}_0) - \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}p(t)]\\ &- B\int_{t_0}^{t}[K(t+\eta_2, s, \overline{\alpha}_0(s)) - K_u(t+\eta_2, s, \overline{\alpha}_0(s))p(s)]ds,\\ &\leq A[F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)]p(t) + B\int_{t_0}^{t} K_u(t+\eta_2, s, \overline{\alpha}_0(s))p(s)ds,\end{aligned}$$

which, in view of Theorem 2, gives $p(t) \leq 0$, that is, $\overline{\alpha}_0(t) \leq \overline{\alpha}_1(t)$ on J.

(**ii**) Next, we set $p(t) = \overline{\alpha}_1(t) - \overline{\beta}_0(t)$ and note that $p(t_0) \leq 0$. Thus

$$\begin{aligned} p'(t) &= \overline{\alpha}_1'(t) - \beta_0'(t) \\ &\leq A[f(t+\eta_2, \overline{\alpha}_0) + \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_1 - \overline{\alpha}_0)] \\ &+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_0(s)) + K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_1(s) - \overline{\alpha}_0(s))]ds \\ &- Af(t+\eta_1, \overline{\beta}_0) - B\int_{t_0}^{t} K(t+\eta_1, s, \overline{\beta}_0(s))ds \\ &\leq A[f(t+\eta_2, \overline{\alpha}_0) - f(t+\eta_2, \overline{\beta}_0(t)) \\ &+ \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_1 - \overline{\alpha}_0)] \\ &+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_0(s)) - K(t+\eta_2, s, \overline{\beta}_0(s)) \\ &+ K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_1(s) - \overline{\alpha}_0(s))]ds \end{aligned}$$

Using (A_3) and (2) together with the fact that $\overline{\beta}_0(t) \geq \overline{\alpha}_0(t)$, we find that

$$f(t+\eta_2, \overline{\alpha}_0) - f(t+\eta_2, \overline{\beta}_0(t)) \leq \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_0 - \overline{\beta}_0),$$

and by virtue of (3), we have

$$K(t+\eta_2, s, \overline{\alpha}_0(s)) - K(t+\eta_2, s, \overline{\beta}_0(s)) \leq K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_0 - \overline{\beta}_0),$$

which, in turn yield that

$$p'(t) \leq A[F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)]p(t) + B\int_{t_0}^{t} K_u(t+\eta_2, s, \overline{\alpha}_0(s))p(s)ds.$$

Hence we obtain that $p(t) \leq 0$, that is, $\overline{\alpha}_1(t) \leq \overline{\beta}_0(t)$ on J. By a similar procedure, we can show that $\overline{\beta}_1(t) \leq \overline{\beta}_0(t)$.

(**iii**) It remains to show that $\alpha_1(t) \leq \beta_1(t)$. For that, we consider

$$\begin{aligned} \overline{\alpha}_1'(t) &= A[f(t+\eta_2, \overline{\alpha}_0) + \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_1 - \overline{\alpha}_0)] \\ &+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_0(s)) + K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_1(s) - \overline{\alpha}_0(s))]ds, \\ &\leq Af(t+\eta_2, \overline{\alpha}_1) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_1(s))ds, \end{aligned}$$

where we have used (2), (3) and $\overline{\alpha}_0(t) \leq \overline{\alpha}_1(t) \leq \overline{\beta}_0(t)$. Similarly, we can prove that

$$\overline{\beta}_1'(t) \geq Af(t+\eta_2, \overline{\beta}_1) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\beta}_1(s))ds.$$

Consequently, by Theorem 2, we have $\overline{\alpha}_1(t) \leq \overline{\beta}_1(t)$ on J. Combining conclusions of (i), (ii) and (iii) proves the validity of (6) on J.

Now, for $n > 1$, we assume that

$$\begin{aligned} \overline{\alpha}_n'(t) &\leq Af(t+\eta_2, \overline{\alpha}_n) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_n(s))ds, \\ \overline{\beta}_n'(t) &\geq Af(t+\eta_2, \overline{\beta}_n) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\beta}_n(s))ds, \end{aligned}$$

and $\overline{\alpha}_0(t) \leq \overline{\alpha}_n(t) \leq \overline{\beta}_n(t) \leq \overline{\beta}_0(t)$ on J. We will show that

$$\overline{\alpha}_n(t) \leq \overline{\alpha}_{n+1}(t) \leq \overline{\beta}_{n+1}(t) \leq \overline{\beta}_n(t), \ t_0 \leq t \leq t_0 + T,$$

where $\overline{\alpha}_{n+1}(t)$ and $\overline{\beta}_{n+1}(t)$ respectively satisfy the following IVPs

$$\begin{aligned} \overline{\alpha}_{n+1}'(t) &= A[f(t+\eta_2, \overline{\alpha}_n) + \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}(\overline{\alpha}_{n+1} - \overline{\alpha}_n)] \\ &+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_n(s)) + K_u(t+\eta_2, s, \overline{\alpha}_n(s))(\overline{\alpha}_{n+1}(s) - \overline{\alpha}_n(s))]ds, \end{aligned}$$

$$\overline{\alpha}_{n+1}(t_0) = u_0, \tag{7}$$

and

$$\begin{aligned} \overline{\beta}_{n+1}'(t) &= A[f(t+\eta_2, \overline{\beta}_n) + \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}(\overline{\beta}_1 - \overline{\beta}_n)] \\ &+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\beta}_n(s)) + K_u(t+\eta_2, s, \overline{\alpha}_n(s))(\overline{\beta}_{n+1}(s) - \overline{\beta}_n(s))]ds, \end{aligned}$$

$$\overline{\beta}_{n+1}(t_0) = u_0, \tag{8}$$

where $\overline{\alpha}_n(t) = \alpha_n(t+\eta_2)$, $\overline{\beta}_n(t) = \beta_n(t+\eta_2)$. As before, we set $p_n(t) = \overline{\alpha}_n(t) - \overline{\alpha}_{n+1}(t)$ so that

$$\begin{aligned} p_n'(t) &\leq Af(t, \overline{\alpha}_n(t)) + B\int_{t_0}^{t} K(t, s, \overline{\alpha}_n(s))ds \\ &- A[f(t+\eta_2, \overline{\alpha}_n) - \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}p_n(t)] \\ &- B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_n(s)) - K_u(t+\eta_2, s, \overline{\alpha}_n(s))p_n(s)]ds, \\ &\leq A[F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)]p_n(t) + B\int_{t_0}^{t} K_u(t+\eta_2, s, \overline{\alpha}_n(s))p_n(s)ds. \end{aligned}$$

Since $p_n(t_0) \leq 0$, therefore, by Theorem 2, we get $p_n(t) \leq 0$, that is, $\overline{\alpha}_n(t) \leq \overline{\alpha}_{n+1}(t)$ on J.

Now, let $p_n(t) = \overline{\alpha}_{n+1}(t) - \overline{\beta}_n(t)$ and using (A_3), (2) and (3) together with $\overline{\beta}_n(t) \geq \overline{\alpha}_n(t)$, we find that

$$\begin{aligned}
p_n'(t) &\leq A[f(t+\eta_2, \overline{\alpha}_0) + \{F_u(t+\eta_2, \overline{\alpha}_0) - \phi_u(t+\eta_2, \overline{\beta}_0)\}(\overline{\alpha}_{n+1} - \overline{\alpha}_n)] \\
&+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_n(s)) + K_u(t+\eta_2, s, \overline{\alpha}_n(s))(\overline{\alpha}_{n+1}(s) - \overline{\alpha}_n(s))]ds \\
&- Af(t+\eta_1, \overline{\beta}_n(t)) - B\int_{t_0}^{t} K(t+\eta_1, s, \overline{\beta}_n(s))ds \\
&\leq A[f(t+\eta_2, \overline{\alpha}_n) - f(t+\eta_2, \overline{\beta}_n(t)) \\
&+ \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}(\overline{\alpha}_{n+1} - \overline{\alpha}_n)] \\
&+ B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_n(s)) - K(t+\eta_2, s, \overline{\beta}_n(s)) \\
&+ K_u(t+\eta_2, s, \overline{\alpha}_0(s))(\overline{\alpha}_{n+1}(s) - \overline{\alpha}_n(s))]ds \\
&\leq A[F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)]p_n(t) + B\int_{t_0}^{t} K_u(t+\eta_2, s, \overline{\alpha}_n(s))p_n(s)ds.
\end{aligned}$$

By the earlier arguments, it follows that $p_n(t) \leq 0$, as $p_n(t_0) \leq 0$, that is, $\overline{\alpha}_{n+1}(t) \leq \overline{\beta}_n(t)$ on J. In a similar way, it can be shown that $\overline{\alpha}_n(t) \leq \overline{\beta}_{n+1}(t) \leq \overline{\beta}_n(t)$ on J.

In view of the inequalities

$$f(t+\eta_2, \overline{\alpha}_n) \leq f(t+\eta_2, \overline{\alpha}_{n+1}(t)) + \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}(\overline{\alpha}_n - \overline{\alpha}_{n+1}),$$

and

$$K(t+\eta_2, s, \overline{\alpha}_n(s)) \leq K(t+\eta_2, s, \overline{\alpha}_{n+1}(s)) + K_u(t+\eta_2, s, \overline{\alpha}_n(s))(\overline{\alpha}_n - \overline{\alpha}_{n+1}),$$

it is not hard to show that

$$\overline{\alpha}_{n+1}'(t) \leq Af(t+\eta_2, \overline{\alpha}_{n+1}) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\alpha}_{n+1}(s))ds.$$

Similarly, we can prove that

$$\overline{\beta}_{n+1}'(t) \geq Af(t+\eta_2, \overline{\beta}_{n+1}) + B\int_{t_0}^{t} [K(t+\eta_2, s, \overline{\beta}_{n+1}(s))ds.$$

Hence, by Theorem 2, we have $\overline{\alpha}_{n+1}(t) \leq \overline{\beta}_{n+1}(t)$ on J. Thus, it has been shown that

$$\overline{\alpha}_n(t) \leq \overline{\alpha}_{n+1}(t) \leq \overline{\beta}_{n+1}(t) \leq \overline{\beta}_n(t), \ t_0 \leq t \leq t_0 + T.$$

Therefore, by induction, for all n, we obtain

$$\overline{\alpha}_0 \leq \overline{\alpha}_1 \leq ... \leq \overline{\alpha}_n \leq \overline{\beta}_n \leq ... \leq \overline{\beta}_1 \leq \overline{\beta}_0,$$

on $[t_0, t_0+T]$. Using the standard arguments [4, 23], it can be shown that the sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of IVP

$$\overline{u}'(t) = Af(t+\eta_2, \overline{u}(t)) + B\int_{t_0}^{t} K(t+\eta_2, s, \overline{u}(s))ds, \ \overline{u}(t_0) = u_0. \tag{9}$$

Introducing the variable $\zeta = t + \eta_2$, it can be shown that (9) is equivalent to the IVP

$$u'(\zeta) = Af(\zeta, u(\zeta)) + B\int_{t_0}^{t} K(\zeta, s, u(s))ds,\ u(\zeta_0) = u_0.$$

In order to prove the quadratic convergence of the sequences, we set $e_{n+1}(t) = \overline{u}(t) - \overline{\alpha}_{n+1}(t) \geq 0,\ e_{n+1}(t_0) = 0$, and $g_{n+1}(t) = \overline{u}(t) - \overline{\beta}_{n+1}(t) \geq 0,\ g_{n+1}(t_0) = 0$. First we consider

$$\begin{aligned} e'_{n+1}(t) &= \overline{u}' - \overline{\alpha}'_{n+1} \\ &= Af(t+\eta_2, \overline{u}) + B\int_{t_0}^{t} K(t+\eta_2, s, \overline{u}(s))ds \\ &- A[f(t+\eta_2, \overline{\alpha}_n) + \{F_u(t+\eta_2, \overline{\alpha}_n) - \phi_u(t+\eta_2, \overline{\beta}_n)\}(\overline{\alpha}_{n+1} - \overline{\alpha}_n)] \\ &- B\int_{t_0}^{t}[K(t+\eta_2, s, \overline{\alpha}_n(s)) + K_u(t+\eta_2, s, \overline{\alpha}_n(s))(\overline{\alpha}_{n+1}(s) - \overline{\alpha}_n(s))]ds, \end{aligned}$$

Using the mean value theorem repeatedly, we get

$$\begin{aligned} e'_{n+1}(t) &\leq A[\{f_u(t+\eta_2, \sigma_1) - f_u(t+\eta_2, \overline{\alpha}_n)\}(\overline{u} - \overline{\alpha}_n) + f_u(t+\eta_2, \overline{\alpha}_n)e_{n+1}] \\ &+ B\int_{t_0}^{t}[K_u(t+\eta_2, s, \sigma_2)(\overline{u} - \overline{\alpha}_n) - K_u(t+\eta_2, s, \overline{\alpha}_n)(\overline{\alpha}_{n+1}(s) - \overline{\alpha}_n(s))]ds \\ &\leq A[\{f_{uu}(t+\eta_2, \rho_1)e_n^2 + f_u(t+\eta_2, \overline{\alpha}_n)e_{n+1}] \\ &+ B\int_{t_0}^{t}[K_{uu}(t+\eta_2, s, \rho_2)e_n^2 + K_u(t+\eta_2, s, \overline{\alpha}_n)e_{n+1}]ds \end{aligned}$$

where $\overline{\alpha}_n \leq \rho_i \leq \sigma i \leq \overline{u},\ i = 1, 2$. In view of $(A_3)$and (A_4), it follows that $|f_{uu}(t+\eta_2, \overline{u})| \leq N_1,\ N_1 \geq 0,\ |f_u(t+\eta_2, \overline{u})| \leq L_1,\ L_1 \geq 0,\ |K_{uu}(t+\eta_2, \overline{u})| \leq M_1,\ |K_u(t+\eta_2, \overline{u})| \leq L_2,\ L_2 \geq 0$. Thus, the above expression takes the form

$$e'_{n+1}(t) \leq AL_1 e_{n+1}(t) + BL_2\int_{t_0}^{t} e_{n+1}(s)ds + \chi(t), \tag{10}$$

where

$$\chi(t) = AN_1 e_n^2(t) + BM_1\int_{t_0}^{t} e_n^2(s)ds \leq (AN_1 + BM_1T)\max e_n^2(t),\ t \in J. \tag{11}$$

Now, by Theorem 2, we find that $e_{n+1}(t) \leq h(t), t \in J$ and $h(t)$ is the solution of related integro-differential equation

$$h'(t) = A_1 h(t) + B_2\int_{t_0}^{t} h(s)ds + \chi(t),$$

$$e_n(t_0) = h(t_0) = 0, \tag{12}$$

where $A_1 = AL_1$ and $B_2 = BL_2$. To find an estimate for $h(t)$, let

$$\psi(t) = \int_{t_0}^{t} h(s)ds,$$

$$\psi(t_0) = 0, \qquad \psi'(t_0) = 0. \tag{13}$$

Then (12) becomes

$$\psi''(t) - A_1\psi'(t) - B_2\psi(t) = \chi(t),$$
$$\psi(t_0) = 0, \qquad \psi'(t_0) = 0.$$

Solving by the method of variation of parameters and using (11) and (13), we find that

$$e_{n+1}(t) \leq h(t) \leq \frac{2e^{A_1 T}}{\sqrt{A_1^2 + 4B_2}}\chi(t).$$

Hence

$$\max_{t\in J} e_{n+1}(t) \leq \delta_1 \max_{t\in J} e_n^2(t),$$

where $\delta_1 = \frac{2e^{A_1T}}{\sqrt{A_1^2+4B_2}}(AN_1 + BM_1T)$. Following a similar procedure, one can arrive at the conclusion

$$\max_{t\in J} g_{n+1}(t) \leq \delta_2 \max_{t\in J} g_n^2(t).$$

This completes the proof.

4. CONCLUDING REMARKS

The integro-differential equation (1) represents a general form of the equation representing distributed-infective (DI) model and in case of $A = 0$, it corresponds to a general type of the distributed-contact (DC) model for a disease spread by the dispersal of infectious individuals [24]. The main result established in the paper offers an approach to study the approximate solution of a general spread disease model with initial time difference. Moreover, the problem (1) reduces to the one dealing with purely integral type of nonlinearity for $A = 0$, $B = 1$ and for a first order nonlinear differential equation, we can put $A = 1$ and $B = 0$.

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