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GLOBAL EXISTENCE FOR RETARDED VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH HILLE-YOSIDA OPERATORS

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ABSTRACT. In this paper, we study the existence and uniqueness of classical global solutions of some integrodifferential equations with infinite delay. We loosen the conditions of the integral term in the equations which are more general than those that have been mentioned in many previous studies. Some sufficient conditions are given which ensure the existence and uniqueness of solutions on $[0,\infty)$. We assume linear part is not necessary to be densely defined and satisfies a Hille-Yosida condition. By using matrix operators and fixed point theorems, we obtain new results for the retarded integrodifferential equations.

Keywords: Hille-Yosida operator, retarded Volterra integrodifferential equation, integrated semigroup, C_0-semigroup, extrapolated semigroup

1. INTRODUCTION

In this paper we consider the following partial functional differential equations with infinite delay

$$(RACP) \qquad \begin{cases} u'(t) = Au(t) + F(t, u_t), & t \geq 0, \\ x_0 = \varphi \in \mathcal{P}, \end{cases}$$

where $A : D(A) \subseteq X \to X$ is a closed linear operator in an infinite-dimensional Banach space $(X, \|\cdot\|)$, the phase space $\mathcal{P}$ is a linear space of function mapping $(-\infty, 0]$ into X satisfying some axioms which will be described later, F is an X-valued function defined on $[0,\infty) \times \mathcal{P}$, and $u_t : (-\infty, 0] \to X$, $t \geq 0$, is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in (-\infty, 0].$$

Moreover, φ represents the initial condition of the given system throughout this paper. The equation (RACP) has been studied by many authors (cf. [1], [2], [3] and [15] etc.).

In the literature devoted to Eq. (RACP) with finite delay, the state space is the space of all continuous function on $[-r, 0]$, $r > 0$, endowed with the uniform

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norm topology. The investigation of functional differential equations with infinite delay in an abstract admissible phase was initialed by Hale and Kato [8], Kapple and Schappacher [13], and Schumacher [18]. The method of using admissible phase spaces enables one to treat a large class of functional differential equations with infinite delay in the same time and obtain more general results. For a detail discussion on this topic, we refer the reader to the book by Hino et, al. [12].

There have been a great deal of works contributed to the study of finite delay by using different methods under different conditions. The most original work is from Travis and Webb [20]. In the study of Eq. (RACP), one assumes that the operator A generates a C_0-semigroup on X. In this case, A must be densely defined and satisfies the Hille-Yosida condition. More recently, in [1], the authors show that the density of domain for A is not necessary to deal with finite or infinite delay. For the study of Eq. (RACP), we refer the reader to Hale and Kato [8] and Hino et, al. [12]. We shall focus on the case that A is not densely defined but satisfies the Hille-Yosida condition. In [1], the authors treated Eq. (RACP) by using variation-of-constant formula and integrated semigroups to extend the results of Henriquez [9], [10] and [11]. In their cases, F satisfies the local Lipschitz condition with respect to the phase space. For a detail discussion on delay equations, we refer the reader to the book by Wu [21].

Based on ideas in [1] and the usage of the extrapolation approach which is introduced by Nagel and Sinestrari [17], we consider the following equation

$$(VID1) \qquad \begin{cases} u'(t) = Au(t) + \int_0^t B(t,\theta,u(\theta))d\theta + F(t,u_t), & t \geq 0, \\ u_0 = \varphi \in \mathcal{P}, \end{cases}$$

where $B \in C(\Gamma \times X, X)$, $\Gamma = \{(s,t) \in \mathbb{R}^2 |\ 0 \leq t \leq s < \infty\}$ and $F \in C([0,\infty) \times \mathcal{P}, X)$. We point out that the Eq. (VID1) can be transformed into

$$\begin{cases} u'(t) = Au(t) + G(t,u_t), & t \geq 0, \\ u_0 = \varphi \in \mathcal{P}, \end{cases}$$

by setting $G(t,u_t) := \int_{-t}^0 B(t,t+\theta,u_t(\theta))d\theta + F(t,u_t)$. However, in order to use this transformation and apply the method in [1], we have to assume that B is a function from $\Gamma \times \mathcal{P}$ into X which is different from our assumptions (in our cases, B is defined on $\Gamma \times X$). We do not treat Eq. (VID1) by this transformation. In general, the conclusions in [1] cannot be applied to our cases.

The purpose of this paper not only consider the Eq. (VID1) but also solve the equation

$$(VID2) \qquad \begin{cases} u'(t) = A[u(t) + \int_0^t B(t-\theta)u(\theta)d\theta] + F(t,u_t), & t \geq 0, \\ u_0 = \varphi \in \mathcal{P} \end{cases}$$

where $B(\cdot)$ is a bounded linear operator from X into X. The Eq. (VID2) without delay was studied in [3], [6] and [16]. We will generalize the method in [6] to solve it.

The obtained results would be an extension of [6]. Finally, the following equation

$$(VID3)\qquad \begin{cases} u'(t) = A[u(t) + \int_{-\infty}^{t} B(t-\theta)u(\theta)d\theta] + F(t, u_t), \quad t \geq 0, \\ u_0 = \varphi \in \mathcal{P} \end{cases}$$

is also studied.

In section 2, we recall some preliminary results about the extrapolated spaces and semigroups. In section 3, we prove the existence and uniqueness of solutions to Eq. (VID1), (VID2), and (VID3), the main results of this paper. Moreover, we give a growth bound of solutions of Eq. (VID1). In the final section we give some examples to show our result are valuable.

2. PRELIMINARY

In this section we give some basic definitions and results of extrapolation spaces that are required in this paper. For more information about extrapolation spaces, see [5], [6], [14] and [17].

Let X be a Banach space and let A be a linear operator with domain $D(A)$.

Definition 2.1. The linear operator A is a Hille-Yosida operator on X if there exists $w \in \mathbb{R}$ and $M \geq 1$ such that $(\omega, +\infty) \subset \rho(A)$ ($\rho(A)$ is the usual resolvent set of A) and satisfies

$$(HY)\qquad \|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all} \quad \lambda > \omega \quad \text{and} \quad n \in \mathbb{N}.$$

Throughout this paper, we assume that A is a Hille-Yosida operator, $T > 0$ is an extended real number and without loss of generality, we set $\omega < 0$. The domain $D(A)$ of A is not necessarily dense in X and we denote its closure in X by X_0. The part A_0 of A in X_0 is the linear operator with domain $D(A_0) = \{x \in D(A) | Ax \in X_0\}$ that defined by $A_0 x = Ax$ for all $x \in D(A_0)$. Here is an elementary property of the Hille-Yosida operator. The proof can be found in [5].

Proposition 2.2. *The part A_0 of A in X_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0. Moreover, $\|T_0(t)\| \leq Me^{\omega t}$ for $t \geq 0$.*

Definition 2.3. For a fixed $\lambda_0 \in \rho(A)$, we define a new norm on X_0 by

$$\|x\|_{-1} = \|R(\lambda_0, A_0)x\| \text{ for } x \in X_0.$$

The completion of $(X_0, \|\cdot\|_{-1})$, denoted by X_{-1}, is called the extrapolation space of X associated with A.

Note that the norm $\|\cdot\|_{-1}$ associated to λ_0 and the norm on X_0 given by $\|R(\lambda, A)x\|$ for different $\lambda \in \rho(A)$ are equivalent. The operator $T_0(t)$ has a bounded linear extension $T_{-1}(t)$ to the Banach space X_{-1} and $T_{-1}(\cdot)$ is a strongly continuous semigroup on X_{-1}. $T_{-1}(\cdot)$ is called the extrapolated semigroup of $T_0(\cdot)$.

Proposition 2.4 ([17], Proposition 2.1). *The following properties hold:*

(i) *The space X_0 is dense in $(X_{-1}, \|\cdot\|_{-1})$. Hence the extrapolation space X_{-1} is also the completion of $(X, \|\cdot\|_{-1})$.*

(ii) *For $f \in L^1_{loc}(\mathbb{R}^+, X)$ (i.e. f is a function locally integrable) and $t > 0$, let*

$$(T_{-1} * f)(t) := \int_0^t T_{-1}(t-s)f(s)ds.$$

Then $(T_{-1} * f)(t) \in X_0$ and $\|(T_{-1} * f)(t)\| \le M_1 \|f\|_{L^1((0,t),X)}$ where M_1 is a constant independent of f and t.

Consider the abstract Cauchy problem

$$(ACP) \qquad \begin{cases} u'(t) = Au(t) + f(t), & t \ge 0, \\ u(0) = x \end{cases}$$

The following result has been proved in [17].

Proposition 2.5. *If $x \in D(A)$, $f \in W^{1,1}([0,T], X)$ and $Ax + f(0) \in X_0$, then there exists a unique solution u of Eq. (ACP) on the interval $[0,T]$, and*

$$\|u(t)\| \le Me^{\omega t}(\|x\| + \int_0^t e^{-\omega s}\|f(s)\|ds)$$

for each $t \in [0,T]$.

For the rest of this paper, M_1 and M always denote the constants in Proposition 2.4 and Proposition 2.5, respectively.

Finally, we give the basic definition and properties of integrated semigroups which can be found in [14], [19] and their references. Let $L(X)$ always denote the set of all bounded linear operators on X.

Definition 2.6. A family $\{S(t); t \in [0, \infty)\} \subset L(X)$ is called an integrated semigroup if the following conditions are satisfied.

(i) $S(0) = 0$.

(ii) For every $x \in X$, $t \mapsto S(t)x$ is a continuous function of $t \ge 0$ with values in X.

(iii) $S(t)S(s) = \int_0^s (S(t+r) - S(r))dr$ for each $t, s \ge 0$.

$S(\cdot)$ is said to be *nondegenerate* if $S(t)x = 0$ for all $t > 0$ implies $x = 0$. The *generator* B of a nondegenerate integrated semigroup $S(\cdot)$ is defined as follows:

$$x \in D(B) \text{ and } Bx = y \text{ if and only if } S(t)x = \int_0^t S(u)ydu + tx \text{ for } t \ge 0.$$

Proposition 2.7. *Let B generate an integrated semigroup $S(\cdot)$. Then for all $x \in X$ and $t \ge 0$, we have*

$$\int_0^t S(s)xds \in D(B) \text{ and } S(t)x = B\int_0^t S(s)xds + tx.$$

In [14], it is shown that A generates an integrated semigroup $S(\cdot)$ on X and $S(t)x = \int_0^t T_0(s)xds$ for $x \in X_0$ and $t \geq 0$. Hence, by Proposition 2.4(i) we derive that

$$S(t)x = \int_0^t T_{-1}(s)xds \tag{2.1}$$

for all $x \in X$ and $t \geq 0$. (2.1) will be used later.

3. GLOBAL EXISTENCE OF SOLUTIONS

In this section, we prove global existence and uniqueness of the solutions to equations (VID1) through (VID3) in an integrated form by using a variation-of-constants formula in the sense of extrapolated semigroups. Then we give a growth bound of the solutions of Eq. (VID1).

Definition 3.1. We say that a function $u : \mathbb{R} \to X$ is a classical solution of Eq. (VID1) on $[0, \infty)$ if u satisfies the following conditions

(i) $u \in C^1([0, \infty), X) \cap C([0, \infty), D(A))$,
(ii) u satisfies $(VID1)$ on $[0, \infty)$.
(iii) $u(t) = \varphi(t)$ for $-\infty < t \leq 0$.

Let us consider the abstract Cauchy problem. Since the mild solution of Eq. (ACP) is given by

$$u(t) = T_0(t)x + \int_0^t T_{-1}(t-s)f(s)ds, \quad t > 0$$

for $x \in X_0$. So, we give the following definition

Definition 3.2. Let $\varphi(0) \in X_0$. We say that a continuous function $u : \mathbb{R} \to X$ is a mild solution of Eq. (VID1) on $[0, \infty)$ if u satisfies the following equation

$$(IE) \qquad \begin{cases} u(t) & = T_0(t)\varphi(0) + \int_0^t T_{-1}(t-s) \int_0^s B(s, \theta, u(\theta))d\theta ds \\ & \qquad + \int_0^t T_{-1}(t-s)F(s, u_s)ds, \\ u_0 = \varphi & \end{cases}$$

on $[0, \infty)$. We denote this solution by $u(\cdot, \varphi)$.

For the rest this paper, we assume that the phase space $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ is a seminormed linear space consisting of functions from $\mathbb{R}^-$ into X satisfying the following axioms introduced by Hale and Kato in [8].

(A1) There exist a positive constant H and functions $K(\cdot)$, $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a \geq 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathcal{P}$ and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold:
   (i) $x_t \in \mathcal{P}$,

(ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{P}}$,

(iii) $\|x_t\|_{\mathcal{P}} \le K(t-\sigma) \sup_{\sigma\le s\le t} \|x(s)\| + M(t-\sigma)\|x_\sigma\|_{\mathcal{P}}$.

(A2) For each function $x(\cdot)$ in (A1), $t \mapsto x_t$ is a $\mathcal{P}$-value continuous function for $t \in [\sigma, \sigma+a]$.

(B) The space $\mathcal{P}$ is complete.

Let C_{00} be the set of continuous functions $\psi : (-\infty, 0] \to X$ with compact support $\mathrm{supp}(\psi)$.

Remark 3.3 ([12].] Any $\psi \in C_{00}$ belongs to $\mathcal{P}$.

If $u \in C([0,\infty), X)$, we define

$$\|u\|_{[0,a]} := \sup_{0\le s\le a} \|u(s)\|$$

for $a \ge 0$.

$(H1)$ $B \in C(\Gamma \times X, X)$. Let $a \ge 0$. There exists M_a^1 dependent of a such that

$$\int_0^t \|B(t,\theta,u(\theta)) - B(t,\theta,v(\theta))\| d\theta \le M_a^1 \|u-v\|_{[0,t]}$$

for $u, v \in C([0,t], X)$, $t \in [0,a]$.

$(H2)$ $F \in C([0,\infty) \times \mathcal{P}, X)$ and $F(t,\cdot)$ satisfies a Lipschitz condition, i.e. for each $a > 0$ there is a constant $L(a)$ such that

$$\|F(t,\psi_1) - F(t,\psi_2)\| \le L(a)\|\psi_1 - \psi_2\|_{\mathcal{P}}$$

for $\psi_1, \psi_2 \in \mathcal{P}$ and $t \in [0,a]$.

Theorem 3.1. *Suppose that $(H1)$ and $(H2)$ hold. If $\varphi \in \mathcal{P}$ and $\varphi(0) \in X_0$, then Eq. (VID1) has a unique mild solution $u(\cdot,\varphi)$ on $[0,\infty)$.*

Proof. Let $a > 0$ be fixed. Consider the set

$$Z_a = \{u : (-\infty, a] \to X;\ u_0 \in \mathcal{P} \text{ and } u : [0,a] \to X \text{ is continuous}\}$$

endowed with the seminorm $\|\cdot\|_{Z_a}$ defined by $\|u\|_{Z_a} = \|u_0\|_{\mathcal{P}} + \|u\|_{[0,a]}$. From the Axiom (B), it follows that $(Z_a, \|\cdot\|_{Z_a})$ is complete. Furthermore, we define the set

$$Z_a(\varphi) = \{u \in Z_a;\ \|u_0 - \varphi\|_{\mathcal{P}} = 0\}.$$

Note that $Z_a(\varphi)$ is a closed subset of Z_a. Define the map

$$P : Z_a(\varphi) \to Z_a(\varphi)$$

by

$$(Pu)(t) = \begin{cases} T_0(t)\varphi(0) + \int_0^t T_{-1}(t-s) \int_0^s B(s,\theta,u(\theta)) d\theta ds \\ + \int_0^t T_{-1}(t-s) F(s,u_s) ds,\ t \in [0,a], \\ \varphi(t),\ t \in (-\infty, 0]. \end{cases}$$

By Proposition 2.4, we know that $\mathcal{P}$ is well defined. Let $u, v \in Z_a(\varphi)$ and $t \in [0, a]$. Then

$$\begin{aligned} \|Pu - Pv\|_{Z_a(\varphi)} &= \|(Pu)_0 - (Pv)_0\|_{\mathcal{P}} + \|Pu - Pv\|_{[0,a]} \\ &= \|Pu - Pv\|_{[0,a]}. \end{aligned} \tag{3.1}$$

Since $\|u_0 - v_0\|_{\mathcal{P}} = 0$, by Proposition 2.4 (ii), Axioms (A1-ii,iii), (H1), and (H2), we have

$$\begin{aligned} \|(Pu - Pv)(t)\| &\leq \| \textstyle\int_0^t T_{-1}(t-s) \int_0^s B(s, \theta, u(\theta)) - B(s, \theta, v(\theta)) d\theta ds\| \\ &\quad + \| \textstyle\int_0^t T_{-1}(t-s)(F(s, u_s) - F(s, v_s)) ds\| \\ &\leq M_1 \| \textstyle\int_0^{\cdot} B(\cdot, \theta, u(\theta)) - B(\cdot, \theta, v(\theta)) d\theta\|_{L^1([0,t),X)} \\ &\quad + M_1 \|F(\cdot, u_{\cdot}) - F(\cdot, v_{\cdot})\|_{L^1([0,t),X)} \\ &\leq M_1 (M_a^1 \textstyle\int_0^t \|u - v\|_{[0,s]} ds + L(a) \int_0^t \|u_s - v_s\|_{\mathcal{P}} ds) \\ &\leq M_1 (M_a^1 \textstyle\int_0^t \|u - v\|_{[0,s]}] ds + L(a) \int_0^t K(s) \sup\limits_{0 \leq r \leq s} \|u(r) - v(r)\| ds) \\ &\leq M_1 [M_a^1 + K_a L(a)] \textstyle\int_0^t \|u - v\|_{[0,s]} ds. \end{aligned}$$

Note that M_1, M_a^1, $L(a)$, and $K_a = \sup\limits_{0 \leq \xi \leq a} K(\xi)$ are constants defined in Proposition 2.4, Axiom (A1), (H1) and (H2). Thus, by equation (3.1), we have

$$\|Pu - Pv\|_{Z_a(\varphi)} \leq M_1 [M_a^1 + K_a L(a)] \|u - v\|_{Z_a(\varphi)} a. \tag{3.2}$$

Applying the method again and by equation (3.1) and (3.2), we have

$$\begin{aligned} \|(P^2 u - P^2 v)(t)\| &\leq \| \textstyle\int_0^t T_{-1}(t-s) \int_0^s B(s, \theta, Pu(\theta)) - B(s, \theta, Pv(\theta)) d\theta ds\| \\ &\quad + \| \textstyle\int_0^t T_{-1}(t-s)(F(s, (Pu)_s) - F(s, (Pv)_s)) ds\| \\ &\leq M_1 [M_a^1 + K_a L(a)] \textstyle\int_0^t \|Pu - Pv\|_{[0,s]} ds \\ &\leq \{M_1 [M_a^1 + K_a L(a)]\}^2 \textstyle\int_0^t \|u - v\|_{[0,s]} s ds. \end{aligned}$$

Thus

$$\|P^2 u - P^2 v\|_{Z_a(\varphi)} \leq \{M_1 [M_a^1 + K_a L(a)]\}^2 \frac{a^2}{2!} \|u - v\|_{Z_a(\varphi)}.$$

Continue this process, we can obtain

$$\|P^n u - P^n v\|_{Z_a(\varphi)} \leq \{M_1 [M_a^1 + K_a L(a)]\}^n \frac{a^n}{n!} \|u - v\|_{Z_a(\varphi)}$$

for all $n \in N$. Since there exists $n \in N$ such that $\{M_1 [M_a^1 + K_a L(a)]\}^n \frac{a^n}{n!} < 1$, it follows that P^n is a strict contraction mapping on the closed set $Z_a(\varphi)$. By the Banach fixed point theorem, there exists a unique $u := u(\cdot, \varphi) \in Z_a(\varphi)$ such that $Pu = u$. The uniqueness of the solution is consequence of the uniqueness of the fixed point of P. This concludes that equation (IE) has a unique solution on $[0, a]$. Since the number a is arbitrary, we proved the existence and uniqueness of the solution of the equation (IE) on $[0, \infty)$. Thus Eq. (VID1) has a unique mild solution on $[0, \infty)$. $\square$

Next, we want to give a sufficient condition for the existence of classical solutions to Eq. (VID1). To do this, we need the differentiability of mild solutions. We give the following more restrictive conditions.

(C) If (ϕ_n) is a Cauchy sequence in $\mathcal{P}$ and if (ϕ_n) converges compactly to ϕ on $(-\infty, 0]$, then $\phi \in \mathcal{P}$ and $\|\phi_n - \phi\|_{\mathcal{P}} \to 0$, as $n \to \infty$.

(D) For a sequence (ϕ_n) in $\mathcal{P}$, if $\|\phi_n\|_{\mathcal{P}} \to 0$ as $n \to \infty$, then $\|\phi_n(\theta)\| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty, 0]$.

(H3) F is continuously differentiable and the derivatives D_1F, D_2F satisfy the following Lipschitz conditions: there is a constant $L_1(\tau) > 0$ such that

$$\|D_1F(t, \psi_1) - D_1F(t, \psi_2)\| \leq L_1(\tau)\|\psi_1 - \psi_2\|_{\mathcal{P}},$$

$$\|D_2F(t, \psi_1) - D_2F(t, \psi_2)\| \leq L_1(\tau)\|\psi_1 - \psi_2\|_{\mathcal{P}},$$

for $\tau \in [0, \infty)$, $t \in [0, \tau]$, and $\psi_1, \psi_2 \in \mathcal{P}$.

The following lemmas have been proved in [2] and [12].

Lemma 3.2 ([12]). *Let $\mathcal{P}$ satisfy axiom (C) and let $f : [0, a] \to \mathcal{P}$, $a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t, \theta) \in [0, a] \times (-\infty, 0]$. Then*

$$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt$$

for $\theta \in (-\infty, 0]$.

Lemma 3.3 ([2]). *Let $\mathcal{P}$ satisfy axiom (D) and let $f : [0, a] \to \mathcal{P}$, $a > 0$, be a continuous function. Then for all $\theta \in (-\infty, 0]$, the function $f(\cdot)(\theta)$ is continuous and*

$$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt$$

for $\theta \in (-\infty, 0]$.

Theorem 3.4. *Let $\mathcal{P}$ satisfy axiom (C) or (D). Assume that (H1), (H2), (H3) and $D_1B \in C(\Gamma \times X, X)$ hold. In addition, assume that $\varphi \in \mathcal{P}$ is continuously differentiable with $\varphi' \in \mathcal{P}$, $\varphi(0) \in D(A)$ and $\varphi'(0) = A\varphi(0) + F(0, \varphi) \in X_0$. If $u(\cdot, \varphi)$ is a mild solution of Eq. (VID1) on $[0, \infty)$, then $u(\cdot, \varphi)$ is continuously differentiable on $[0, \infty)$. Furthermore, $u(\cdot, \varphi)$ is a classical solution of Eq. (VID1) on $[0, \infty)$.*

Proof. Let $a \in [0, \infty)$. Consider the equation

$$(3.3) \quad \begin{cases} y(t) &= T_0(t)(A\varphi(0) + F(0, \varphi)) + \int_0^t T_{-1}(t - s)B(s, s, u(s))ds \\ & \quad + \int_0^t T_{-1}(t - s) \int_0^s D_1B(s, \theta, u(\theta))d\theta ds \\ & \quad + \int_0^t T_{-1}(t - s)(D_1F(s, u_s) + D_2F(s, u_s)y_s)ds, \qquad t \in [0, a], \\ y(t) &= \varphi'(t), \qquad t \in (-\infty, 0], \end{cases}$$

where $u(\cdot) = u(\cdot, \varphi)$. With a similar argument to that used to establish the Theorem 3.4, it can be shown that Eq. (3.3) has a unique solution $y(\cdot, \varphi')$. Define the function z by

$$z(t) = \begin{cases} \varphi(0) + \int_0^t y(s)ds, & t \in [0, a] \\ \varphi(t), & t \in (-\infty, 0]. \end{cases} \tag{3.4}$$

We shall show $u = z$. By (2.1) and Fubini's Theorem, we obtain

$$\begin{aligned} &\int_0^t \int_0^r T_{-1}(r-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)ds\, dr \\ &\quad = \int_0^t \int_s^t T_{-1}(r-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)dr\, ds \\ &\quad = \int_0^t S(t-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)ds, \qquad t \in [0,a], \end{aligned}$$

and

$$\int_0^t T_0(s)(A\varphi(0) + F(0,\varphi))ds = \int_0^t T_0(s)\varphi'(0)ds = S(t)\varphi'(0),$$

where $S(\cdot)$ is the integrated semigroup generated by A. Therefore, z becomes

$$\begin{aligned} z(t) \; &= \varphi(0) + S(t)\varphi'(0) + \int_0^t T_{-1}(t-s) \int_0^s B(s,\theta,u(\theta))d\theta ds \\ &\quad + \int_0^t S(t-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)ds, \qquad t \in [0,a]. \end{aligned} \tag{3.5}$$

By Lemma 3.5 or Lemma 3.6, we obtain

$$z_t = \varphi + \int_0^t y_s ds \quad \text{ for } t \in [0,a].$$

By the elementary properties of $S(\cdot)$, $\varphi' \in \mathcal{P}$, $\varphi(0) \in D(A)$ and $\varphi'(0) = A\varphi(0) + F(0,\varphi) \in X_0$, we know that

$$S(t)\varphi'(0) = T_0(t)\varphi(0) - \varphi(0) + S(t)F(0,\varphi) \text{ for } t \in [0,a]. \tag{3.6}$$

Moreover, using the integration by parts formula, we have

$$\begin{aligned} &\int_0^t T_{-1}(t-s)F(s,z_s)ds \\ &= S(t)F(0,\varphi) + \int_0^t S(t-s)(D_1F(s,z_s) + D_2F(s,z_s)y_s)ds \text{ for } t \in [0,a]. \end{aligned} \tag{3.7}$$

From (3.7), we deduce that

$$\begin{aligned} S(t)F(0,\varphi) \; &= -(\int_0^t S(t-s)(D_1F(s,z_s) + D_2F(s,z_s)y_s)ds) \\ &\quad + \int_0^t T_{-1}(t-s)F(s,z_s)ds \text{ for } t \in [0,a]. \end{aligned} \tag{3.8}$$

Consequently, by (3.5), (3.6), and (3.8), we have

$$\begin{aligned} z(t) = T_0(t)\varphi(0) + S(t)F(0,\varphi) + \int_0^t T_{-1}(t-s) \int_0^s B(s,\theta,u(\theta))d\theta ds \\ + \int_0^t S(t-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)ds \end{aligned}$$

$$
\begin{aligned}
&= T_0(t)\varphi(0) + \int_0^t T_{-1}(t-s)\int_0^s B(s,\theta,u(\theta))d\theta ds \\
&\quad + \int_0^t S(t-s)(D_1F(s,u_s) + D_2F(s,u_s)y_s)ds \\
&\quad - (\int_0^t S(t-s)(D_1F(s,z_s) + D_2F(s,z_s)y_s)ds \\
&\quad + \int_0^t T_{-1}(t-s)F(s,z_s)ds \text{ for } t\in[0,a].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|u(t)-z(t)\| &\le \|\int_0^t T_{-1}(t-s)(F(s,u_s)-F(s,z_s))ds\| \\
&\quad + \|\int_0^t S(t-s)(D_1F(s,u_s)-D_1F(s,z_s)ds\| \\
&\quad + \|\int_0^t S(t-s)(D_2F(s,u_s)y_s - D_2F(s,z_s)y_s)ds\| \\
&\le M_1L(a)\int_0^t \|u_s-z_s\|_{\mathcal{P}}ds \\
&\quad + M_1aL_1(a)(\int_0^t \|u_s-z_s\|_{\mathcal{P}}ds + \int_0^t \|u_s-z_s\|_{\mathcal{P}}\|y_s\|_{\mathcal{P}}ds) \\
&\le K_a(M_1L(a) + M_1aL_1(a) + M_1aL_1(a)\times \max_{0\le s\le a}\|y(s)\|_{\mathcal{P}}) \\
&\quad \times \int_0^t \sup_{0\le\zeta\le s}\|u(\zeta)-z(\zeta)\|ds
\end{aligned}
$$

where $K_a = \sup_{0\le s\le a}\max\{K(t)\}$ and $K(t)$ is defined in Axiom (A1-iii). By Gronwall's Lemma, we get $u = z$. So, we derive that u is continuously differentiable on $[0,a]$. Since a is arbitrary, we complete the proof. □

Next, we consider the solutions of Eq. (VID2) and (VID3).

Definition 3.4. We say that a function $u : \mathbb{R} \to X$ is a classical solution of Eq. (VID2) on $[0,\infty)$ if u satisfies the following conditions

(i) $u(t) + \int_0^t B(t-\theta)u(\theta)d\theta \in D(A)$ for $t\in[0,\infty)$.
(ii) $u \in C^1([0,\infty),X)$.
(iii) u satisfies Eq. (VID2) on $[0,\infty)$ and $u(t)=\varphi(t)$ for $-\infty < t \le 0$.

Lemma 3.5. *Suppose that x_1 and x_2 are two vectors of X, then there is a function $\psi\in\mathcal{P}$ such that ψ is continuously differentiable with $\psi'\in\mathcal{P}$, $\psi(0)=x_1$ and $\psi'(0)=x_2$.*

Proof. Let

$$C_c^\infty(\mathbb{R}) := \{f:\mathbb{R}\to\mathbb{R};\ f \text{ is infinitely differentiable with compact support}\}.$$

By Urysohn's Lemma, there is a function $h \in C_c^\infty$ such that $h(t) = 1$ for $t \in [-1, 0]$. It follows that the function ψ defined by $\psi(t) = h(t - 0.5)x_1 + th(t - 0.5)x_2$ for $t \in (-\infty, 0]$ is the desired function from Remark 3.3. □

The following is an extension of Theorem 3.2 in [6].

Theorem 3.6. *Let $\mathcal{P}$ satisfy axiom (C) or (D). Assume that*

(1) *$B(\cdot) \in L(X)$, the strong derivative $B'(\cdot)x$ and $B''(\cdot)x$ exist and are continuous on $[0, \infty)$ for each $x \in X$. F satisfies the condition (H2) and (H3).*
(2) *$\varphi \in \mathcal{P}$ is continuously differentiable with $\varphi' \in \mathcal{P}$, $\varphi(0) \in D(A)$, $\varphi'(0) = A\varphi(0) + F(0, \varphi) \in X_0$ and $A\varphi(0) + B(0)\varphi(0) + F(0, \varphi) \in X_0$.*

Then the equation

$$\begin{cases} u'(t) = A[u(t) + \int_0^t B(t-s)u(s)ds] + F(t, u_t), & t \in \mathbb{R}^+, \\ u_0 = \varphi \in \mathcal{P}, \end{cases}$$

has a unique classical solution on $[0, \infty)$.

Proof. Let $\vec{A} = \begin{bmatrix} 0 & A \\ B(0) & A \end{bmatrix}$. Grimmer and Liu in [6] showed that $\vec{A}$ is a Hille-Yosida operator on $X \times X$. According to Lemma 3.9, we can choose a $\varphi_1 \in \mathcal{P}$ such that $\varphi_1' \in \mathcal{P}$ with $\varphi_1(0) = \varphi(0)$ and $\varphi_1'(0) = A\varphi(0) + B(0)\varphi(0) + F(0, \varphi)$. Let $\vec{\varphi} = \begin{bmatrix} \varphi \\ \varphi_1 \end{bmatrix}$. Therefore, $\vec{\varphi} \in \mathcal{P} \times \mathcal{P}$. We consider the following equation in the Banach space $X \times X$ and phase space $\mathcal{P} \times \mathcal{P}$

$$\begin{cases} \frac{d}{dt}\vec{w}(t) = \vec{A}\vec{w}(t) + \int_0^t \vec{B}(t-\theta)\vec{w}(\theta)d\theta + \vec{F}(t, \vec{w}_t), \ t \in \mathbb{R}^+, \\ \vec{w}_0 = \begin{bmatrix} \varphi \\ \varphi_1 \end{bmatrix}, \end{cases} \tag{3.12}$$

where

$$\vec{w}(t) := \begin{bmatrix} u(t) \\ w(t) \end{bmatrix},$$

$$\vec{B}(t-\theta)\begin{bmatrix} u(\theta) \\ w(\theta) \end{bmatrix} := \begin{bmatrix} 0 \\ B'(t-\theta)u(\theta) \end{bmatrix}$$

and

$$\vec{F}(t, \vec{w}_t) := \begin{bmatrix} F(t, u_t) \\ F(t, u_t) \end{bmatrix}.$$

We can apply Theorem 3.7 to solve Eq. (3.12). By assumption (1) and Principle of Uniform Boundedness, we know that $\vec{B}$ satisfies the hypothesis (H1). Moreover, it is easy to see that $\vec{F}$ satisfies the hypothesis (H2) and (H3) form assumption (1). Finally, by assumption (2),

$$\left(\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \varphi_1(0) \end{bmatrix} + \begin{bmatrix} F(0, \varphi) \\ F(0, \varphi) \end{bmatrix} \right)$$

$$= \begin{bmatrix} A\varphi(0) + F(0,\varphi) \\ A\varphi(0) + B(0)\varphi(0) + F(0,\varphi) \end{bmatrix}$$
$$= \begin{bmatrix} \varphi'(0) \\ \varphi_1'(0) \end{bmatrix} \in X_0 \times X_0.$$

It follows that $\vec{w}_0' \in \mathcal{P} \times \mathcal{P}$, $\vec{w}_0(0) \in D(\vec{A})$ and $\vec{w}_0'(0) = \vec{A}\vec{\varphi}(0) + \vec{F}(0,\varphi) \in X_0 \times X_0$. Thus, by Theorem 3.7, Eq. (3.12) has a unique classical solution on $[0,\infty)$. On the other hand, we rewrite equation (3.12) in the following component form

$$u'(t) = Aw(t) + F(t,u_t),$$
$$w'(t) = Aw(t) + \int_0^t B'(t-\theta)u(\theta)d\theta + B(0)u(t) + F(t,u_t)$$
$$= u'(t) + \frac{d}{dt}\int_0^t B(t-\theta)u(\theta)d\theta.$$

Since $w(0) = \varphi(0) = \varphi_1(0) = u(0)$, we conclude that $w(t) = u(t) + \int_0^t B(t-\theta)u(\theta)d\theta$. So, u is the unique classical solution Eq. (VID2). □

Definition 3.5. We say that a function $u : \mathbb{R} \to X$ is a classical solution of Eq. (VID3) on $[0,\infty)$ if u satisfies the following conditions

(i) $u(t) + \int_{-\infty}^t B(t-\theta)u(\theta)d\theta \in D(A)$ for $t \in [0,\infty)$.
(ii) $u \in C^1([0,\infty), X)$.
(iii) u satisfies Eq. (VID3) on $[0,\infty)$ and $u(t) = \varphi(t)$ for $-\infty < t \le 0$.

Theorem 3.7. *Let $\mathcal{P}$ satisfy axiom (C) or (D). Assume that*

(1) $B(\cdot) \in L(X)$. *$B'(\cdot)x$ and $B''(\cdot)x$ exist and are continuous on $[0,\infty)$ for each $x \in X$. F satisfies the condition (H2) and (H3).*
(2) *$\varphi \in \mathcal{P}$ is continuously differentiable with $\varphi' \in \mathcal{P}$, and $\int_{-\infty}^0 B(t-\theta)\varphi(\theta)d\theta \in X$ for each $t \in [0,\infty)$.*
(3) *$G \in C^2([0,\infty), X)$ where G is defined by $G(t) = \int_{-\infty}^0 B(t-\theta)\varphi(\theta)d\theta$.*
(4) *$\varphi(0) + \int_{-\infty}^0 B(-\theta)\varphi(\theta)d\theta \in D(A)$,*

$$\varphi'(0) = A[\varphi(0) + \int_{-\infty}^0 B(-\theta)\varphi(\theta)d\theta] + F(0,\varphi) \in X_0$$

and

$$A[\varphi(0) + \int_{-\infty}^0 B(-\theta)\varphi(\theta)d\theta] + B(0)\varphi(0) + G'(0) + F(0,\varphi) \in X_0.$$

Then the equation

$$\begin{cases} u'(t) = A[u(t) + \int_{-\infty}^t B(t-s)u(s)ds] + F(t,u_t), \quad t \in [0,\infty), \\ u_0 = \varphi \in \mathcal{P} \end{cases}$$

has a unique classical solution on $[0,\infty)$.

Proof. Let $\vec{A} = \begin{bmatrix} 0 & A \\ B(0) & A \end{bmatrix}$. According to Lemma 3.9, we can choose a $\varphi_1 \in \mathcal{P}$ such that $\varphi_1' \in \mathcal{P}$ with $\varphi_1(0) = \varphi(0) + \int_{-\infty}^0 B(-\theta)\varphi(\theta)d\theta$ and $\varphi_1'(0) = A[\varphi(0) + \int_{-\infty}^0 B(-\theta)\varphi(\theta)d\theta] + B(0)\varphi(0) + F(0,\varphi) + G'(0)$. Let $\vec{\varphi} = \begin{bmatrix} \varphi \\ \varphi_1 \end{bmatrix}$. Therefore, $\vec{\varphi} \in \mathcal{P}\times\mathcal{P}$. We consider the following equation in the Banach space $X \times X$ and phase space $\mathcal{P}\times\mathcal{P}$

$$\begin{cases} \frac{d}{dt}\vec{w}(t) = \vec{A}\vec{w}(t) + \int_0^t \vec{B}(t-\theta)\vec{w}(\theta)d\theta + \vec{F}(t,\vec{w}_t), \ t \in \mathbb{R}^+, \\ \vec{w}_0 = \begin{bmatrix} \varphi \\ \varphi_1 \end{bmatrix}, \end{cases} \tag{3.12}$$

where

$$\vec{w}(t) := \begin{bmatrix} u(t) \\ w(t) \end{bmatrix},$$

$$\vec{B}(t-\theta)\begin{bmatrix} u(\theta) \\ w(\theta) \end{bmatrix} := \begin{bmatrix} 0 \\ B'(t-\theta)u(\theta) \end{bmatrix}$$

and

$$\vec{F}(t,\vec{w}_t) := \begin{bmatrix} F(t,u_t) \\ F(t,u_t) + G'(t) \end{bmatrix}.$$

The rest of the proof is the same as Theorem 3.10. □

We enclose this section by discussing the asymptotical behavior of solutions of Eq. (VID1). Let us assume that (H1) (H2) and (H3) hold. Note that $K(s)$, $M(s)$, M_s^1 and $L(s)$ are defined in Axiom (A1(iii)), (H1), and (H2). Define $K_t = \sup_{0\le s\le t} K(s)$, $M_t = \sup_{0\le s\le t} M(s)$, and $L_t = \sup_{0\le s\le t} L(s)$.

Theorem 3.8. *Let v and w be two solutions of Eq. $(VID1)$ with initial conditions v_0 and w_0 respectively. Then*

$$\|v-w\|_{[0,t]} \le M\gamma(t)e^{M\delta(t)t}\|v_0-w_0\|_{\mathcal{P}}$$

where $\gamma(t) = H + L_tM_t\int_0^t e^{-\omega s}ds$ and $\delta(t) = M_t^1 + L_tK_t$.

Proof. Set $u(t) = v(t) - w(t)$ and $u_0 = v_0 - w_0$. By Proposition 2.5, Axiom (A1)(iii), (H1), and (H2) we have

$$\begin{aligned} \|u(t)\| &\le M[\|u(0)\|e^{\omega t} + \int_0^t e^{\omega(t-s)}\|\int_0^s B(s,\theta,v(\theta)) - B(s,\theta,w(\theta))d\theta\|ds \\ &\quad + \int_0^t e^{\omega(t-s)}\|F(s,v_s) - F(s,w_s)\|ds] \\ &\le M[\|u_0\|_{\mathcal{P}}He^{\omega t} + M_t^1e^{\omega t}\int_0^t e^{-\omega s}\|u\|_{[0,s]}ds \\ &\quad + L_tK_te^{\omega t}\int_0^t e^{-\omega s}\|u\|_{[0,s]}ds + L_tM_te^{\omega t}\int_0^t e^{-\omega s}\|u_0\|_{\mathcal{P}}ds] \end{aligned}$$

$$\leq Me^{\omega t}\gamma(t)\|u_0\|_{\mathcal{P}} + M\delta(t)\int_0^t e^{\omega(t-s)}\|u\|_{[0,s]}ds$$

where $\gamma(t) = H + L_t M_t \int_0^t e^{-\omega s}ds$ and $\delta(t) = M_t^1 + L_t K_t$. Thus,

$$\|u\|_{[0,t]} \leq M\gamma(t)\|u_0\|_{\mathcal{P}} + M\delta(t)\int_0^t \|u\|_{[0,s]}ds.$$

By applying Gronwall's lemma, we derive

$$\|u\|_{[0,t]} \leq M\gamma(t)e^{M\delta(t)t}\|u_0\|_{\mathcal{P}}. \qquad \square$$

4. EXAMPLES

In this section, we improve the conclusions about the partial integrodifferential equation which had been studied in [1]:

$$(PDED) \qquad \begin{cases} \frac{\partial}{\partial t}\omega(t,\xi) = d\Delta\omega(t,\xi) + \int_{-\infty}^t k(t,\theta,\omega(\theta,\xi))d\theta \\ \qquad\qquad +h(t,\omega(t,\xi),\omega(t-\tau,\xi)), \qquad t \geq 0,\ \xi \in \Omega, \\ \omega(t,\xi) = 0, \quad t \geq 0,\ \xi \in \partial\Omega, \\ \omega(\theta,\xi) = \omega_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ \xi \in \overline{\Omega}, \end{cases}$$

where $d > 0$, $\tau > 0$, Ω is a bounded open set in $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $\Delta = \sum_{i=0}^n \frac{\partial^2}{\partial \xi_i^2}$ and $k : \Gamma\times\mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ and $\omega_0 : (-\infty,0]\times\overline{\Omega} \to \mathbb{R}^n$ are functions.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. Following the notations in [1], the Banach space X, Hille-Yosida operator A, and the phase space $\mathcal{P}$ are given by $X := C(\overline{\Omega},\mathbb{R}^n)$, $A : D(A) \subset X \to X$ with

$$D(A) = \{u \in X; \Delta u \in X \text{ and } u|_{\partial\Omega} = 0\}$$
$$Au = d\Delta u,$$

and

$$\mathcal{P} := \{\phi \in C((-\infty,0],X); \lim_{\theta\to-\infty}\frac{|\phi(\theta)|}{g(\theta)} = 0\},$$

endowed with the following norm:

$$\|\phi\|_{\mathcal{P}} = \sup_{-\infty<\theta\leq 0}\frac{\|\phi(\theta)\|}{g(\theta)},$$

where $g : (-\infty,0] \to (0,\infty)$ is a continuous function satisfying

(g1) $g(0) = 1$.

(g2) The function $G : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$G(t) = \sup_{-\infty<\theta\leq -t}\frac{g(t+\theta)}{g(\theta)},$$

is locally bounded.

It is has been shown that $X_0 = \{u \in X; u|_{\partial\Omega} = 0\}$ and $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ satisfies the axioms (A1), (A2), (B) and (D) (c.f. [1]).

Let a be an arbitrary positive number We transform the equation (PDED) into

$$(DACP) \qquad \begin{cases} u'(t) = Au(t) + \int_0^t B(t,\theta,u(\theta))d\theta + F(t,u_t), \quad 0 \le t \le a, \\ u_0 = \varphi \in \mathcal{P} \end{cases}$$

by setting

$$\begin{aligned} u(t)(\xi) &= \omega(t,\xi), \quad t \in \mathbb{R}^+,\ \xi \in \overline{\Omega}, \\ B(t,\theta,\psi(\theta))(\xi) &= k(t,\theta,\psi(\theta)(\xi)) \text{ where } t \in [0,a],\ \psi \in C([0,a]\times\overline{\Omega},\mathbb{R}^n),\ \xi \in \overline{\Omega}, \\ \varphi(\theta) &= \omega_0(\theta,\xi) \text{ where } \theta \in (-\infty,0], \\ F(t,\phi)(\xi) &= h(t,\phi(0)(\xi),\phi(-\tau)(\xi)) + \int_{-\infty}^0 B(t,\theta,\varphi(\theta)(\xi))d\theta, \\ &\qquad\qquad t \in \mathbb{R}^+,\ \phi \in \mathcal{P},\ \xi \in \overline{\Omega}. \end{aligned}$$

Let $L : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. We assume that $k : \Gamma \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions that satisfy the following conditions.

(i) Let $T \in [0,a]$. $|k(t,\theta,x) - k(t,\theta,y)| \le L(T)|x-y|$ for $x,\ y \in \mathbb{R}^n$ and $(t,\theta) \in \{(t,\theta) \in \mathbb{R}^2; 0 \le t \le T,\ \theta \le t\}$.

(ii) $\int_{-\infty}^0 k(t,\theta,\varphi(\theta)(\xi))d\theta$ exists for each $t \in [0,a]$.

(iii) Let $T \in [0,a]$. $|h(t,x_1,y_1) - h(t,x_2,y_2)| \le L(T)(|x_1-x_2|+|y_1-y_2|)$ for $t \in [0,T]$ and $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}^n$.

(iv) $\omega_0 \in C^2((-\infty,0]\times\overline{\Omega},\mathbb{R}^n)$, with $\lim_{\theta\to-\infty} |\omega_0(\theta)|/g(\theta) = 0$ and $\omega_0(0,\xi) = 0$ for $\xi \in \partial\Omega$.

From the assumption (i), we know that B satisfies the hypothesis (H1). Let $\{(t_n,\phi_n)\}$ be convergent sequence of $[0,\infty)\times\mathcal{P}$ with limit (t,ϕ). Then

$$\|F(t_n,\phi_n) - F(t,\phi)\| \le \|F(t_n,\phi_n) - F(t_n,\phi)\| + \|F(t_n,\phi) - F(t,\phi)\|. \tag{4.1}$$

From the definition of $\mathcal{P}$, we know that

$$|\phi(\tau)| \le g(\tau)\|\phi\|_{\mathcal{P}} \tag{4.2}$$

for $\phi \in \mathcal{P}$. By the continuity of h, Axiom (A1), (4.1) and (4.2), we know that F is a continuous function . Moreover, we have

$$\|F(t,\phi_1) - F(t,\phi_2)\| \le L(T)(1+g(\tau))\|\phi_1-\phi_2\|_{\mathcal{P}} \tag{4.3}$$

by Axiom (A1), (4.2) and assumptions (ii) and (iii) for each $t \in [0,T]$. Finally, we know that F satisfies the hypothesis (H2). It follows that Eq. (DACP) has a unique mild solution by Theorem 3.4.

Under more restrictive conditions, we obtain the existence of classical solution.

(v) $\omega_0 \in C^2((-\infty,0]\times\overline{\Omega},\mathbb{R}^n)$, with $\lim_{\theta\to-\infty} \frac{1}{g(\theta)}\|\frac{\partial}{\partial\theta}\omega_0(\theta,\cdot)\| = 0$.

(vi) $\Delta\omega_0(0,\xi)=0$ for $\xi\in\partial\Omega$.

(vii) $\omega_0(\theta,\xi)=0$ for $\theta\in(-\infty,0]$ and $\xi\in\partial\Omega$.

(viii) $\frac{\partial}{\partial\theta}\omega_0(0,\xi)=d\Delta\omega_0(0,\xi)+h(0,\omega_0(0,\xi),\omega_0(-\tau,\xi))+\int_{-\infty}^0 k(0,\theta,\omega_0(\theta,\xi)d\theta$ for $\xi\in\overline{\Omega}$.

(ix) $\int_{-\infty}^0 k(t,\theta,\varphi(\theta))d\theta$ and $k(t,\cdot,\cdot)$ are continuously differentiable on $[0,a]$.

(x) There is a function $L_1:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the following condition:

$$|D_1h(t,x_1,y_1)-D_1h(t,x_2,y_2)|+|D_2h(t,x_1,y_1)-D_2h(t,x_2,y_2)| \\ +|D_3h(t,x_1,y_1)-D_3h(t,x_2,y_2)|\le L_1(T)(|x_1-x_2|+|y_1-y_2|)$$

for $t\in[0,T]$ and $x_1,\ x_2,\ y_1,\ y_2\in\mathbb{R}^n$.

From assumptions (ix) and (x), we know that both D_1F and D_2F exist on $[0,a]\times\mathcal{P}$. Moreover, it is also easy to see that

$$D_1F(t,\phi)(\xi)=\frac{d}{dt}\{\int_{-\infty}^0 B(t,\theta,\varphi(\theta))d\theta\}(\xi)+D_1h(t,\phi(0)(\xi),\phi(-\tau)(\xi))$$

and

$$D_2(F(t,\phi))(\psi)(\xi) \\ =D_2h(t,\phi(0)(\xi),\phi(-\tau)(\xi))\psi(0)(\xi)+D_3h(t,\phi(0)(\xi),\phi(-\tau)(\xi))\psi(-\tau)(\xi)$$

for $\phi,\ \psi\in\mathcal{P}$. By assumption (x) and a similar computation to (4.3), we know that F satisfies the hypothesis (H3). It follows that Eq. (4.2) has a unique classical solution on $[0,a]$ by Theorem 3.7. Since a is arbitrary, we derive that Eq. (PDED) has a solution on $[0,\infty)$.

Remark 4.1. In [1], the authors transform the equation (PDED) into

$$\begin{cases} u'(t)=Au(t)+F(t,u_t), & 0\le t,\\ u_0=\varphi\in\mathcal{P} \end{cases}$$

by setting

$$F(t,\phi)(\xi)=h(t,\phi(0)(\xi),\phi(-\tau)(\xi))+\int_{-\infty}^0 k(t,t+\theta,\phi(\theta)(\xi))d\theta$$

for $\phi\in\mathcal{P}$ and $0\le t$. In this situation, the more growth bound and differentiability for k and h are needed.

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REFERENCES

[1] M. Adimy, H. Bouzahir and K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, *Nonlinear Anal. TMA.* **46** (2001) 91–112.

[2] K. Ezzinbi, Existence and stability for some partial functional differential equations, *Electron J. Differential Equations,* **116** (2005) 1–13.

[3] J. -C. Chang, On the Volterra integrodifferential equations and applications, *Semigroup Forum* **66** (2003) 68–80.

[4] P. Eloe, M. Islam, and B. Zhang, Uniform asymptotic stability in linear Volterra integrodifferential equations with application to delay systems, *Dynam. Systems Appl.* **9** (2000) 331–344.

[5] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.

[6] R. Grimmer and J. H. Liu, Integrated semigroups and integrodifferential equations, *Semigroup Forum* **48** (1994) 79–95.

[7] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, 1985.

[8] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978) 11–41.

[9] H. R. Henriquez, Regularity of solutions of abstract retarded functional differential equations with unbounded delay, *Nonlinear Anal. TMA* **28**(3) (1997) 513–531.

[10] H. R. Henriquez, Periodic solutions of quasi-linear partial functional differential equations with unbounded delay, *Funkcial. Ekvac.* **37**(2) (1994) 329–343.

[11] H. R. Henriquez, Approximation of abstract functional differential equations with unbounded delay, *Indian. J. Pure Appl. Math.* **27**(4) (1996) 357–368.

[12] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Vol. 1473, Springer, Berlin, 1991.

[13] F. Kappel and W. Schappacher, Some considerations to the fundamental theory of infinite delay equations, *J. Differential Equations* **37** (1980) 141–183.

[14] H. Kellermann and M. Hieber, Integrated semigroups, *J. Funct. Anal.* **15** (1989) 160–180.

[15] J. Liang, T. J. Xiao and J. van Casteren, A note on semilinear abstract functional differential and integrodifferential equations with infinite delay, *Applied Mathematics Letters* **17** (2004) 473–477.

[16] R. K. Miller, Volterra integral equations in Banach space, *Funkcial. Ekvac.* **18** (1975) 163–194.

[17] R. Nagel and E. Sinestrari, Inhomogenous Volterra integrodifferential equations for Hille-Yosida operators, *Deckker Lecture Notes in Pure and Applied Mathematics* **150** (1994) 51–70.

[18] K. Schumacher, Existence and continuous dependence for differential equations with unbounded delay, *Arch. Rational Mech. Anal.* **64** (1978) 313–335.

[19] H. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations* **3**(6) (1990) 1035–1066.

[20] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974) 395–418.

[21] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, 1996.