Dynamic Systems and Applications 16 (2007) 533-550

TRIPLE POSITIVE SOLUTIONS OF NONLINEAR SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper presents a new theorem on the existence of triple nontrivial fixed points and then investigates the existence of triple positive solutions of a class of nonlinear singular boundary value problem by using this new fixed point theorem. Meanwhile, an example is worked out to demonstrate the main result.

Keywords: Fixed point, Singularity, Cone, Positive solution, Boundary value problem.

AMS (MOS) Subject Classification. 34B16.

1. INTRODUCTION

In recent years, singular boundary value problems(SBVP, for short) have been studied extensively (see, for instance, [2, 8, 9, 13, 16, 17] and references therein) since they arise quite naturally in physics, fluid theory and the study of radially symmetric solutions to elliptic problems (see [14, 15], for example). The applicable approaches to study such problem are mainly as follows: fixed point theorems (see e.g. [2, 13]), shooting method (see e.g. [16, 17]), upper and lower solutions method (see e.g. [8, 9]) etc. The fixed point theorems used to study SBVP are the Leray-Schauder continuation theorem, the nonlinear alternative of Leray-Schauder, or Krasnoselskii's fixed point theorem. The results obtained in the literatures are the existence of one or two positive solutions.

Work establishing the existence of three solutions of nonlinear equations using a degree theoretic approach traces back to the Leggett-Williams multiple fixed point theorem [12]. And lately, this theorem together with other two triple fixed point theorems due to Avery [4] and Avery and Peterson [5] has bee applied to obtain triple solutions of integral equations, certain boundary value problems for ordinary differential equations as well as for their discrete analogues (see, for instance, [1, 3, 6, 10, 11] and reference therein). However, to our best knowledge, there is no paper to consider SBVP by using the Leggett-Williams multiple fixed point theorem or its generalizations when the nonlinear term is singular (in particular, the nonlinear term

Received May 5, 2006

$f(t, x)$ is singular at $x = 0$, for detail, please see Section 3). The main reason lies in that such fixed point theorem can only solve the existence of fixed point for bounded operator. Unfortunately, the operator converted into by SBVP is often unbounded.

Motivated by Leggett-Williams multiple fixed point theorem and Krasnoselskii's fixed point theorem, this paper investigates how to extend the Leggett-Williams theorem and how to solve SBVP by the obtained extension of Leggett-Williams theorem. Theorem 2.2 obtained in Section 2 can solve SBVP not only when the nonlinear term is sublinear at $+\infty$ but also when it is superlinear at $+\infty$. And in both cases, same results is obtained.

The paper is organized as follows. Section 2 is devoted to the extension of Leggett-Williams multiple fixed point theorem. Section 3 is concerned with the existence of triple positive solutions of SBVP by applying the result obtained in Section 2. Meanwhile, an examples is worked out to demonstrate the main result.

2. AN EXTENSION OF LEGGETT-WILLIAMS FIXED POINT THEOREM

Let E be a real Banach space with norm $\|\cdot\|$ and $P \subset E$ be a cone of E, $P_r = \{x \in P : \ \|x\| < r\}$ $(r > 0)$. Then $\partial P_r = \{x \in P : \ \|x\| = r\}$ and $\overline{P}_r = \{x \in P : \ \|x\| \leq r\}$. Consider nonnegative continuous and concave functional $\alpha(x)$ defined on P, i.e., $\alpha : \ P \to \mathbf{R}^+ = [0, \ +\infty)$ is continuous and satisfies

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \quad \text{for } x, y \in P \text{ and } t \in [0, 1]. \tag{2.1}$$

Let

$$P(\alpha, a, b) =: \ \{x \in P \big| \ a(x) \geq a, \ \|x\| \leq b\}, \tag{2.2}$$

where $0 < a < b$. It is not difficult to see $P(\alpha, a, b)$ is a bounded closed convex subset of P.

For convenience of comparing with our result, we state the well-known Leggett-Williams multiple fixed point theorem.

Theorem 2.1 (Leggett-Williams fixed point theorem)**.** *Let operator* $A : \ \overline{P}_c \to P_c$ *be completely continuous and let* α *be a nonnegative continuous concave functional on* P *such that* $\alpha(x) \leq \|x\|$ *for every* $x \in \overline{P}_c$. *Suppose that there exist* $0 < d < a < b \leq c$ *such that*

(C1) $\{x \big| \ x \in P(\alpha, a, b), \ \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for each* $x \in P(\alpha, a, b)$;
(C2) $\|Ax\| < d$ *for* $x \in \overline{P}_d$;
(C3) $\alpha(Ax) > a$ *for* $x \in P(\alpha, a, c)$ *with* $\|Ax\| > b$.

Then, A *has at least three fixed points* x_1, x_2 *and* x_3 *satisfying*

$$x_1 \in P_d, \quad x_2 \in U, \quad \text{and } x_3 \in \overline{P}_c \setminus (P_d \cup U),$$

where

$$U = \{x : \ x \in P(\alpha, a, c), \ \alpha(x) > a\}.$$

Now we are ready to give the main result of this section.

Theorem 2.2. *Let operator $A : \ \overline{P}_c \setminus P_e \to P$ be completely continuous, where $c > e > 0$. Suppose there exists a nonnegative continuous and concave functional $\alpha(x)$ defined on P with $\alpha(x) \leq \|x\|$ for every $x \in \overline{P}_c$. Also suppose there exist three positive numbers a, b, and d with $e < d < a < b \leq c$ such that*

(A1) $\{x \big| \ x \in P(\alpha, a, b), \ \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *for each* $x \in P(\alpha, a, b)$.
(A2) $\alpha(Ax) > a$ *for* $x \in \{x \in P \big| \ \alpha(x) > a, \ \|x\| < c$, *and* $\|Ax\| > b\}$.
(A3) $Ax \not\leq x$ *for* $x \in \partial P_e$, $Ax \not\geq x$ *for* $x \in \partial P_d$, *and either* $x \neq \mu Ax$ *for* $x \in \partial P_c$ *and* $\mu \in [0, 1]$ *or* $\|Ax\| > \|x\|$ *for* $x \in \partial P_c$.

Then the operator A has at least three positive fixed points x_1, x_2, and x_3 satisfying

$$x_1 \in P_d \setminus \overline{P}_e, \quad x_2 \in U, \quad \text{and } x_3 \in P_c \setminus \overline{(P_d \cup U)},$$

where

$$U = \{x \in P \big| \ \alpha(x) > a \text{ and } \|x\| < c\}.$$

Remark 2.3. In Theorem 2.1, the operator A is not supposed to map $\overline{P}_c$ into $\overline{P}_c$ as in Theorem 2.1. Thus, A may be unbounded on some neighbour fields of $x = \theta$, where $x = \theta$ is the zero element of Banach space E. So, it is natural that $Ax \not\leq x$ for $x \in \partial P_e$ in condition (A3). On the other hand, the fixed point obtained in Theorem 2.1 may be trivial.

Remark 2.4. Conditions (A1) and (A2) are similar to conditions (C1) and (C3) of Theorem 2.1. If c is sufficiently large, condition (A3) shows that the operator A is allowed to belong to one of two cases, i.e., sublinear or superlinear at $+\infty$. However, condition (C2) means that A must be sublinear at $+\infty$ if d is sufficiently large (for detail, see Remark 3.4 in Section 3).

Proof of Theorem 2.2. By the extension theorem (see [7, Theorem A.5.1]), the operator A has a completely continuous extension $\widehat{A}$ from $\overline{P}_c$ into P, which satisfies $\widehat{A} = Ax$ for $x \in \overline{P}_c \setminus P_e$. For conveniens, still denote $\widehat{A}$ by A. Let

$$U =: \ \{x \in P \big| \ \alpha(x) > a \text{ and } \|x\| < c\}. \tag{2.3}$$

Then by continuity of $\alpha(\cdot)$ and $\|\cdot\|$, we know U is an open subset of P.

Now we show $Ax \neq x$ for each $x \in \partial U$. Also we have $U \subset P_c \setminus \overline{P}_e$ since $\alpha(x) \leq \|x\|$ for every $x \in \overline{P}_c$.

Suppose, on the contrary, there exists a $x_0 \in \partial U$ such that $Ax_0 = x_0$. This together with condition (A3) guarantees $\alpha(x_0) = a$. If $\|x_0\| \leq b$, then by (2.2) we have

$x_0 \in P(\alpha, a, b)$, and consequently, by assumption (A1) we know $\alpha(x_0) = \alpha(Ax_0) > a$. This is a contradiction. If $\|x_0\| > b$, that is, $\|Ax_0\| > b$, then from condition (A2) it follows that $\alpha(Ax_0) > a$, i.e., $\alpha(x_0) > a$, which contradicts with $\alpha(x_0) = a$.

Therefore, $Ax \neq x$ for each $x \in \partial U$. This means the fixed point index $i(A, U, P)$ is well defined. Next we show

$$i(A,\ U,\ P) = 1. \tag{2.4}$$

By assumption (A1), we can choose $z_0 \in P(\alpha, a, b)$ such that $\alpha(z_0) > a$. From (2.3) we know $z_0 \in U$. Let

$$h(t, x) = tz_0 + (1 - t)Ax, \quad \text{for } t \in [0, 1] \text{ and } x \in \overline{U}. \tag{2.5}$$

Obviously, $h:\ [0, 1]\times\overline{U} \to P$ is completely continuous. We now prove that $x \neq h(t, x)$ for every pair $(t, x) \in [0, 1] \times \partial U$. Indeed, if there exists $(t_0, x_0) \in [0, 1] \times \partial U$ such that $h(t_0, x_0) = x_0$, then $\|x_0\| = c$ or $\alpha(x_0) = a$ holds since $x_0 \in \partial U$. In case $\|x_0\| = c$, by the definition of U and the condition (A3), we know $t_0 \in (0, 1)$. This together with (2.5) guarantees that

$$c = \|x_0\| \leq t\|z_0\| + (1 - t)\|Ax_0\| < tb + (1 - t)c \leq c,$$

which is a contradiction. If $\alpha(x_0) = a$ holds, we have the following two cases to consider. One case is that $\|Ax_0\| > b$ holds. In this case by condition (A2) we know $\alpha(Ax_0) > a$. From (2.1) it follows that

$$\alpha(x_0) = \alpha(h(t_0, x_0)) = \alpha(t_0z_0 + (1 - t_0)Ax_0) \geq t_0\alpha(z_0) + (1 - t_0)\alpha(Ax_0) > a,$$

which is a contradiction. The other case is that $\|Ax_0\| \leq b$ holds. In this case we know

$$\|x_0\| = \|h(t_0, x_0)\| = \|t_0z_0 + (1 - t_0)Ax_0\| \leq t_0\|z_0\| + (1 - t_0)\|Ax_0\| \leq b,$$

which means $x_0 \in P(\alpha, a, b)$. This together with condition (A1) guarantees that $\alpha(Ax_0) > a$. Therefore, similar as above, one can get $\alpha(x_0) > a$, which contradicts with $\alpha(x_0) = a$.

Consequently, we obtain that $h(t, x) \neq x$ for each pair $(t, x) \in [0, 1] \times \partial U$. By virtue of homotopy invariance and normality property of fixed point index we have

$$i(A,\ U,\ P) = i(z_0,\ U, P) = 1,$$

which implies that (2.4) holds.

On the other hand, from assumption (A3) and the proof of [7, Theorem 2.3.3-2.3.4], it is not difficult to see

$$i(A,\ P_e,\ P) = 0, \tag{2.6}$$

$$i(A,\ P_d,\ P) = 1, \tag{2.7}$$

either

$$i(A,\ P_c,\ P) = 1 \tag{2.8}$$

or

$$i(A,\ P_c,\ P) = 0. \tag{2.9}$$

Obviously, by (2.3) and the fact that $\alpha(x) \leq \|x\|$ for $x \in \overline{P}_c$ we know $P_a \bigcap U = \emptyset$. Notice that

$$P_e \subset P_d \subset P_a \subset P_c \quad \text{and} \quad U \subset P_c.$$

Using the additivity property of fixed point index, (2.4), and (2.6)-(2.9), we obtain that

$$i(A,\ P_d \setminus \overline{P}_e,\ P) = i(A,\ P_d,\ P) - i(A,\ P_e,\ P) = 1 - 0 = 1,$$

either

$$i(A,\ P_c \setminus \overline{(P_d \cup U)},\ P) = i(A,\ P_c,\ P) - i(A,\ P_d,\ P) - i(A,\ U,\ P) = 1 - 1 - 1 = -1$$

or

$$i(A,\ P_c \setminus \overline{(P_d \cup U)},\ P) = i(A,\ P_c,\ P) - i(A,\ P_d,\ P) - i(A,\ U,\ P) = 0 - 1 - 1 = -2.$$

These together with (2.4) and the solution property of fixed point index guarantee that there exist three positive fixed points x_1, x_2 and x_3 of operator A satisfying

$$x_1 \in P_d \setminus \overline{P}_e, \quad x_2 \in U, \quad \text{and } x_3 \in P_c \setminus \overline{(P_d \cup U)}.$$ □

From the proof of Theorem 2.2 and [7, Theorem 2.3.4], we can get the following Corollary.

Corollary 2.5. *Let operator $A: \ \overline{P}_c \setminus P_e \to P$ be completely continuous, where $c > e > 0$. Suppose the assumptions* (A1) *and* (A2) *of Theorem 2.2 hold. Then the operator A has at least one positive fixed point.*

Corollary 2.6. *Let operator $A: \ \overline{P}_c \to P$ be completely continuous. Suppose the assumptions* (A1) *and* (A2) *of Theorem 2.2 hold. In addition assume*

(A4) *$Ax \nleq x$ for $x \in \partial P_c$.*

Then operator A has at least two nonnegative fixed points.

Corollary 2.7. *Let operator $A: \ \overline{P}_c \setminus P_e \to P$ be completely continuous, where $c > e > 0$. Suppose the assumptions* (A1) *and* (A2) *of Theorem 2.2 hold. In addition assume*

(A5) *$Ax \nleq x$ for $x \in \partial P_e$, $Ax \ngeq x$ for $x \in \partial P_d$.*

Then operator A has at least two positive fixed points.

Corollary 2.8. *Let operator $A: \overline{P}_c \to P$ be completely continuous. Suppose the assumptions* (A1) *and* (A2) *of Theorem 2.2 hold. In addition assume*

(A6) *$Ax \ngeq x$ for $x \in \partial P_d$ and either $Ax \ngeq x$ for $x \in \partial P_c$ or $Ax \nleq x$ for $x \in \partial P_c$.*

Then operator A has at least three nonnegative fixed points.

Remark 2.9. Corollary 2.8 is quite similar to Leggett-Williams theorem when $Ax \ngeq x$ for $x \in \partial P_c$. However, the conditions of Corollary 2.8 are more extensive since Corollary 2.8 demands that $Ax \ngeq x$ or $Ax \nleq x$ only for $x \in \partial P_c$.

Remark 2.10. The assumption (A3) is used to guarantee (2.6)-(2.9) hold. Therefore, we may replace (A3) with other suitable conditions to guarantee (2.6)-(2.9) hold. For example, we may replace the condition that $x \neq \mu Ax$ for $x \in \partial P_c$ and $\mu \in [0,1]$ in Theorem 2.2 with the condition that $x \nleq Ax$ for $x \in \partial P_c$.

3. APPLICATIONS TO NONLINEAR SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

In this section we use Theorem 2.2 obtained in Section 2 to investigate the existence of triple positive solutions for the following nonlinear singular Sturm-Liouville boundary value problem

$$\begin{cases} (p(t)x'(t))' + f(t,x(t)) = 0, \quad t \in (0,1); \\ \alpha_1 x(0) - \beta_1 p(0)x'(0) = 0, \\ \alpha_2 x(1) + \beta_2 p(1)x'(1) = 0, \end{cases} \tag{3.1}$$

where $p \in C[[0,1],(0,+\infty)]$, $\alpha_i, \beta_i \geq 0$ $(i = 1,2)$, $\beta_1\alpha_2 + \alpha_1\alpha_2 + \alpha_1\beta_2 > 0$; $f \in C[(0,1) \times (0,+\infty), \mathbf{R}^+]$, that is, $f(t,x)$ may be singular at $t = 0$, $t = 1$, and $x = 0$, $\mathbf{R}^+ = [0,+\infty)$. Let

$$\tau_0(t) = \int_0^t \frac{ds}{p(s)}, \quad \tau_1(t) = \int_t^1 \frac{ds}{p(s)}, \quad \rho^2 = \beta_1\alpha_2 + \alpha_1\beta_2 + \alpha_1\alpha_2 \int_0^1 \frac{dt}{p(t)}, \ \rho > 0,$$

where $p(t)$, α_1, β_1, α_2, and β_2 are the same as in (3.1). Also define

$$u(t) = \frac{1}{\rho}[\beta_2 + \alpha_2\tau_1(t)], \quad v(t) = \frac{1}{\rho}[\beta_1 + \alpha_1\tau_0(t)].$$

Then $\alpha_2 v + \alpha_1 u \equiv \rho$.

The basic space used in this section is $C[0,1]$. Obviously, it is a Banach space if it is endowed with the norm $\|x\| = \max_{t\in J} |x(t)|$ for every $x \in C[0,1]$, where $J = [0,1]$. $x \in C[0,1]$ is said to be a positive solution of SBVP(3.1) if $x \in C[0,1]$ satisfies (3.1) and $x(t) > 0$ for $t \in (0,1)$. Now we list the following lemma from the literature.

Lemma 3.1 ([13], Lemma 2.1)**.** *Assume that a function x satisfies*

$$\left\{\begin{array}{ll} (p(t)x'(t))' + Q(t) = 0, & t \in (0,1); \\ \alpha_1 x(0) - \beta_1 p(0)x'(0) = 0, & \\ \alpha_2 x(1) + \beta_2 p(1)x'(1) = 0, & \end{array}\right. \tag{3.2}$$

where $Q \in L^1[0,1]$, $Q \geq 0$. Then

$$x(t) = \int_0^1 G(t,s)Q(s)ds \geq q(t)\|x\|, \quad \text{for } t \in (0,1) \tag{3.3}$$

and if the maximum of x on $[0,1]$ occurs at $\sigma \in [0,1]$, it is certain that $x'(\sigma) = 0$; conversely, if $x'(\sigma) = 0$ for some $\sigma \in [0,1]$, then the function x on $[0,1]$ will get its maximum at $t = \sigma$, where

$$G(t,s) = \left\{\begin{array}{ll} u(t)v(s), & 0 \leq s \leq t \leq 1; \\ u(s)v(t), & 0 \leq t \leq s \leq 1, \end{array}\right. \tag{3.4}$$

$$q(t) = \min\left\{\frac{\beta_1 + \alpha_1\tau_0(t)}{\beta_1 + \alpha_1\tau_0(1)}, \quad \frac{\beta_2 + \alpha_2\tau_1(t)}{\beta_2 + \alpha_2\tau_1(0)}\right\}, \tag{3.5}$$

$u(t)$ and $v(t)$ are the same as above.

From Lemma 3.1 we can get that $q(t) > 0$ for $t \in (0,1)$. For the sake of obtaining the existence of positive solutions for SBVP (3.1), let

$$P = \{x \in C[0,1] \big| \ x(t) \geq q(t)\|x\|, \ \forall t \in J\}, \tag{3.6}$$

where $J = [0,1]$, $q(t)$ is the same as in (3.5). It is easy to see that P is a nonempty, convex and closed subset of $C[0,1]$. Furthermore, one can prove that P is a cone of Banach space $C[0,1]$.

For convenience, let us list the following assumptions.

(H1) There exist functions $g,\ \hat{g} \in C[(0,1),\ \mathbf{R}^+]$, $h,\ \hat{h} \in C[(0,+\infty),\ \mathbf{R}^+]$ satisfying

$$\hat{g}(t)\hat{h}(x) \leq f(t,x) \leq g(t)h(x), \quad \text{for } t \in (0,1) \text{ and } x \in (0,\ +\infty)$$

and

$$\int_0^1 G(t,t)g(t)h_{r,R}(t)dt < +\infty, \quad \text{for } \forall R \geq r > 0,$$

where $h_{r,R}(t) =: \ \max\{h(u) \big| \ u \in [rq(t),\ R]\}$ for each $t \in (0,1)$, $G(t,s)$ is the same as in (3.4).

(H2) There exist two positive numbers a and b with $\bar{q}b \geq a$ such that

$$a < \bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)ds \cdot \min_{u \in [a,b]} \hat{h}(u),$$

where $\bar{q} = \min\{q(\frac{1}{4}),\ q(\frac{3}{4})\}$, q is the same as in (3.5).

(H3) There exists a positive number d with $d < a$ such that

$$\int_0^1 G(s,s)g(s)h_{d,d}(s)ds < d.$$

(H4) $\varliminf_{x\to 0+} \frac{h(x)}{x} > l$ and either $\varlimsup_{x\to+\infty} \frac{h(x)}{x} < L$ or $\varliminf_{x\to+\infty} \frac{\hat{h}(x)}{x} > \hat{L}$, where

$$l =: \Big(\max_{t\in J} q(t)\cdot \int_0^1 G(s,s)\hat{g}(s)q(s)ds\Big)^{-1}, \quad \hat{L} =: \Big(\bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)q(s)ds\Big)^{-1},$$

$$L =: \begin{cases} p_0\Big(\min\{\int_0^{\frac{1}{2}} sg(s)ds, \quad \int_{\frac{1}{2}}^1 (1-s)g(s)ds\}\Big)^{-1}, & \beta_1=\beta_2=0;\\ \Big(p_0\int_0^1 sg(s)ds\Big)^{-1}, & \beta_1=0,\ \beta_2\neq 0;\\ \Big(p_0\int_0^1 (1-s)g(s)ds\Big)^{-1}, & \beta_1\neq 0,\ \beta_2=0;\\ \Big(\int_0^1 G(s,s)g(s)ds\Big)^{-1}, & \beta_1\beta_2\neq 0, \end{cases}$$

$p_0 = \min_{t\in J} p(t).$

(H5) $\varliminf_{x\to 0+} \frac{h(x)}{x} > l$ and either $\lim_{x\to+\infty} \frac{h(x)}{x} = 0$ or $\lim_{x\to+\infty} \frac{\hat{h}(x)}{x} = +\infty$, where

$$l =: \Big(\max_{t\in J} q(t)\cdot \int_0^1 G(s,s)\hat{g}(s)q(s)ds\Big)^{-1}.$$

Now we are ready to state the main result in this section.

Theorem 3.2. *Assume that* (H1)-(H4) *are satisfied. Then SBVP*(3.1) *has at least three positive solutions.*

Since assumption (H5) means assumption (H4), we have the following corollary.

Corollary 3.3. *Assume that* (H1)-(H3) *and* (H5) *are satisfied. Then SBVP* (3.1) *has at least three positive solutions.*

Remark 3.4. Condition $\lim_{x\to+\infty} \frac{h(x)}{x} = 0$ in (H5) means $f(t,x)$ is sublinear at $+\infty$. Condition $\lim_{x\to+\infty} \frac{\hat{h}(x)}{x} = +\infty$ in (H5) means $f(t,x)$ is superlinear at $+\infty$.

For the sake of proving Theorem 3.2, we first define an operator on $P\setminus\{\theta\}$ by

$$(Ax)(t) =: \int_0^1 G(t,s)f(s,x(s))ds, \quad \text{for } x\in P\setminus\{\theta\}, \tag{3.7}$$

where the function $G(t,s)$ is the same as in (3.4), θ is the zero element of $C[0,1]$. Then we have the following Lemma.

Lemma 3.5. *Assume that* (H1) *holds. Then the operator A is well defined on $P\setminus\{\theta\}$ and for each $\delta > 0$, $A: \ P \setminus P_\delta \to P$ is completely continuous, where $P_\delta = \{x \in P \big| \ \|x\| < \delta\}$.*

Proof. Notice from (3.4) and (3.5) that

$$G(s,s) \geq G(t,s) \geq q(t)G(\tau,s), \quad \text{for } \forall t,s,\tau \in J. \tag{3.8}$$

For each $x \in P\setminus\{\theta\}$, by (3.6) we know that $x(t) \in [q(t)\|x\|, \ \|x\|]$ for each $t \in (0, \ 1)$. So from (H1) it follows that

$$0 \leq f(t,x(t)) \leq g(t)h(x(t)) \leq g(t)h_{\|x\|,\|x\|}(t), \quad \text{for } t \in (0,1).$$

This together with (3.7)-(3.8), (H1), and Lebesgue dominant convergence theorem implies that $(Ax)(t)$ is well defined and $Ax \in P$.

Now we show for each $\delta > 0$, $A: \ P \setminus P_\delta \to P$ is completely continuous.

For every bounded subset V of $P \setminus P_\delta$, let $AV =: \ \{Ax\big|x \in V\}$, $(AV)(t) =: \{(Ax)(t)\big|x \in V\}$ and $(AV)'(t) =: \ \{(Ax)'(t)\big|x \in V\}$ for $t \in (0,1)$.

First we show AV is relatively compact. Since there exists a positive number M such that $\|x\| \leq M$ for each $x \in V$, applying (H1) we obtain

$$0 \leq f(t,x(t)) \leq g(t)h(x(t)) \leq g(t)h_{\delta,M}(t), \quad \text{for } t \in (0,1). \tag{3.9}$$

This together with (3.7)-(3.8) and (H1) guarantees that there exists $\hat{M} > 0$ such that $\|Ax\| \leq \hat{M}$ for each $x \in V$. So, AV is bounded. Now we show $(AV)(t)$ are equicontinuous on $[0,1]$.

To see this according to boundary conditions of SBVP (3.1), we need to consider the following four cases.

Case (i) $\beta_1\beta_2 \neq 0$. In this case, $G(t,s) \geq \dfrac{\beta_1\beta_2}{\rho^2} > 0$. This together with (H1) and (3.9) guarantees that

$$\int_0^1 f(s,x(s))ds \leq \int_0^1 g(s)h_{\delta,M}(s)ds < +\infty, \quad \forall x \in V.$$

Combining the uniform continuity of $G(t,s)$ on $J \times J$, we obtain that $(AV)(t)$ are equicontinuous on J.

Case (ii) $\beta_1 = 0$, $\beta_2 \neq 0$. In this case we may choose $\alpha_1 = 1$. Then $v(t) = \dfrac{1}{\rho}\tau_0(t) = \dfrac{1}{\rho}\displaystyle\int_0^t \frac{ds}{p(s)}$, $v(0) = 0$, and $v(t)$ is increasing on J. We need to prove only that $\lim\limits_{t\to 0+}(Ax)(t) = 0$ uniformly with respect to $x \in V$ and $(AV)(t)$ are equicontinuous on $[\bar{\delta}, \ 1]$ for each $\bar{\delta} \in (0,1)$.

Notice that for $x \in V$, we have

$$(Ax)(t) = \int_0^1 G(t,s)f(s,x(s))ds = \int_0^t u(t)v(s)g(s)h_{\delta,M}(s)ds + \int_t^1 u(s)v(t)g(s)h_{\delta,M}(s)ds. \tag{3.10}$$

For arbitrary $\varepsilon > 0$, by (H1) there exists $\delta_1 > 0$ such that

$$\Big|\int_{t_1}^{t_2} G(s,s)g(s)h_{\delta,M}(s)ds\Big| < \varepsilon, \quad \text{for } \forall t_1, t_2 \in J \text{ with } |t_1 - t_2| < \delta_1.$$

Choose a positive number δ_2 such that $\delta_2 \le \delta_1$ and $v(\delta_2) \le \frac{\varepsilon}{c} v(\delta_1)$, where

$$c =: \int_0^1 G(s,s)g(s)h_{\delta,M}(s)ds < +\infty.$$

Then for each $t \in (0, \delta_2)$ we have by the monotonicity of $u(t)$ and $v(t)$ on J that

$$\int_0^t u(t)v(s)g(s)h_{\delta,M}(s)ds \le \int_0^t G(s,s)g(s)h_{\delta,M}(s)ds < \varepsilon$$

and

$$\begin{aligned}
& \int_t^1 u(s)v(t)g(s)h_{\delta,M}(s)ds \\
= & \int_t^{\delta_1} u(s)v(t)g(s)h_{\delta,M}(s)ds + \int_{\delta_1}^1 u(s)v(t)g(s)h_{\delta,M}(s)ds \\
\le & \int_t^{\delta_1} G(s,s)g(s)h_{\delta,M}(s)ds + \frac{v(t)}{v(\delta_1)} \int_{\delta_1}^1 u(s)v(s)g(s)h_{\delta,M}(s)ds \\
\le & \ \varepsilon + \frac{v(\delta_2)}{v(\delta_1)} \int_{\delta_1}^1 G(s,s)g(s)h_{\delta,M}(s)ds \\
\le & \ \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Combining the above two inequalities with (3.10) we obtain that $\lim_{t\to 0+}(Ax)(t) = 0$ uniformly with respect to $x \in V$.

To see the equicontinuity of $(AV)(t)$ on $[\bar{\delta},\ 1]$ for each $\bar{\delta} \in (0,1)$, notice from (3.7)-(3.9), and (H1) that

$$\begin{aligned}
& u(1)\Big[\int_0^t v(s)f(s,x(s))ds + \int_t^1 v(t)f(s,x(s))ds\Big] \\
\le & \int_0^t u(t)v(s)f(s,x(s))ds + \int_t^1 u(s)v(t)f(s,x(s))ds \\
= & \int_0^1 G(t,s)f(s,x(s))ds \\
\le & \int_0^1 G(s,s)g(s)h_{\delta M}(s)ds < +\infty, \quad \text{for } t \in (0,1) \text{ and } x \in V.
\end{aligned}$$

This together with $u(1) > 0$ and the fact that $v(t) \geq v(\bar{\delta})$ for $t \in [\bar{\delta},\ 1]$ guarantees that $\int_0^t v(s)f(s,x(s))ds$ and $\int_t^1 f(s,x(s))ds$ are uniformly bounded with respect to $t \in [\bar{\delta},\ 1]$ and $x \in V$.

Notice that for $t \in (0,1)$ and $x \in V$, we have

$$(Ax)'(t) = u'(t)\int_0^t v(s)f(s,x(s))ds + v'(t)\int_t^1 u(s)f(s,x(s))ds.$$

Taking into account the boundedness of $u'(t),\ v'(t)$, and $u(t)$ on J, we obtain that $(AV)'(t)$ are uniformly bounded on $[\bar{\delta},\ 1]$, which means $(AV)(t)$ are equicontinuous on $[\bar{\delta},\ 1]$.

Thus, $(AV)(t)$ are equicontinuous on J in this case.

Case (iii) $\beta_1 \neq 0,\ \beta_2 = 0$. Similar as in case (ii) we can prove in this case that $\lim\limits_{t\to 1-0}(Ax)(t) = 0$ uniformly with respect to $x \in V$ and $(AV)(t)$ are equicontinuous on $[0,\bar{\delta}]$ for each $\bar{\delta} \in (0,1)$. So we omit it.

Case (iv) $\beta_1 = 0,\ \beta_2 = 0$. Similar as in case (ii) and (iii) we can prove in this case that $\lim\limits_{t\to 0+}(Ax)(t) = 0$ and $\lim\limits_{t\to 1-0}(Ax)(t) = 0$ uniformly with respect to $x \in V$ and $(AV)(t)$ are equicontinuous on $[\bar{\delta},\ 1-\bar{\delta}]$ for each $\bar{\delta} \in (0,\frac{1}{2})$.

Consequently, applying the well-known Ascoli-Arzela theorem, we get that $A : P \setminus P_\delta \to P$ is relatively compact.

Finally we prove the operator $A:\ P \setminus P_\delta \to P$ is continuous. Suppose $x_n, x \in P \setminus P_\delta$ with $\|x_n - x\| \to 0$ as $n \to +\infty$. Then by (H1), (3.10), and Lebesgue dominant convergence theorem we obtain that

$$\lim_{n\to+\infty}(Ax_n)(t) = (Ax)(t), \quad \text{for } t \in J.$$

From above, one can see that $\{(Ax_n)(t)\}$ are equicontinuous and uniformly bounded on J(choosing $V = \{x_n\}$). By virtue of Ascoli-Arzela theorem we know $\{Ax_n\}$ is relatively compact. Now it remains to show $\lim\limits_{n\to+\infty}\|Ax_n - Ax\| = 0$.

If not, then there exist some $\varepsilon_0 > 0$ and $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\|Ax_{n_i} - Ax\| \geq \varepsilon_0, \quad \text{for } i = 1, 2, \cdots.$$

Since $\{Ax_n\}$ is relatively compact, there is a subsequence of $\{Ax_{n_i}\}$ converging to some $y \in C[J,P]$. we may still set, without loss of generality, that $\lim\limits_{i\to\infty} Ax_{n_i} = y$, namely, $\lim\limits_{i\to\infty}\|Ax_{n_i} - y\| = 0$. Then we know $y = Ax$. This is a contradiction.

Hence A is continuous on $P \setminus P_\delta$.

In conclusion, the proof is complete. □

Proof of Theorem 3.2. Define a functional $\alpha(x)$ on P by

$$\alpha(x) = \min\{x(t) \big|\ t \in [\frac{1}{4},\ \frac{3}{4}]\}, \quad \forall x \in P, \tag{3.11}$$

where P is defined as in (3.6). It is easy to see that the functional $\alpha(x)$ is nonnegative continuous and concave on P. Furthermore, we have $\alpha(x) \leq \|x\|$ for each $x \in P$.

In the following we prove that there exist positive numbers e and c such that conditions (A1)-(A3) of Theorem 2.2 hold. Taking into account (3.2) and (3.7), we have

$$\big(p(t)(Ax)'(t)\big)' = -f(t, x(t)) \leq 0, \quad \text{for } t \in (0,1) \text{ and } x \in P \setminus \{\theta\}. \tag{3.12}$$

Suppose $(Ax)(t)$ attains its maximum at $\sigma \in [0,1]$. Then by Lemma 3.1 we know

$$p(t)(Ax)'(t) \geq 0, \quad \text{for } t \in [0,\sigma]; \quad p(t)(Ax)'(t) \leq 0, \quad \text{for } t \in [\sigma, 1]. \tag{3.13}$$

Combining the fact that $p(t) > 0$ for $t \in J$, we obtain

$$\alpha(Ax) = \min\{(Ax)(\frac{1}{4}),\ (Ax)(\frac{3}{4})\}, \quad \text{for } x \in P. \tag{3.14}$$

Notice from (2.2) that $\frac{a+b}{2} \in P(\alpha, a, b)$ and $\alpha(\frac{a+b}{2}) > a$, which means $\{x \big|\ x \in P(\alpha,a,b),\ \alpha(x) > a\} \neq \emptyset$. On the other hand, for each $x \in P(\alpha, a, b)$, we have by (2.2) and (3.11) that $x(t) \in [a,\ b]$ for $t \in [\frac{1}{4},\ \frac{3}{4}]$. Therefore, applying (3.14), (3.8) , and conditions (H1)-(H2) we get

$$\begin{aligned}
\alpha(Ax) &= \min\Big\{\int_0^1 G(\frac{1}{4}, s) f(s, x(s))ds, \quad \int_0^1 G(\frac{3}{4}, s) f(s, x(s))ds\Big\} \\
&\geq \min\big\{q(\frac{1}{4}),\ q(\frac{3}{4})\big\} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s) f(s, x(s))ds \\
&\geq \bar{q} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)ds \cdot \min_{u \in [a,b]} \hat{h}(u) \\
&> a, \quad \text{for } x \in P(\alpha, a, b),
\end{aligned}$$

which implies that condition (A1) of Theorem 2.2 is satisfied.

Next for each $x \in \{x \in P \big|\ \alpha(x) > a,\ \|x\| < c, \text{ and } \|Ax\| > b\}$, we know by Lemma 3.1 that $Ax \in P$. Then from (3.14) and condition (H2), we conclude that

$$\alpha(Ax) \geq \min\{q(\frac{1}{4}),\ q(\frac{3}{4})\}\|Ax\| > \bar{q}b \geq a.$$

This means condition (A2) of Theorem 2.2 is met.

Now we are in position to show that condition (A3) of Theorem 2.2 is satisfied.

First choose a positive number with $l_1 > l$ such that $\varliminf_{x \to 0+} \frac{h(x)}{x} > l_1$. From (H4) we know there exists a positive number e with $e < d$ such that $\hat{h}(u) > l_1 u$ for $u \in (0, e)$. Now we show

$$Ax \not\leq x, \quad \text{for } x \in \partial P_e. \tag{3.15}$$

Suppose, on the contrary, there exists a $x \in \partial P_e$ such that $Ax \leq x$, that is,

$$\begin{aligned} x(t) &\geq (Ax)(t) \geq \int_0^1 G(t,s)\hat{g}(s)\hat{h}(x(s))ds \geq q(t)\int_0^1 G(s,s)\hat{g}(s)\big(l_1|x(s)|\big)ds \\ &\geq q(t)\int_0^1 G(s,s)\hat{g}(s)q(s)ds \cdot l_1\|x\|, \quad \text{for } t \in J, \end{aligned}$$

which is a contradiction. This means (3.15) holds.

Next we show

$$Ax \not\geq x, \quad \text{for } x \in \partial P_d. \tag{3.16}$$

If this is false, then there exists a $x \in \partial P_d$ such that $Ax \geq x$, which implies

$$x(t) \leq \int_0^1 G(t,s)g(s)h(x(s))ds \leq \int_0^1 G(s,s)g(s)h_{d,d}(s)ds < d, \quad \text{for } t \in J.$$

in contradiction with $\|x\| = d$. So, (3.16) holds.

In the following we first suppose $\overline{\lim\limits_{x\to+\infty}}\dfrac{h(x)}{x} < L$ according to condition (H4).

Without loss of generality, assume that $h(x) \not\equiv 0$. Then there exists a $x_0 \in (0,+\infty)$ such that $h(x_0) > 0$. Let

$$D_1(x) = \begin{cases} \max\limits_{u\in[x,x_0]} h(u), & x \in (0,x_0); \\ h(x_0), & x \geq x_0. \end{cases}$$

Thus $D_1(x) > 0$ is nonincreasing on $(0,+\infty)$. Again, let

$$D(x) = \begin{cases} \frac{xh(x_0)}{x_0 D_1(x)}, & x \in (0,x_0); \\ 1, & x \geq x_0. \end{cases} \tag{3.17}$$

Then $D \in C[\mathbf{R}^+,\mathbf{R}^+]$ is a nondecreasing function satisfying $D(x) > 0$ as $x > 0$. Notice that $0 \leq h(x) \leq D_1(x)$ for $x \in (0,x_0)$. Thus, $\lim\limits_{x\to 0+} h(x)D(x) = 0$. And consequently, $D(x)h(x) \in C[\mathbf{R}^+,\mathbf{R}^+]$. Moreover,

$$\overline{\lim_{x\to+\infty}}\frac{D(x)h(x)}{x} = \overline{\lim_{x\to+\infty}}\frac{h(x)}{x} < L. \tag{3.18}$$

Now we show that there exists a positive number c with $c > b$ sufficiently large such that

$$x \neq \mu Ax, \quad \text{for } \mu \in [0,1] \text{ and } x \in \partial P_c. \tag{3.19}$$

To see this we need to consider the following four cases.

Case (i) $\beta_1 = \beta_2 = 0$. In this case, from (H4) we can choose a positive number L_1 with $L_1 < L$ such that $\overline{\lim\limits_{x\to+\infty}}\dfrac{h(x)}{x} < L_1$. This together with (3.18) guarantees that there exists a positive number M_1 such that $D(x)h(x) \le L_1 x + M_1$ for each $x \in \mathbf{R}^+$. From (3.17) and condition (H4) we know there exists $c \ge b$ satisfying

$$p_0 \int_0^c D(u)du > (cL_1 + M_1) \min\{\int_0^{\frac{1}{2}} sg(s)ds, \quad \int_{\frac{1}{2}}^1 (1-s)g(s)ds\}. \tag{3.20}$$

Now we show the positive number c chosen as above satisfies our requirement, that is, (3.19) holds.

Indeed, suppose, on the contrary, (3.19) does not hold. Then there exist $\mu \in [0,1]$ and $x \in \partial P_c$ such that $x = \mu Ax$. Using (3.7) we can obtain

$$\big(p(t)x'(t)\big)' + \mu f(t, x(t)) = 0, \quad \forall t \in (0,1). \tag{3.21}$$

Notice that $\beta_1 = \beta_2 = 0$ yields that $x(0) = x(1) = 0$. Then $x(t)$ on $[0,1]$ takes its maximum at $\sigma \in (0,1)$. By Lemma 3.1 and its proof (see [13, Lemma 2.1]) we know $x'(\sigma) = 0$, $x'(t) \ge 0$ on $(0,\sigma)$, and $x'(t) \le 0$ on $(\sigma, 1)$. Integrating (3.19) from t $(t \in (0,\sigma))$ to σ together with the monotonicity of $D(x)$ yields that

$$\begin{aligned} p(t)x'(t) &= \mu \int_t^\sigma f(s, x(s))ds \le \int_t^\sigma g(s)h(x(s))ds \\ &\le \int_t^\sigma g(s)\frac{h(x(s))D(x(s))}{D(x(s))}ds \le \frac{1}{D(x(t))}\int_t^\sigma g(s)[L_1 x(s) + M_1]ds. \end{aligned}$$

Therefore,

$$p(t)D(x(t))x'(t) \le \int_t^\sigma g(s)[L_1 x(s) + M_1]ds \le (cL_1 + M_1)\int_t^\sigma g(s)ds.$$

Integrate this inequality from 0 to σ to obtain

$$\begin{aligned} p_0 \textstyle\int_0^c D(u)du &\le (cL_1 + M_1) \textstyle\int_0^\sigma \int_t^\sigma g(s)dsdt \\ &\le (cL_1 + M_1) \textstyle\int_0^\sigma sg(s)ds. \end{aligned} \tag{3.22}$$

Very similarly, integrating (3.19) from σ to t $(t \in (\sigma, 1))$, one can conclude

$$p_0 \int_0^c D(u)du \le (cL_1 + M_1)\int_\sigma^1 (1-s)g(s)ds.$$

From this and (3.22) we get a contradiction with (3.20). Therefore, (3.19) holds.

Case (ii) $\beta_1 = 0,\ \beta_2 \ne 0$. In this case, from (3.17) and condition (H4) it follows that there exists $c \ge b$ satisfying

$$p_0 \int_0^c D(u)du > (cL_1 + M_1)\int_0^1 sg(s)ds, \tag{3.23}$$

where L_1 is the same as in Case (i). Then (3.19) holds for such c.

If not, then there exist $\mu \in [0,1]$ and $x \in \partial P_c$ such that $x = \mu Ax$. Notice that $\beta_1 = 0$ and $\beta_2 \neq 0$ implies that $x(0) = 0$ and $x(1) \neq 0$. So, $x(t)$ on $[0,1]$ takes its maximum at $\sigma \in (0,1]$. By a process similar to proving (3.22), one can get

$$p_0 \int_0^c D(u)du \leq (cL_1 + M_1) \int_0^\sigma \int_t^\sigma g(s)dsdt \leq (cL_1 + M_1) \int_0^1 sg(s)ds$$

in contradiction with (3.23).

Case (iii) $\beta_1 \neq 0$, $\beta_2 = 0$. The proof is similar to Case (ii).

Case (iv) $\beta_1\beta_2 \neq 0$. As in Case (i), still choose L_1 with $L_1 < L$ such that $\overline{\lim_{x\to+\infty}} \frac{h(x)}{x} < L_1$. Then there exists a positive number c such that

$$\frac{h(x)}{x} < L_1, \quad \text{for } x > \bar{l}c, \tag{3.24}$$

where $\bar{l} =: \min\left\{\frac{\beta_1}{\beta_1 + \alpha_1\tau_0(1)}, \quad \frac{\beta_2}{\beta_2 + \alpha_2\tau_1(0)}\right\}$. We show (3.19) also holds for such c.

If not, then there exists a $\mu \in [0,1]$ and $x \in \partial P_c$ such that $x = \mu Ax$. Notice by (3.5) that $q(t) \geq \bar{l}$ for $t \in J$. Therefore, from (3.6) it follows that $x(t) \geq \bar{l}c$ for $t \in J$. Taking into account (3.24) and $L_1 \int_0^1 G(s,s)g(s)ds < 1$, we conclude that

$$\begin{aligned} x(t) &= \mu \int_0^1 G(t,s)f(s,x(s))ds \leq \int_0^1 G(t,s)g(s)h(x(s))ds \\ &\leq \int_0^1 G(s,s)g(s) \max_{u\in[c\bar{l},c]} h(u))ds \leq cL_1 \int_0^1 G(s,s)g(s)ds < c, \end{aligned}$$

which is in contradiction with $x \in \partial P_c$. Consequently, (3.19) holds.

Finally, suppose $\overline{\lim_{x\to+\infty}} \frac{\hat{h}(x)}{x} > \hat{L}$ according to condition (H4). Choose a positive number M with $M > \hat{L}$ such that $\overline{\lim_{x\to+\infty}} \frac{\hat{h}(x)}{x} > M$. Therefore, by (H4) we have

$$M\bar{q} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)q(s)ds > 1. \tag{3.25}$$

Since $\lim_{x\to+\infty} \frac{\hat{h}(x)}{x} > M$, there exists $c > 0$ satisfying

$$\hat{h}(x) \geq Mx, \quad \text{for } u \geq c\bar{q}. \tag{3.26}$$

Observe that for each $x \in \partial P_c$, we have

$$x(t) \geq q(t)\|x\| \geq c\bar{q}, \quad \text{as } t \in [\frac{1}{4}, \ \frac{3}{4}].$$

This together with (3.25) and (3.26) guarantees that

$$\begin{aligned}(Ax)(t) &= \int_0^1 G(t,s)f(s,x(s))ds \geq \bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)\hat{h}(x(s))ds \\ &\geq \bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)\cdot Mx(s)ds \geq M\bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)q(s)ds\cdot\|x\| \\ &> \|x\|, \quad \text{for } t\in[\frac{1}{4},\ \frac{3}{4}] \text{ and } x\in\partial P_c,\end{aligned}$$

which means $\|Ax\| > \|x\|$ for each $x \in \partial P_c$.

Thus, the condition (A3) of Theorem 2.2 holds.

In conclusion, by Theorem 2.2, SBVP(3.1) has at least three positive solutions.□

Example 3.6. Consider the following SBVP:

$$\begin{cases} x''(t) + f(t,x(t)) = 0, & t\in(0,1); \\ x(0) = x(1) = 0, \end{cases} \tag{3.27}$$

where

$$f(t,x) = \begin{cases} \dfrac{1}{4\sqrt{t(1-t)}}\Big(\dfrac{1}{x} + x^{\beta}\Big), & x\in(0,1)\cup(9,+\infty); \\[2ex] \dfrac{258x-256}{4\sqrt{t(1-t)}}, & x\in[1,2]; \\[2ex] \dfrac{260}{4\sqrt{t(1-t)}}, & x\in[2,8]; \\[2ex] \dfrac{\Big(\big(\frac{1}{9}+9^{\beta}-260\big)(x-8)+260\Big)}{4\sqrt{t(1-t)}}, & x\in(8,9), \end{cases} \tag{3.28}$$

for $t \in (0,1)$, $\beta > 0$, $\beta \neq 1$.

Then SBVP(3.27) has at least three positive solutions.

Proof. SBVP(3.27) can be regard as a SBVP of the form (3.1), where $p(t) \equiv 1$, $\alpha_1 = \alpha_2 = 1$, and $\beta_1 = \beta_2 = 0$. Obviously, $f \in C[(0,1)\times(0,+\infty), \mathbf{R}^+]$, and $f(t,x)$ is singular at $t=0$, $t=1$, and $x=0$. Let

$$g(t) = \hat{g}(t) = \frac{1}{4\sqrt{t(1-t)}} \tag{3.29}$$

and

$$h(x) = \hat{h}(x) = \begin{cases} \frac{1}{x} + x^{\beta}, & x \in (0,1) \cup (9,+\infty); \\ 258x - 256, & x \in [1,2]; \\ 260, & x \in [2,8]; \\ \big(\frac{1}{9} + 9^{\beta} - 260\big)(x-8) + 260, & x \in (8,9), \end{cases} \tag{3.30}$$

Notice here that

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1; \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

$q(t) = \min\{t,\ 1-t\}$, $\bar{q} = \frac{1}{4}$. Combining (3.28)-(3.30), it is not difficult to see (H1) is satisfied.

Next we show (H2) is met. Choose $a = 2$ and $b = 8$. Then

$$\bar{q}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)\hat{g}(s)ds \cdot \min_{u\in[a,b]} \hat{h}(u) = \frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{s(1-s)}{4\sqrt{s(1-s)}}ds \cdot 260 > \frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}\times 260 > 2,$$

which means (H2) is satisfied.

Thirdly, we show (H3) is satisfied. Notice that

$$\int_0^1 \sqrt{s(1-s)}ds = \frac{\pi}{8}, \quad \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi, \quad h_{d,d}(s) = h_{1,1}(s) \le \frac{1}{s(1-s)} + 1.$$

Then, one can see that

$$\int_0^1 G(s,s)g(s)h_{d,d}(s)ds \le \int_0^1 s(1-s)\frac{1}{4\sqrt{s(1-s)}}\Big(\frac{1}{s(1-s)} + 1\Big)ds \le \frac{1}{4}\big(\pi + \frac{\pi}{8}\big) < 1.$$

This implies condition (H3) is satisfied for $d = 1$.

Finally, from (3.30) we obtain that

$$\lim_{x\to 0+} \frac{h(x)}{x} = +\infty, \quad \lim_{x\to+\infty} \frac{h(x)}{x} = 0 \text{ if } \beta < 1, \quad \text{and } \lim_{x\to+\infty} \frac{\hat{h}(x)}{x} = +\infty \text{ if } \beta > 1.$$

So, condition (H4) is satisfied.

Consequently, by Theorem (3.1), SBVP(3.27) has at least three positive solutions. □

Acknowledgment. The author wishes to thank Professor Ravi P. Agarwal for his suggestion. The Project was supported by NNSF of P.R.China (10571111), China Scholarship Council and Natural Science Foundation of Shandong Province (Y2006A22).

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