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ON A CLASS OF SECOND ORDER INFINITE HORIZON VARIATIONAL PROBLEMS

ALEXANDER J. ZASLAVSKI

Department of Mathematics, Technion-Israel Institute of Technology, 32000, Haifa, Israel (ajzasltx.technion.ac.il)

ABSTRACT. In this paper we consider a class of one-dimensional variational problems arising in continuum mechanics which are defined on infinite intervals. We are interested in the existence of non-constant periodic minimizers for these problems.

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1. INTRODUCTION

In this paper we consider a class of one-dimensional variational problems arising in continuum mechanics which was studied in [2-13]. Given $x \in R^2$ we study the infinite horizon problem of minimizing the expression $\int_0^T f(w(t), w'(t), w''(t))dt/T$ as T grows to infinity where

$$w \in A_x = \{v \in W^{2,1}_{loc}([0, \infty))\colon (v(0), v'(0)) = x\}.$$

Here

$$W^{2,1}_{loc}([0, \infty)) = \{f : \ [0, \infty) \to R : \ f \in W^{2,1}[0, T], \ \forall T > 0\}$$

[1] and f belongs to a space of functions to be described below. Namely, we study the following variational problem

$$(P_\infty) \qquad \text{Minimize } \liminf_{T\to\infty} \int_0^T f(w(t), w'(t), w''(t))dt/T, \ w \in A_x,$$

where $x \in R^2$.

Now we describe a space of integrands which will be considered in the paper.

Denote by $\mathfrak{A}$ the set of all continuous functions $f\colon R^3 \to R^1$ such that for each $N > 0$ the function $|f(x, y, z)| \to \infty$ as $|z| \to \infty$ uniformly on the set $\{(x, y) \in R^2\colon |x|, |y| \le N\}$. For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base:

$$E(N, \epsilon, \Gamma) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A} :$$
$$|f(x_1, x_2, x_3) - g(x_1, x_2, x_3)| \le \epsilon \ (x_i \in R^1, \ |x_i| \le N, \ i = 1, 2, 3),$$
$$(|f(x_1, x_2, x_3)| + 1)(|g(x_1, x_2, x_3)| + 1)^{-1} \in [\Gamma^{-1}, \Gamma]$$

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$$(x_1, x_2, x_3) \in R^3,\ |x_1|, |x_2| \le N)\},$$

where $N > 0$, $\epsilon > 0$, $\Gamma > 1$. Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable. It is not difficult to verify that the uniform space $\mathfrak{A}$ is complete.

Let $a = (a_1, a_2, a_3, a_4) \in R^4$, $a_i > 0$ $(i = 1, 2, 3, 4)$ and let α, β, γ be positive numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha, \beta, \gamma, a)$ the set of all functions $f \in \mathfrak{A}$ such that:

$$f(w, p, r) \ge a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w, p, r) \in R^3; \tag{1.1}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w, p, r) > 0 \text{ for all } (w, p, r) \in R^3;$$

there is a monotone increasing function $M_f\colon [0, \infty) \to [0, \infty)$ such that for every $(w, p, r) \in R^3$

$$\sup\{f(w, p, r),\ |\partial f/\partial w(w, p, r)|,\ |\partial f/\partial p(w, p, r)|,\ |\partial f/\partial r(w, p, r)|\} \le$$

$$M_f(|w| + |p|)(1 + |r|^\gamma).$$

Denote by $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ the closure of $\mathfrak{M}(\alpha, \beta, \gamma, a)$ in $\mathfrak{A}$ and consider any $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{\liminf_{T \to +\infty} T^{-1} \int_0^T f(w(t), w'(t), w''(t))dt\colon w \in A_x\}. \tag{1.2}$$

It was shown in [3] that $\mu(f)$ is well defined and is independent of the initial vector x. A function $w \in W^{2,1}_{loc}([0, \infty))$ is called (f)-good if the function

$$\phi^f_w\colon T \to \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt,\ T \in (0, \infty)$$

is bounded.

Leizarowitz and Mizel [3] established that for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ satisfying

$$\mu(f) < \inf\{f(w, 0, s)\colon (w, s) \in R^2\}$$

there exists a periodic (f)-good function. In [9] it was shown that this resut is valid for every $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$.

Let $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. It is easy to see that

$$\mu(f) \le \inf\{f(t, 0, 0) :\ t \in R^1\}.$$

If

$$\mu(f) = \inf\{f(t, 0, 0) :\ t \in R^1\}, \tag{1.3}$$

then there is an (f)-good function v which is a constant function. If

$$\mu(f) < \inf\{f(t, 0, 0) :\ t \in R^1\}, \tag{1.4}$$

then there exists a periodic (f)-good function which is not a constant function. It was shown in [8] that if inequality (1.4) is valid, then extremals of (P_∞) have important asymptotic properties. In [12] we showed that inequality (1.4) holds for most of integrands. More precisely, in [12] we denote by $\mathcal{F}$ the set of all $f \in \bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ which satisfy (1.4) and establish that $\mathcal{F}$ (respectively $\mathcal{F} \cap \mathfrak{M}(\alpha, \beta, \gamma, a)$) is an open everywhere dense subset of $\bar{\mathfrak{M}}(\alpha, \beta, \gamma, a)$ (respectively, $\mathfrak{M}(\alpha, \beta, \gamma, a)$). The main ingredient of the proof of the main result of [12] is the following proposition [12, Proposition 2.3].

Proposition 1.1. *Let* $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ *satisfy*

$$\mu(f) = \inf\{f(t, 0, 0) : \ t \in R^1\}$$

and let ϵ *be a positive number. Then there exists a nonnegative function* $\phi \in C^\infty(R^1)$ *such that*

$$\phi(x) = \epsilon \text{ if } |x| \text{ is large enough,}$$

$$\sup\{\phi(x) : \ x \in R^1\} \le \epsilon$$

and the function

$$g(x_1, x_2, x_3) = f(x_1, x_2, x_3) + \phi(x_2), \ (x_1, x_2, x_3) \in R^3$$

belongs to $\mathfrak{M}(\alpha, \beta, \gamma, a)$ *and satisfies*

$$\mu(g) < \inf\{g(t, 0, 0) : \ t \in R^1\}.$$

Surely, the functions f and g from Proposition 1.1 satisfy $|f(x) - g(x)| \le \epsilon$ for all $x \in R^3$ and are close in the C^0-topology.

In [13, Theorem 1.1] we generalized Proposition 1.1 and showed that the functions f and g can be close in the C^1-topology. In this paper we study if the functions f and g can be close in the C^2-topology and obtain two main results. Our first main result (Theorem 2.1) establishes that if f satisfies certain assumptions, then f and g are close in the C^2-topology. Theorem 2.1 is stated in Section 2 and is proved in Section 3. In Section 4 we state our second main result (Theorem 4.1) which establishes that if f belongs to a certain subset of $\mathfrak{M}(\alpha, \beta, \gamma, a)$, then the functions f and g cannot be close in the C^2-topology. Theorem 4.1 is proved in Section 5.

In the sequel we use the following notation.

For each function $h : R^1 \to R^1$ set

$$||h|| = \sup\{|h(t)| : \ t \in R^1\}. \tag{1.5}$$

For each function $f \in C^1(R^3)$ denote by $\nabla f(z)$ the gradient of the function f at the point $z \in R^3$.

We denote by $||\cdot||$ the Euclidean norm of the space R^n and by $< \cdot, \cdot >$ the scalar product in R^n. (Here n is a natural number).

2. THE FIRST MAIN RESULT

In this paper we will establish the following result which shows that if an integrand f satisfies certain conditions (see (2.1)-(2.3)), then there exists and integrand g which is close to f in C^2-topology and which satisfies $\mu(g) < \inf\{g(t,0,0) : t \in R^1\}$.

Theorem 2.1. *Let a function*

$$f(x_1,x_2,x_3) = h(x_1) + H(x_2,x_3),\ (x_1,x_2,x_3) \in R^3 \tag{2.1}$$

belong to $\mathfrak{M}(\alpha,\beta,\gamma,a)$ *where* $h : R^1 \to R^1$ *and* $H : R^2 \to R^1$.

Assume that $\epsilon \in (0,1)$, $\theta > 0$ *and* $t_0 \in R^1$ *satisfy*

$$f(t_0,0,0) = \inf\{f(t,0,0) : t \in R^1\} \tag{2.2}$$

and

$$2h''(t_0) + \theta^2(\partial^2 H/\partial x_2^2)(0,0) + \theta^4(\partial^2 H/\partial x_3^2)(0,0) \le 0. \tag{2.3}$$

Then there exist a nonnegative function $\phi \in C^\infty(R^1)$ *and* $\epsilon_0 \in [0,\epsilon)$ *such that*

$$\phi(x) = \epsilon_0 \textit{ if } |x| \textit{ is large enough,} \tag{2.4}$$

$$\phi(x) = \epsilon_0 \textit{ in a neighborhood of zero,}$$

$$\sup\{\phi(t) : t \in R^1\} \le \epsilon_0, \tag{2.5}$$

$$\sup\{|\phi'(t)|, |\phi''(t)| : t \in R^1\} \le \epsilon \tag{2.6}$$

and the function

$$g(x_1,x_2,x_3) = f(x_1,x_2,x_3) + \phi(x_2),\ (x_1,x_2,x_3) \in R^3 \tag{2.7}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma,a)$ *and satisfies*

$$\mu(g) < \inf\{g(t,0,0) : t \in R^1\}. \tag{2.8}$$

Corollary 2.1. *Let* $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ *satisfy (2.1) where* $h : R^1 \to R^1$ *and* $H : R^2 \to R^1$ *and let* $\epsilon \in (0,1)$. *Assume that* $t_0 \in R^1$ *satisfies (2.2) and*

$$h''(t_0) = 0,\ \partial^2 H/\partial x_2^2(0,0) < 0.$$

Then there exist a nonnegative function $\phi \in C^\infty(R^1)$ *and* $\epsilon_0 \in [0,\epsilon)$ *such that (2.4)-(2.6) hold and that the function* $g : R^3 \to R^1$ *defined by (2.7) belongs to* $\mathfrak{M}(\alpha,\beta,\gamma,a)$ *and satisfies (2.8).*

Corollary 2.2. *Let* $f \in \mathfrak{M}(\alpha, \beta, \gamma, a)$ *satisfy (2.1) where* $h : R^1 \to R^1$ *and* $H : R^2 \to R^1$ *and let* $\epsilon \in (0, 1)$. *Assume that* $t_0 \in R^1$ *satisfies (2.2) and*

$$h''(t_0) > 0,\ \partial^2 H/\partial x_2^2(0,0) < 0,$$

$$(\partial^2 H/\partial x_2^2(0,0))^2 \geq 8h''(t_0)(\partial^2 H/\partial x_3^2)(0,0).$$

Then there exist a nonnegative function $\phi \in C^\infty(R^1)$ *and* $\epsilon_0 \in [0, \epsilon)$ *such that (2.4)-(2.6) hold and that the function* $g : R^3 \to R^1$ *defined by (2.7) belongs to* $\mathfrak{M}(\alpha, \beta, \gamma, a)$ *and satisfies (2.8).*

3. PROOF OF THEOREM 2.1

If $\mu(f) < \inf\{f(t, 0, 0) :\ t \in R^1\}$, then the assertion of the theorem holds with $\phi(t) = 0$ for all $t \in R^1$.

Assume that

$$\mu(f) = \inf\{f(t, 0, 0) :\ t \in R^1\}. \tag{3.1}$$

There exists a nonnegative function $\psi \in C^\infty(R^1)$ such that

$$0 \leq \psi(x) \leq 1 \text{ for all } x \in R^1, \tag{3.2}$$

$$\psi(x) = 0 \text{ for all } x \in R^1 \text{ satisfying } |x| \geq 1,$$

$$\psi(x) = 1 \text{ for all } x \in [-1/2, 1/2].$$

Relations (2.1) and (2.2) imply that

$$h'(t_0) = 0,\ h''(t_0) \geq 0. \tag{3.3}$$

In view of (1.1) and the Taylor's theorem

$$\lim_{z \to 0} ||z||^{-2}[f((t_0, 0, 0) + z) - f(t_0, 0, 0) - < \nabla f(t_0, 0, 0), z >$$

$$-2^{-1} \sum_{i,j=1}^{3} \partial^2 f/\partial x_i \partial x_j(t_0, 0, 0) z_i z_j] = 0. \tag{3.4}$$

Fix a positive number δ_0 such that

$$\delta_0 < (\epsilon/8)(||\psi'|| + ||\psi''|| + 1)^{-1}(2\pi)^{-1}(\pi/2 - \arcsin(3/4))(1 + \theta^2)^{-2} \min\{1, \theta^2\}. \tag{3.5}$$

By (3.4) there exists $\Delta \in (0, 1)$ such that for each $z = (z_1, z_2, z_3) \in R^3$ satisfying

$$|z_1|, |z_2|, |z_3| \leq 2\Delta(1 + \theta^2)(\theta + \theta^{-1}) \tag{3.6}$$

the following inequality holds:

$$|f((t_0, 0, 0) + z) - f(t_0, 0, 0) - \langle \nabla f(t_0, 0, 0), z \rangle$$

$$- 2^{-1} \sum_{i,j=1}^{3} \partial^2 f/\partial x_i \partial x_j(t_0, 0, 0) z_i z_j| \leq (\delta_0/4)||z||^2. \tag{3.7}$$

Set

$$\epsilon_0 = (\epsilon/8)(||\psi'|| + ||\psi''|| + 1)^{-1}\Delta^2. \tag{3.8}$$

Define

$$\phi(x) = \epsilon_0 - \psi(2\Delta^{-1}x - 2)\epsilon_0, \ x \in R^1. \tag{3.9}$$

Clearly ϕ is nonnegative, $\phi \in C^\infty(R^1)$ and relations (2.5) and (2.4) hold. It follows from (3.9) that for each $x \in R^1$

$$\phi(x) = 0 \text{ if and only if } \psi(2\Delta^{-1}x - 2) = 1. \tag{3.10}$$

By (3.10) and (3.2)

$$\phi(x) = 0 \ \text{ for each } x \in [(3/4)\Delta, (5/4)\Delta]. \tag{3.11}$$

Define a function $g : R^3 \to R^1$ by (2.7). Clearly $g \in \mathfrak{M}(\alpha, \beta, \gamma, a)$. In view of (2.7), (3.9) and (3.2)

$$\inf\{g(t,0,0) : \ t \in R^1\} = \inf\{f(t,0,0) : \ t \in R^1\} + \phi(0)$$

$$= \inf\{f(t,0,0) : \ t \in R^1\} + \epsilon_0 - \epsilon_0\psi(-2) = \inf\{f(t,0,0) : \ t \in R^1\} + \epsilon_0. \tag{3.12}$$

Relations (3.8) and (3.9) imply that for each $t \in R^1$

$$|\phi'(t)| = 2|\psi'(2\Delta^{-1}x - 2)|\epsilon_0\Delta^{-1} \le 2||\psi'||\epsilon_0\Delta^{-1} < \epsilon,$$

$$|\phi''(t)| = |\psi''(2\Delta^{-1}x - 2)|\epsilon_0(2\Delta^{-1})^2 \le ||\psi''||\epsilon_0(2\Delta^{-1})^2 < \epsilon.$$

Therefore (2.6) holds.

By (2.2) and (3.12), in order to complete the proof of the theorem we need only to show that

$$\mu(g) < \epsilon_0 + f(t_0, 0, 0). \tag{3.13}$$

Set

$$v(t) = t_0 + \Delta\theta^{-1}\cos(\theta t), \ t \in R^1. \tag{3.14}$$

Then for each $t \in R^1$

$$v'(t) = -\Delta\sin(\theta t), \ v''(t) = -\Delta\theta\cos(\theta t). \tag{3.15}$$

It is clear that for each $t \in R^1$

$$|v(t) - t_0|, \ |v'(t)|, \ |v''(t)| \le \Delta\max\{\theta^{-1}, \theta\}. \tag{3.16}$$

For each $t \in R^1$ set

$$z(t) = (z_1(t), z_2(t), z_3(t)) = (v(t), v'(t), v''(t)) - (t_0, 0, 0). \tag{3.17}$$

By (3.17), (3.16) and the choice of Δ (see (3.6), (3.7)), for each $t \in R^1$

$$|f(v(t), v'(t), v''(t)) - f(t_0, 0, 0) - < \nabla f(t_0, 0, 0), (v(t) - t_0, v'(t), v''(t)) >$$

$$-2^{-1}\sum_{i,j=1}^{3}(\partial^2 f/\partial x_i\partial x_j)(t_0,0,0)z_i(t)z_j(t)| \le (\delta_0/4)3\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}. \tag{3.18}$$

Relation (3.14) implies that for all $t \in R^1$

$$v(t+2\pi\theta^{-1}) = t_0+\theta^{-1}\Delta\cos(\theta t+2\pi) = t_0+\theta^{-1}\Delta\cos(\theta t) = v(t). \tag{3.19}$$

It follows from (3.18), (2.1), (3.3), (3.15) and (3.17) that for each $t \in R^1$

$$f(v(t),v'(t),v''(t)) \le f(t_0,0,0) + (\partial H/\partial x_2)(0,0)v'(t) + (\partial H/\partial x_3)(0,0)v''(t)$$
$$+2^{-1}h''(t_0)(v(t)-t_0)^2 + 2^{-1}(\partial^2 H/\partial x_2^2)(0,0)(v'(t))^2 + 2^{-1}(\partial^2 H/\partial x_3^2)(0,0)(v''(t))^2$$
$$+(\partial^2 H/\partial x_2\partial x_3)(0,0)v'(t)v''(t) + (3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}$$
$$\le f(t_0,0,0) + (\partial H/\partial x_2)(0,0)(-\Delta\sin(\theta t)) + (\partial H/\partial x_3)(0,0)(-\Delta\theta\cos(\theta))$$
$$+2^{-1}h''(t_0)(v(t)-t_0)^2 + 2^{-1}(\partial^2 H/\partial x_2^2)(0,0)\Delta^2(\sin(\theta t))^2$$
$$+2^{-1}(\partial^2 H/\partial x_3^2)(0,0)\Delta^2\theta^2(\cos(\theta t))^2$$
$$+(\partial^2 H/\partial x_2\partial x_3)(0,0)\Delta^2\theta\sin(\theta t)\cos(\theta t) + (3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}.$$

Together with (3.14) and (3.3), this inequality implies that

$$(2\pi)^{-1}\theta\int_0^{2\pi/\theta} f(v(t),v'(t),v''(t))dt \le f(t_0,0,0)$$
$$+(\partial H/\partial x_2)(0,0)(2\pi)^{-1}\theta\int_0^{(2\pi)/\theta} -\Delta\sin(\theta t)dt$$
$$+(\partial H/\partial x_3)(0,0)(-\Delta\theta)(2\pi)^{-1}\theta\int_0^{(2\pi)/\theta}\cos(\theta t)dt + 2^{-1}h''(t_0)\Delta^2\theta^{-2}$$
$$+2^{-1}(\partial^2 H/\partial x_2^2)(0,0)\Delta^2(2\pi)^{-1}\theta\int_0^{2\pi/\theta}(\sin(\theta t))^2 dt$$
$$+2^{-1}(\partial^2 H/\partial x_3^2)(0,0)\Delta^2\theta^2(2\pi)^{-1}\theta\int_0^{2\pi/\theta}(\cos(\theta t))^2 dt$$
$$+(\partial^2 H/\partial x_2\partial x_3)(0,0)\Delta^2\theta(2\pi)^{-1}\theta\int_0^{2\pi/\theta}\sin(\theta t)\cos(\theta t)dt + (3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}$$
$$\le f(t_0,0,0) + 2^{-1}h''(t_0)\Delta^2\theta^{-2} + 4^{-1}(\partial^2 H/\partial x_2^2)(0,0)\Delta^2$$
$$+4^{-1}(\partial^2 H/\partial x_3^2)(0,0)\Delta^2\theta^2 + (3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}. \tag{3.20}$$

By (3.19), (2.7) and (3.20)

$$\mu(g) \le (2\pi)^{-1}\theta\int_0^{2\pi/\theta} g(v(t),v'(t),v''(t))dt$$
$$= (2\pi)^{-1}\theta\int_0^{2\pi/\theta} f(v(t),v'(t),v''(t))dt + (2\pi)^{-1}\theta\int_0^{2\pi/\theta}\phi(v'(t))dt$$
$$\le f(t_0,0,0) + 2^{-1}h''(t_0)\Delta^2\theta^{-2} + 4^{-1}(\partial^2 H/\partial x_2^2)(0,0)\Delta^2$$
$$+4^{-1}(\partial^2 H/\partial x_3^2)(0,0)\Delta^2\theta^2 + (3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}$$

$$+(2\pi)^{-1}\theta\int_0^{2\pi/\theta}\phi(v'(t))dt. \tag{3.21}$$

We estimate $(2\pi)^{-1}\theta\int_0^{2\pi/\theta}\phi(v'(t))dt$. In view of (2.5), (3.9) and (3.2)

$$0\le\phi(v'(t))\le\epsilon_0 \text{ for all } t\in R^1. \tag{3.22}$$

Now let

$$t\in[-\pi(2\theta)^{-1},-\arcsin(3/4)\theta^{-1}]. \tag{3.23}$$

By (3.15) and (3.23)

$$v'(t)=-\Delta\sin(t\theta)\in[3\Delta/4,\Delta].$$

Combined with (3.11), this relations implies that $\phi'(v(t))=0$. Thus

$$\phi(v'(t))=0 \text{ for all } t\in[-\pi(2\theta)^{-1},-\arcsin(3/4)\theta^{-1}]. \tag{3.24}$$

It follows from (3.24), (3.22) and (3.19) that

$$\theta(2\pi)^{-1}\int_0^{2\pi/\theta}\phi(v'(t))dt\le\theta(2\pi)^{-1}[0\cdot(\pi(2\theta)^{-1}-\arcsin(3/4)\theta^{-1})]$$

$$+\theta(2\pi)^{-1}\epsilon_0[2\pi\theta^{-1}-(\pi(2\theta)^{-1}-\arcsin(3/4)\theta^{-1})]=(2\pi)^{-1}\epsilon_0[(3/2)\pi+\arcsin(3/4)].$$

Combined with (3.21), (2.3), (3.5) and (3.8) this relation implies that

$$\mu(g)\le f(t_0,0,0)+2^{-1}h''(t_0)\Delta^2\theta^{-2}+4^{-1}(\partial H/\partial x_2^2)(0,0)\Delta^2$$

$$+4^{-1}(\partial^2H/\partial x_3^2)(0,0)\Delta^2\theta^2+(3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}$$

$$+(2\pi)^{-1}\epsilon_0[(3/2)\pi+\arcsin(3/4)]\le f(t_0,0,0)+(3\delta_0/4)\Delta^2(\max\{1,\theta^2\})^2\theta^{-2}$$

$$+(2\pi)^{-1}\epsilon_0[(3/2)\pi+\arcsin(3/4)]<f(t_0,0,0)$$

$$+(3/4)\Delta^2(\epsilon/8)(||\psi'||+||\psi''||+1)^{-1}(2\pi)^{-1}(\pi/2-\arcsin(3/4))$$

$$+(2\pi)^{-1}\epsilon_0[(3/2)\pi+\arcsin(3/4)]$$

$$=f(t_0,0,0)+(3/4)\epsilon_0(2\pi)^{-1}(\pi/2-\arcsin(3/4))$$

$$+(2\pi)^{-1}\epsilon_0[(3/2)\pi+\arcsin(3/4)]<f(t_0,0,0)+\epsilon_0.$$

Thus (3.13) holds. This completes the proof of the theorem.

4. THE SECOND MAIN RESULT

In this section, we state our second main result which shows that there exist integrands f, such that for each integrand g which is close to f in the C^2-topology the equality $\mu(g) = \inf\{g(t,0,0) :\ t \in R^1\}$ holds.

We use the notation and definitions from Sections 1 and 2.

The following result will be proved in Section 5.

Theorem 4.1. *Let* $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ *and*

$$f(x_1,x_2,x_3) = f_1(x_1) + f_2(x_2) + f_3(x_3),\ x = (x_1,x_2,x_3) \in R^3 \tag{4.1}$$

where $f_i : R^1 \to R^1$, $i = 1,2,3$ *and* $t_0 \in R^1$ *satisfy*

$$\inf\{f_3''(t) :\ t \in R^1\} > 0, \tag{4.2}$$

$$\mu(f) = \inf\{f(t,0,0) :\ t \in R^1\} = f(t_0,0,0). \tag{4.3}$$

Then there exists $\lambda_0 > 0$ *such that the following assertion holds.*

Let $\lambda \geq \lambda_0$ *and*

$$f_\lambda(x_1,x_2,x_3) = f(x_1,x_2,x_3) + \lambda(x_1 - t_0)^2,\ (x_1,x_2,x_3) \in R^3. \tag{4.4}$$

Then $f_\lambda \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ *and there exists* $\delta > 0$ *such that if the functions* $\phi_1,\phi_2,\phi_3 \in C^2(R^1)$ *satisfy*

$$|\phi_i(t)|,\ |\phi_i'(t)|,\ |\phi_i''(t)| \leq \delta \text{ for all } t \in R^1 \text{ and } i = 1,2,3 \tag{4.5}$$

and if the function $g : R^3 \to R^1$ *defined by*

$$g(x_1,x_2,x_3) = f_\lambda(x_1,x_2,x_3) + \phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3),\ x = (x_1,x_2,x_3) \in R^3 \tag{4.6}$$

belongs to $\mathfrak{M}(\alpha,\beta,\gamma,a)$, *then* g *possesses a unique periodic* (g)*-good function which is constant.*

For each $v \in W^{2,1}_{loc}([0,\infty))$ we denote by $\Omega(v)$ the set of all limit points of $(v(t), v'(t))$ as $t \to \infty$.

We say that an integrand $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ has the asymptotic turnpike property, or briefly (ATP), if $\Omega(v_1) = \Omega(v_2)$ for each pair of (f)-good functions v_1 and v_2.

In the sequel, we use the following two results.

Proposition 4.1. *Let* $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ *and* $t_0 \in R^1$ *satisfy*

$$\mu(f) = \inf\{f(t,0,0) :\ t \in R^1\} = f(t_0,0,0),$$

$\lambda > 0$ *and let*

$$g(x_1,x_2,x_3) = f(x_1,x_2,x_3) + \lambda(x_1 - t_0)^2,\ (x_1,x_2,x_3) \in R^3. \tag{4.7}$$

Then

$$g \in \mathfrak{M}(\alpha,\beta,\gamma,a),\ \mu(g) = g(t_0,0,0) = f(t_0,0,0)$$

and g has (ATP).

Proof. It is easy to see that (4.8) is valid. We can show that g has (ATP) by arguing as in the proof of Theorem 3.2 of [7]. □

The next result follows from Theorem 2.2 of [7].

Proposition 4.2. *Let $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ possess (ATP) $\epsilon > 0$ and let $t_0 \in R^1$ satisfy $\mu(f) = f(t_0,0,0)$. Then there exists $\delta > 0$, such that for each $h \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ satisfying $|f(x) - h(x)| \le \delta$ for all $x \in R^3$ and each (h)-good function v, we have $|(v(t),v'(t)) - (t_0,0)| \le \epsilon$ for all large enough $t \in [0,\infty)$.*

5. PROOF OF THEOREM 4.1

By (4.2), there exists $\bar{c} \in (0,1)$ such that

$$f_3''(t) \ge \bar{c} \text{ for all } t \in R^1. \tag{5.1}$$

By Lemma 4.10 of [1], there exists

$$\lambda_0 > 4 + 8|f_1''(t_0)| \tag{5.2}$$

such that for each $T \ge 1$ and each $\xi \in C^2([0,T])$

$$\int_0^T (\xi'(t))^2 dt \le 4^{-1}(2+|f_2''(0)|)^{-1}\bar{c}\int_0^T (\xi''(t))^2 dt + (2+|f_2''(0)|)^{-1}8^{-1}\lambda_0 \int_0^T |\xi(t)|^2 dt. \tag{5.3}$$

Let $\lambda \ge \lambda_0$ and let f_λ be defined by (4.4). In view of Proposition 4.1

$$f_\lambda \in \mathfrak{M}(\alpha,\beta,\gamma,a),\ \mu(f_\lambda) = f(t_0,0,0) \tag{5.4}$$

and f_λ possesses (ATP).

By (5.1) and the Taylor's theorem for each $t \in R^1$ there is $s \in R^1$ such that

$$f_3(t) = f_3(0) + f_3'(0)t + f_3''(s)t^2/2 \ge f_3(0) + f_3'(0)t + (\bar{c}/2)t^2.$$

Thus

$$f_3(t) \ge f_3(0) + f_3'(0)t + (\bar{c}/2)t^2 \text{ for all } t \in R^1. \tag{5.5}$$

Fix $\epsilon_0 \in (0,1)$ and choose

$$\epsilon_1 \in (0, \epsilon_0/2) \tag{5.6}$$

such that

$$|f_1''(t) - f_1''(t_0)| \le \lambda_0/16 \text{ for each } t \in [t_0-\epsilon_1, t_0+\epsilon_1], \tag{5.7}$$

$$|f_2''(t) - f_2''(0)| \le 1 \text{ for each } t \in [-\epsilon_1, \epsilon_1]. \tag{5.8}$$

Since f_λ possesses (ATP), it follows from Proposition 4.2, (4.4) and (5.4) that there is $\delta_1 \in (0,1)$ such that the following property holds:

(P1) If $h \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ satisfies

$$|f_\lambda(x_1,x_2,x_3) - h(x_1,x_2,x_3)| \le 4\delta_1$$

for all $(x_1,x_2,x_3) \in R^3$, then for each (h)-good function v the inequality

$$|(v(t),v'(t)) - (t_0,0)| \le \epsilon_1/8$$

holds for all sufficiently large $t \in [0,\infty)$.

Choose a positive number δ such that

$$\delta < \min\{\delta_1, 16^{-1}\epsilon_1^2, \bar{c}/4\}. \tag{5.9}$$

Assume that $\phi_1,\phi_2,\phi_3 \in C^2(R^1)$ satisfy (4.5) and the function $g: R^3 \to R^1$ defined by (4.6) belongs to $\mathfrak{M}(\alpha,\beta,\gamma,a)$.

In order to complete the proof of the theorem, it is sufficient to show that the function g possesses a unique periodic (g)-good function which is constant. By (4.5), (4.6), property (P1) and the inequality $\delta < \delta_1$ the following property holds:

(P2) For each (g)-good periodic function v we have

$$|(v(t),v'(t)) - (t_0,0)| \le \epsilon_1/8 \text{ for all } t \in R^1.$$

Set

$$g_1(t) = f_1(t) + \lambda(t-t_0)^2 + \phi_1(t),\ t \in R^1. \tag{5.10}$$

Consider the restriction of the function g_1 to the interval $[-\epsilon_1,\epsilon]$. By (4.3)

$$f_1'(t_0) = 0,\ f_1''(t_0) \ge 0. \tag{5.11}$$

It follows from the Taylor's theorem that for each $t \in [t_0-\epsilon_1, t_0+\epsilon_1]$ there is $\xi \in [t_0-\epsilon_1, t_0+\epsilon_1]$ such that

$$f_1(t) + \lambda(t-t_0)^2 + \phi_1(t) = f_1(t_0) + \phi_1(t_0) + (f_1'(t_0) + \phi_1'(t_0))(t-t_0)$$

$$+2^{-1}(f_1''(\xi) + 2\lambda + \phi_1''(\xi))(t-t_0)^2. \tag{5.12}$$

Combined with (5.11), (5.7), (5.9) and (5.2), this inequality implies that for each $t \in [t_0-\epsilon_1, t_0+\epsilon_1]$

$$f_1(t) + \lambda(t-t_0)^2 + \phi_1(t) \ge f_1(t_0) + \phi_1(t_0) - \delta|t-t_0| + 2^{-1}\lambda(t-t_0)^2. \tag{5.13}$$

We will show that there is $t_g \in [t_0-\epsilon_1, t_0+\epsilon_1]$ such that

$$g_1(t_g) < g(t) \text{ for all } t \in R^1 \setminus \{t_g\}.$$

By (5.13) and (5.10), for each real number t which satisfies

$$t \in [t_0-\epsilon_1, t_0+\epsilon_1] \text{ and } |t-t_0| \ge 8\delta\lambda_0^{-1} \tag{5.14}$$

we have

$$g_1(t) = f_1(t)+\lambda(t-t_0)^2+\phi_1(t) \ge f_1(t_0)+\phi_1(t_0)-(t-t_0)^2\lambda_0 8^{-1}+2^{-1}\lambda(t-t_0)^2 \tag{5.15}$$

$$\ge f_1(t_0) + \phi_1(t) + 4^{-1}\lambda(t-t_0)^2 = g_1(t_0) + 4^{-1}\lambda(t-t_0)^2.$$

Since (5.15) holds for each real t satisfying (5.14), we conclude that

(5.16)

$$\{\tau \in [t_0-\epsilon_1, t_0+\epsilon_1] : \ g_1(\tau) \le g_1(t) \text{ for all } t \in [t_0-\epsilon_1, t_0+\epsilon_1]\} \subset (t_0-8\delta\lambda_0^{-1}, t_0+8\delta\lambda_0^{-1}).$$

In view of (5.10), (5.11), (5.7) and (5.9), for each $t \in [t_0-\epsilon_1, t_0+\epsilon_1]$

$$g_1''(t) = f_1''(t) + 2\lambda + \phi_1''(t) \ge 2\lambda - \lambda_0/16 - \delta \ge \lambda. \tag{5.17}$$

There is

$$t_g \in [t_0-\epsilon_1, t_0+\epsilon_1] \tag{5.18}$$

such that

$$g_1(t_g) = \inf\{g_1(t) : \ t \in [t_0-\epsilon_1, t_0+\epsilon_1]\}. \tag{5.19}$$

By (5.19) and (5.16)

$$|t_g - t_0| < 8\delta\lambda_0^{-1}. \tag{5.20}$$

It follows from (5.19), (5.20), (5.9) and (5.2) that

$$g_1'(t_g) = 0. \tag{5.21}$$

Let

$$t \in [t_0-\epsilon_1, t_0+\epsilon_1] \setminus \{t_g\}. \tag{5.22}$$

By the Taylor's theorem there exists $\xi \in [t_0-\epsilon_1, t_0+\epsilon_1]$ such that

$$g_1(t) = g_1(t_g) + g_1'(t_g)(t-t_g) + 2^{-1}g_1''(\xi)(t-t_g)^2.$$

Combined with (5.21), (5.17) and (5.22), this relation implies that

$$g_1(t) = g_1(t_g) + 2^{-1}g_1''(\xi)(t-t_g)^2 \ge g_1(t_g) + 2^{-1}\lambda(t-t_g)^2 > g_1(t_g).$$

Therefore

$$g_1(t) \ge 2^{-1}\lambda|t-t_g|^2 + g_1(t_g) > g_1(t_g) \text{ for each } t \in [t_0-\epsilon_1, t_0+\epsilon_1] \setminus \{t_g\}. \tag{5.23}$$

Assume that $t \in R^1$ satisfies

$$|t - t_0| > \epsilon_1. \tag{5.24}$$

By (5.10), (5.24), (4.5), (4.3), (4.1), (5.9), (5.2) and (5.19)

$$g_1(t) = f_1(t) + \lambda(t-t_0)^2 + \phi_1(t) \ge f_1(t) + \lambda\epsilon_1^2 - \delta$$

$$\ge f_1(t_0) + \lambda\epsilon_1^2 - \delta \ge f_1(t_0) + \phi_1(t_0) + \lambda\epsilon_1^2 - 2\delta$$

$$= g_1(t_0) + \lambda\epsilon_1^2 - 2\delta > g_1(t_0) \ge g_1(t_g).$$

Therefore $g_1(t) > g_1(t_g)$ for all $t \in R^1$ satisfying (5.24). Together with (5.23), this implies that

$$g_1(t_g) < g_1(t) \text{ for each } t \in R^1 \setminus \{t_g\}. \tag{5.25}$$

It is clear that

$$\mu(g) \le g_1(t_g) = g(t_g, 0, 0). \tag{5.26}$$

Assume that $w \in W^{2,1}_{loc}([0,\infty))$ is a (g)-good periodic function. There is $T > 0$ such that

$$w(t+T) = w(t) \text{ for all } t \in [0,\infty). \tag{5.27}$$

We may assume that $T \ge 4$. By [9, Proposition 4.1] $w \in C^4([0,\infty))$. We will show that $w(t) = t_g$ for all $t \ge 0$. In view of (P2)

$$|(w(t), w'(t)) - (t_0, 0)| \le \epsilon_1/8 \text{ for all } t \in R^1. \tag{5.28}$$

Relations (5.28) and (5.23) imply that for each $t \in [0,\infty)$

$$g_1(w(t)) \ge g_1(t_g) + 2^{-1}\lambda(w(t) - t_g)^2. \tag{5.29}$$

Let $t \in [0,\infty)$. By (5.28) and the Taylor's theorem there is

$$\xi \in [-4^{-1}\epsilon_1, 4^{-1}\epsilon_1] \tag{5.30}$$

such that

$$(f_2+\phi_2)(w'(t)) = (f_2+\phi_2)(0) + (f_2+\phi_2)'(0)w'(t) + 2^{-1}(f_2+\phi_2)''(\xi)(w'(t))^2.$$

Together with (4.5), (5.30), (5.8) and (5.9), this relation implies that

$$\begin{aligned}(f_2+\phi_2)(w'(t)) &\ge (f_2+\phi_2)(0) + (f_2+\phi_2)'(0)w'(t) - 2^{-1}(\delta + 1 + |f_2''(0)|)(w'(t))^2 \\ &\ge (f_2+\phi_2)(0) + (f_2+\phi_2)'(0)w'(t) - (2 + |f_2''(0)|)(w'(t))^2.\end{aligned}$$

Therefore for each $t \in [0,\infty)$

$$(f_2+\phi_2)(w'(t)) \ge (f_2+\phi_2)(0) + (f_2+\phi_2)'(0)w'(t) - (2 + |f_2''(0)|)(w'(t))^2. \tag{5.31}$$

Let $t \in [0,\infty)$. By (5.1), (4.5) and the Taylor's theorem there is $\xi \in R^1$ such that

$$\begin{aligned}(f_3+\phi_3)(w''(t)) &= (f_3+\phi_3)(0) + (f_3+\phi_3)'(0)w''(t) + 2^{-1}(f_3+\phi_3)''(\xi)(w''(t))^2 \\ &\ge (f_3+\phi_3)(0) + (f_3+\phi_3)'(0)w''(t) + 2^{-1}(\bar{c} - \delta)(w''(t))^2 \\ &\ge (f_3+\phi_3)(0) + (f_3+\phi_3)'(0)w''(t) + 4^{-1}\bar{c}(w''(t))^2. \qquad (5.32)\end{aligned}$$

In view of (5.29)

$$T^{-1}\int_0^T g_1(w(t))dt \ge g_1(t_g) + 2^{-1}\lambda T^{-1}\int_0^T (w(t) - t_g)^2 dt. \tag{5.33}$$

By (5.31) and (5.27)

$$T^{-1}\int_0^T (f_2+\phi_2)(w'(t))dt \geq (f_2+\phi_2)(0) + T^{-1}(f_2+\phi_2)'(0)\int_0^T w'(t)dt$$

$$-T^{-1}(2+|f_2''(0)|)\int_0^T (w'(t))^2dt = (f_2+\phi_2)(0) - T^{-1}(2+|f_1''(0)|)\int_0^T (w'(t))^2dt. \tag{5.34}$$

Relations (5.32) and (5.27) imply that

$$T^{-1}\int_0^T (f_3+\phi_3)(w''(t))dt \geq (f_3+\phi_3)(0) + T^{-1}\int_0^T (f_3+\phi_3)'(0)w''(t)dt$$

$$+(4T)^{-1}\bar{c}\int_0^T (w'(t))^2dt \geq (f_3+\phi_3)(0) + (4T)^{-1}\bar{c}\int_0^T (w''(t))^2dt. \tag{5.35}$$

Relations (5.10), (4.6), (4.4), (4.1), (5.26), (5.33), (5.34) and (5.35) imply that

$$g_1(t_g) + (f_2+\phi_2)(0) + (f_3+\phi_3)(0) = g(t_g,0,0) \geq \mu(g)$$

$$= T^{-1}\int_0^T g(w(t),w'(t),w''(t))dt = T^{-1}\int_0^T g_1(w(t))dt$$

$$+T^{-1}\int_0^T (f_2+\phi_2)(w'(t))dt + T^{-1}\int_0^T (f_3+\phi_3)(w''(t))dt$$

$$\geq g_1(t_g) + 2^{-1}\lambda T^{-1}\int_0^T (w(t)-t_g)^2dt + (f_2+\phi_2)(0)$$

$$-T^{-1}(2+|f_2''(0)|)\int_0^T (w'(t))^2dt + (f_3+\phi_3)(0) + (4T)^{-1}\bar{c}\int_0^T (w''(t))^2dt.$$

This relation implies that

$$(2+|f_2''(0)|)\int_0^T (w'(t))^2dt \geq 2^{-1}\lambda\int_0^T (w(t)-t_g)^2dt + 4^{-1}\bar{c}\int_0^T (w''(t))^2dt$$

and

$$\int_0^T (w'(t))^2dt \geq 2^{-1}\lambda_0(2+|f_2''(0)|)^{-1}\int_0^T (w(t)-t_g)^2dt + 4^{-1}(2+|f_2''(0)|)^{-1}\bar{c}\int_0^T (w''(t))^2dt. \tag{5.36}$$

Applying (5.3) to the function $\xi = w(\cdot) - t_g$, we obtain that

$$\int_0^T (w'(t))^2dt \leq 8^{-1}\lambda_0(2+|f_2''(0)|)^{-1}\int_0^T (w(t)-t_g)^2dt + 4^{-1}(2+|f_2''(0)|)^{-1}\bar{c}\int_0^T (w''(t))^2dt.$$

Together with (5.36) this inequality implies that

$$\int_0^T (w(t)-t_g)^2dt = 0$$

and $w(t) = t_g$ for all $t \geq 0$. We have shown that if w is a periodic (g)-good function, then $w(t) = t_g$ for all $t \in [0,\infty)$. This completes the proof of Theorem 4.1.

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