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DOMAIN INVARIANCE THEOREMS FOR CONTRACTIVE TYPE MAPS

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ABSTRACT. New domain invariance theorems are presented for nonlinear contractions on spaces with two metrics.

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1. INTRODUCTION

One of the most important (and useful) consequences of Banach's contraction principle is the domain invariance theorem for contractive maps (see [3, 4] and the references therein). This result automatically produces an 'open mapping' or 'Fredholm alternative' type result for contractive maps. In this paper we present new domain invariance theorems for nonlinear contractions on spaces with two metrics. Our theory relies on fixed point results presented in [1, 5, 6] for nonlinear contractions on spaces with two metrics.

To conclude the introduction we present the results of [1, 5, 6] with some consequences which will be needed in Section 2. We discuss nonlinear contractions and we present both a local and global result. Throughout this paper (X, d') will be a complete metric space and d will be another metric on X. If $x_0 \in X$ and $r > 0$ let

$$B(x_0, r) = \{x \in X : \ d(x, x_0) < r\},$$

and we let $\overline{B(x_0, r)}^{d'}$ denote the d'–closure of $B(x_0, r)$.

THEOREM 1.1. Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $F : \overline{B(x_0, r)}^{d'} \to X$. Suppose there exists a continuous,

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nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x,\, y \in \overline{B(x_0,r)}^{d'}$ we have

$$d(F\,x, F\,y\,) \le \phi\,(\,d(x,y)\,). \tag{1.1}$$

In addition assume the following three properties hold:

$$d(x_0, F\,x_0) \;<\; r - \phi(r) \tag{1.2}$$

(1.3)

if $d \not\geq d'$ assume F is uniformly continuous from $(B(x_0,r), d)$ into (X, d')

and

$$\text{if } d \neq d' \text{ assume } F \text{ is continuous from } (\overline{B(x_0,r)}^{d'}, d') \text{ into } (X, d'). \tag{1.4}$$

Then F has a fixed point. That is there exists $x \in \overline{B(x_0,r)}^{d'}$ with $x = F\,x$.

It is worth stating the special case of Theorem 1.1 when $d = d'$.

THEOREM 1.2. Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$, and $F : \overline{B(x_0,r)}^{d} \to X$. Suppose there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x,\, y \in \overline{B(x_0,r)}^{d}$ we have

$$d(F\,x, F\,y\,) \le \phi(\,d(x,y)\,).$$

Also suppose

$$d(x_0, F\,x_0) \;<\; r - \phi(r).$$

Then there exists $x \in \overline{B(x_0,r)}^{d}$ with $x = F\,x$.

THEOREM 1.3. Let (X,d') be a complete metric space, d another metric on X, and $F : X \to X$. Suppose there exists a continuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x,\, y \in X$ we have

$$d(F\,x, F\,y\,) \le \phi\,(\,d(x,y)\,). \tag{1.5}$$

In addition assume the following two properties hold:

$$\text{if } d \not\geq d' \text{ assume } F \text{ is uniformly continuous from } (X,d) \text{ into } (X,d') \tag{1.6}$$

and

$$\text{if } d \neq d' \text{ assume } F \text{ is continuous from } (X,d') \text{ into } (X,d'). \tag{1.7}$$

Then F has a fixed point.

Theorem 1.3 immediately yields the following result of Boyd and Wong [2].

THEOREM 1.4. Let (X,d) be a complete metric space and $F : X \to X$. Suppose there exists a continuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x,\, y \in X$ we have

$$d(F\,x, F\,y\,) \le \phi\,(\,d(x,y)\,).$$

Then F has a fixed point.

REMARK 1.1. Theorem 1.2 follows from Theorem 1.4 if we notice that $F : \overline{B(x_0, r)}^d \to \overline{B(x_0, r)}^d$. If we assume $\lim_{x\to\infty}[x - \phi(x)] = \infty$ then Theorem 1.4 (with ϕ nondecreasing) can be deduced from Theorem 1.2 since if we fix $x_0 \in X$ then there exists $r > 0$ with $d(F\,x_0, x_0) < r - \phi(r)$.

For our domain invariance theorem we will need the following results for nonlinear contractions.

THEOREM 1.5. Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$, and $F : B(x_0, r) \to X$. Suppose there exists a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in B(x_0, r)$ we have

$$d(F\,x, F\,y\,) \le \phi(\,d(x, y)\,). \tag{1.8}$$

Also suppose

$$d(x_0, F\,x_0) \;<\; r - \phi(r). \tag{1.9}$$

Then F has a fixed point in $B(x_0, r)$.

PROOF. Now (1.9) and the fact that ϕ is continuous guarantees that there exists r_0, $0 < r_0 < r$ with

$$d(x_0, F\,x_0) \;<\; r_0 - \phi(r_0). \tag{1.10}$$

Now $F : \overline{B(x_0, r_0)}^d \to X$, so we may apply Theorem 1.2 to deduce the existence of an $x \in \overline{B(x_0, r_0)}^d$ with $x = F(x)$. ∎

THEOREM 1.6. Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$, and $F : B(x_0, r) \to X$. Suppose there exists a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in B(x_0, r)$ we have

$$d(F\,x, F\,y\,) \le \phi\,(\,d(x, y)\,). \tag{1.11}$$

In addition assume there exists r_0, $0 < r_0 < r$ with

$$\overline{B(x_0, r_0)}^{d'} \subseteq B(x_0, r) \;\text{ and }\; d(x_0, F\,x_0) \;<\; r_0 - \phi(r_0) \tag{1.12}$$

holding. Finally suppose the following two conditions hold:

(1.13)

if $d \not\geq d'$ assume F is uniformly continuous from $(B(x_0, r_0), d)$ into (X, d')

and

(1.14) if $d \neq d'$ assume F is continuous from $(\overline{B(x_0, r_0)}^{d'}, d')$ into (X, d').

Then F has a fixed point.

PROOF. Now $F : \overline{B(x_0, r_0)}^{d'} \to X$ and we may apply Theorem 1.1 to deduce the result. ∎

REMARK 1.2. In (1.13) we could replace $(B(x_0,r_0),d)$ with $(B(x_0,r),d)$ and in (1.14) we could replace $(\overline{B(x_0,r_0)}^{d'},d')$ with $(B(x_0,r),d')$.

A special case of Theorem 1.6 is the following result.

THEOREM 1.7. Let (X,d') be a complete metric space, d another metric on X, $x_0 \in X$, $r>0$, and $F: B(x_0,r) \to X$. Suppose there exists a continuous, nondecreasing function $\phi:[0,\infty) \to [0,\infty)$ satisfying $\phi(z)<z$ for $z>0$ such that for $x,\, y \in B(x_0,r)$ we have

$$d(F\,x, F\,y\,) \le \phi\,(\,d(x,y)\,). \tag{1.15}$$

In addition suppose the following conditions hold:

$$\exists\ M>0 \ \text{ with } \ d(x,y) \le M\,d'(x,y)\ \forall\ x,\, y \in X \tag{1.16}$$

$$d(x_0, F\,x_0)\ <\ r-\phi(r) \tag{1.17}$$

(1.18)

if $d \not\geq d'$ assume F is uniformly continuous from $(B(x_0,r),d)$ into (X,d')

and

(1.19) if $d \neq d'$ assume F is continuous from $(B(x_0,r),d')$ into (X,d').

Then F has a fixed point.

PROOF. The result follows immediately from Theorem 1.6 once we show (1.12) holds. To see this notice (1.17) guarantees that there exists r_0, $0<r_0<r$ with $d(x_0,F\,x_0)\ <\ r_0-\phi(r_0)$. Also there exists $\epsilon>0$ with $M\,\epsilon+r_0<r$. Let $x \in \overline{B(x_0,r_0)}^{d'}$. We must show $d(x,x_0)<r$. Now there exists a sequence $\{x_n\} \subseteq B(x_0,r_0)$ with $d'(x_n,x) \to 0$ as $n \to \infty$. In particular there exists $N \in \{1,2,...\}$ with $d'(x_n,x)<\epsilon$ for $n \ge N$. Now

$$d(x,x_0) \le d(x,x_N)+d(x_N,x_0) \le M\,d'(x,x_N)+r_0 \le M\,\epsilon+r_0\ <\ r,$$

so (1.12) holds. ∎

2. DOMAIN INVARIANCE THEOREMS

We begin with a result for nonlinear contractions when $d=d'$.

THEOREM 2.1. Suppose $E=(E,\|\,.\,\|)$ is a Banach space, U is an open subset of E, and $F:U \to E$ is such that

$$\|F(x)-F(y)\| \le \phi(\|x-y\|) \quad \text{for all } \ x,\, y \in U; \tag{2.1}$$

here $\phi:[0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function satisfying $\phi(z)<z$ for $z>0$. Then

(a). $I-F:U \to E$ is an open mapping,

and

(b). $I - F : U \to (I - F)(U)$ is a homeomorphism.

PROOF. Let $f = I - F$ and let d be the metric induced by the norm $\|\,.\,\|$. We claim for any $u \in U$ that

$$\text{(2.2)} \qquad \text{if} \quad B(u, r) \subseteq U \quad \text{then} \quad B(f(u), r - \phi(r)) \subseteq f(B(u, r)).$$

If (2.2) is true then (a) follows. To see this let V be an open subset of U and let $x \in f(V)$. Then $\exists\, u \in V$ with $f(u) = x$, and $\exists\, r > 0$ with $B(u, r) \subseteq V$. Now (2.2) implies

$$B(x, r - \phi(r)) = B(f(u), r - \phi(r)) \subseteq f(B(u, r)) \subseteq f(V),$$

so $f(V)$ is open.

It remains to show (2.2). Fix $x_0 \in B(f(u), r - \phi(r))$ and define the map $H : B(u, r) \to E$ by $H(x) = x_0 + F(x)$. Notice for $x,\, y \in B(u, r)$ that we have

$$d(H(x), H(y)) = d(x_0 + F(x), x_0 + F(y)) = \|F(x) - F(y)\| \le \phi(d(x, y)).$$

Also

$$d(H(u), u) = \|H(u) - u\| = \|x_0 - f(u)\| \,<\, r - \phi(r).$$

Theorem 1.5 guarantees that there exists $y_0 \in B(u, r)$ with $y_0 = H(y_0)$ i.e. $f(y_0) = x_0$. Thus (2.2) holds.

Now since $f : U \to f(U)$ is open and continuous, to show (b) it suffices to show f is injective. Given $u,\, v \in U$ notice

$$\|f(u) - f(v)\| \ge \|u - v\| - \|F(u) - F(v)\| \ge \|u - v\| - \phi(\|u - v\|).$$

Thus if $f(u) = f(v)$, then $\|u - v\| \le \phi(\|u - v\|)$ which forces $\|u - v\| = 0$ since $\phi(z) < z$ for $z > 0$. ∎

THEOREM 2.2. Suppose $E = (E, \|\,.\,\|)$ is a Banach space and $F : E \to E$ is such that

$$\text{(2.3)} \qquad \|F(x) - F(y)\| \le \phi(\|x - y\|) \quad \text{for all} \ \ x,\, y \in E;$$

here $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function satisfying $\phi(z) < z$ for $z > 0$. Then $I - F : E \to E$ is a homeomorphism.

PROOF. Let $f = I - F$. From Theorem 2.1 it remains to show $f(E) = E$. Let $x_0 \in E$ and let $H : E \to E$ be given by $H(x) = x_0 + F(x)$. Notice for $x,\, y \in E$ that we have

$$d(H(x), H(y)) = \|F(x) - F(y)\| \le \phi(d(x, y)).$$

Theorem 1.4 implies that there exists $y_0 \in E$ with $y_0 = H(y_0)$. That is $f(y_0) = x_0$. ∎

Next we discuss the case when $d \neq d'$.

THEOREM 2.3. Let $(E, \|\,.\,\|')$ be a Banach space and $(E, \|\,.\,\|)$ a normed linear space, and let d' (respectively d) be the metric induced by $\|\,.\,\|'$ (respectively $\|\,.\,\|$). Let U be $\|\,.\,\|$–open subset of E and $F : U \to E$ is such that

$$\|F(x) - F(y)\| \leq \phi(\|x - y\|) \quad \text{for all} \quad x,\, y \in U; \tag{2.4}$$

here $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function satisfying $\phi(z) < z$ for $z > 0$. Fix $u \in U$ and $r > 0$ so that $B(u, r) \subseteq U$. Suppose there exists r_0, $0 < r_0 < r$ with $\overline{B(u, r_0)}^{d'} \subseteq B(u, r)$. In addition suppose the following two conditions hold:

(2.5)
if $d \not\geq d'$ assume F is uniformly continuous from $(B(u, r_0), d)$ into (E, d')

and

$$\text{if } d \neq d' \text{ assume } F \text{ is continuous from } (\overline{B(u, r_0)}^{d'}, d') \text{ into } (E, d'). \tag{2.6}$$

Then if $f = I - F$ we have

$$B(f(u), r_0 - \phi(r_0)) \subseteq f(B(u, r)). \tag{2.7}$$

PROOF. Fix $x_0 \in B(f(u), r_0 - \phi(r_0))$ and define the map $H : B(u, r) \to E$ by $H(x) = x_0 + F(x)$. Notice for $x,\, y \in B(u, r)$ that we have

$$d(H(x), H(y)) = \|F(x) - F(y)\| \leq \phi(d(x, y))$$

and

$$d(H(u), u) = \|x_0 - f(u)\| \,<\, r_0 - \phi(r_0).$$

Theorem 1.6 guarantees that there exists $y_0 \in \overline{B(u, r_0)}^{d'} \subseteq B(u, r)$ with $y_0 = H(y_0)$ i.e. $f(y_0) = x_0$. Thus (2.7) holds. ∎

REMARK 2.1. In Theorem 2.3 it is easy to check that $I - F : U \to (I - F)(U)$ is injective.

REMARK 2.2. If $U = E$ in Theorem 2.3 and if (2.5) and (2.6) are replaced by

$$\text{if } d \not\geq d' \text{ assume } F \text{ is uniformly continuous from } (E, d) \text{ into } (E, d') \tag{2.8}$$

and

$$\text{if } d \neq d' \text{ assume } F \text{ is continuous from } (E, d') \text{ into } (E, d'), \tag{2.9}$$

then $f(E) = E$. To see this let $x_0 \in E$ and let $H : E \to E$ be given by $H(x) = x_0 + F(x)$. Notice for $x,\, y \in E$ that we have

$$d(H(x), H(y)) = \|F(x) - F(y)\| \leq \phi(d(x, y)).$$

Theorem 1.3 guarantees that there exists $y_0 \in E$ with $y_0 = H(y_0)$ i.e. $f(y_0) = x_0$.

REMARK 2.3. In Theorem 2.3 suppose we replace $\exists\, r_0,\ 0 < r_0 < r$ with $\overline{B(u,r_0)}^{d'} \subseteq B(u,r)$ with

(2.10) $$\exists\ M > 0 \ \text{ with } \ d(x,y) \le M\, d'(x,y)\ \forall\ x,\, y \in E,$$

and if we replace (2.5) and (2.6) with

(2.11)
$$\text{if } \ d \not\geq d' \ \text{ assume } \ F \ \text{ is uniformly continuous from } \ (B(u,r),d) \ \text{ into } \ (E,d')$$

and

(2.12) $$\text{if } \ d \neq d' \ \text{ assume } \ F \ \text{ is continuous from } \ (B(u,r),d') \ \text{ into } \ (E,d'),$$

then in the conclusion of Theorem 2.3 one can replace (2.7) with

(2.13) $$B(f(u), r - \phi(r)) \subseteq f(B(u,r)).$$

To see this apply Theorem 1.7 (notice in this case that if $x_0 \in B(f(u), r-\phi(r))$ then $d(H(u),u) < r - \phi(r))$ instead of Theorem 1.6.

THEOREM 2.4. Let $(E, \|\,.\,\|')$ be a Banach space and $(E, \|\,.\,\|)$ a normed linear space, and let d' (respectively d) be the metric induced by $\|\,.\,\|'$ (respectively $\|\,.\,\|$). Let U be $\|\,.\,\|$–open subset of E and $F : U \to E$ is such that

(2.14) $$\|F(x) - F(y)\| \le \phi(\|x-y\|) \quad \text{for all} \ \ x,\, y \in U;$$

here $\phi : [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function satisfying $\phi(z) < z$ for $z > 0$. Suppose the following three conditions are satisfied:

(2.15) $$\begin{gathered}\text{for any } \ u \in U \ \text{ and } \ r > 0 \ \text{ with } \ B(u,r) \subseteq U,\ \exists\, r_0(r),\ 0 < r_0(r) < r \\ \text{with } \ \overline{B(u,r_0(r))}^{d'} \subseteq B(u,r)\end{gathered}$$

(2.16) $$\text{if } \ d \not\geq d' \ \text{ assume } \ F \ \text{ is uniformly continuous from } \ (U,d) \ \text{ into } \ (E,d')$$

and

(2.17) $$\text{if } \ d \neq d' \ \text{ assume } \ F \ \text{ is continuous from } \ (U,d') \ \text{ into } \ (E,d').$$

Then $I - F : (U,d) \to (E,d)$ is an open mapping.

PROOF. Let $f = I - F$ and $u \in U$. From (2.15) (and Theorem 2.3) we have that

(2.18)
$$\text{if } \ B(u,r) \subseteq U \ \text{ then } \ \exists\, r_0,\ 0 < r_0 < r \text{ with } \ B(f(u), r_0 - \phi(r_0)) \subseteq f(B(u,r)).$$

Let V be a $\|\,.\,\|$–open subset of U and let $x \in f(V)$. Then there exists $u \in V$ with $f(u) = x$, and there exists $r > 0$ with $B(u,r) \subseteq V$. Now (2.18) implies

$$B(x, r_0 - \phi(r_0)) \subseteq f(B(u,r)) \subseteq f(V),$$

so $f(V)$ is $\|\,.\,\|$–open. ∎

THEOREM 2.5. Let $(E, \|\,.\,\|')$ be a Banach space and $(E, \|\,.\,\|)$ a normed linear space, and let d' (respectively d) be the metric induced by $\|\,.\,\|'$ (respectively $\|\,.\,\|$). Let U be $\|\,.\,\|$–open subset of E and $F : U \to E$ is such that

$$\|F(x) - F(y)\| \leq \phi(\|x - y\|) \quad \text{for all} \quad x,\, y \in U; \tag{2.19}$$

here $\phi : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function satisfying $\phi(z) < z$ for $z > 0$. Suppose the following three conditions are satisfied:

$$\exists\; M > 0 \;\text{ with }\; d(x, y) \leq M\, d'(x, y)\; \forall\; x,\, y \in E \tag{2.20}$$

$$\text{if }\; d \not\geq d' \;\text{ assume }\; F \;\text{ is uniformly continuous from }\; (U, d) \;\text{ into }\; (E, d') \tag{2.21}$$

and

$$\text{if }\; d \neq d' \;\text{ assume }\; F \;\text{ is continuous from }\; (U, d') \;\text{ into }\; (E, d'). \tag{2.22}$$

Then $I - F : (U, d) \to (E, d)$ is an open mapping.

PROOF. The result follows from Theorem 2.4 once we show (2.15) holds. To see this fix $u \in U$ and $r > 0$ with $B(u, r) \subseteq U$. Choose $0 < r_0 < r$. Then there exists $\epsilon > 0$ with $M\,\epsilon + r_0 < r$. Let $x \in \overline{B(u, r_0)}^{d'}$. We must show $d(x, u) < r$. Now there exists a sequence $\{x_n\} \subseteq B(u, r_0)$ with $d'(x_n, x) \to 0$ as $n \to \infty$. In particular there exists $N \in \{1, 2, ...\}$ with $d'(x_n, x) < \epsilon$ for $n \geq N$. Now

$$d(x, u) \leq d(x, x_N) + d(x_N, u) \leq M\, d'(x, x_N) + r_0 \leq M\,\epsilon + r_0 \;<\; r,$$

and as a result (2.15) holds. ∎

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