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ON A CLASS OF BACKWARD MCKEAN-VLASOV STOCHASTIC EQUATIONS IN HILBERT SPACE: EXISTENCE AND CONVERGENCE PROPERTIES

NAZIM I. MAHMUDOV AND MARK A. MCKIBBEN

Eastern Mediterranean University, Department of Mathematics, Gazimagusa, TRNC, Mersin 10, Turkey (nazim.mahmudov@emu.edu.tr)
Goucher College, Mathematics and Computer Science Department, Baltimore, MD 21204, USA (mmckibben@goucher.edu)

ABSTRACT. This investigation is devoted to the study of a class of abstract first-order backward McKean-Vlasov stochastic evolution equations in a Hilbert space. Results concerning the existence and uniqueness of solutions and the convergence of an approximating sequence of solutions (and corresponding probability measures) are established. Examples that illustrate the abstract theory are also provided.

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1. INTRODUCTION

This investigation is devoted to the study of a class of abstract backward stochastic evolution equations of McKean-Vlasov type of the general form

$$\begin{aligned} dx(t) + Ax(t)dt &= f(t, \mu(t), x(t), y(t))dt + (g(t, x(t)) + y(t))dW(t), 0 \le t \le T, \\ x(T) &= \xi \end{aligned} \tag{1.1}$$

in a separable Hilbert space H, where $\mu(t)$is the probability distribution of $(x(t), z(t))$, where $z(t) = \int_t^T y(s)dW(s)$. Here, W is a given K-valued cylindrical Wiener process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ equipped with a natural filtration $\{\mathfrak{F}_t\}_{t\ge0}$ (that is, $\mathfrak{F}_t$ is the σ-algebra generated by $\{W(s) : 0 \le s \le t\}$, for all $t \ge 0$); the linear operator $A : D(A) \subset H \to H$ generates a strongly continuous semigroup on H; $f : [0, T] \times \mathfrak{P}_{\lambda^2}(H) \times H \times L_2(K; H) \to H$ and $g : [0, T] \times H \to L_2(K; H)$ are given mappings (where K is a real separable Hilbert space, $\mathfrak{P}_{\lambda^2}(H)$denotes a particular subset of probability measures on H, and $L_2(K; H)$is the collection of all Hilbert-Schmidt operators from K to H); and $\xi \in L^2(\Omega; H)$. (The function spaces will be made precise in Section 2.)

The study of backward stochastic differential equations (BSDEs) was initiated by Pardoux and Peng [28] and subsequently investigated (mainly in the finite-dimensional

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case) by several authors – see [16, 22, 26, 30]. A flourish of activity on infinite dimensional BSDEs has since continued by many authors, including Confortola [5], Fuhrman & Tessitore [10, 11], Guatten & Tessitore [12], and Mahmudov & McKibben [24]. Such equations arise naturally in a wide variety of applications, including stochastic control and financial mathematics.

It is known that if the nonlinearities f and g do not depend on the probability distribution $\mu(t)$ of the state process, then the process described by a forward SDE is a standard Markov process [1]. Numerous papers and books devoted to the formulation of theory of such equations have been written during the past two decades (see [6, 8, 9, 27]). Allowing for the dependence of the nonlinearities on $\mu(t)$ is not artificial and, in fact, such problems arise naturally in the study of diffusion processes and have been studied extensively in the finite dimensional setting. Regarding the infinite-dimensional setting, Ahmed and Ding [1] established an abstract formulation of such problems in a Hilbert space, and subsequently, Keck and McKibben [19] extended this theory to a class of integro-differential stochastic evolution equations. Recently, Mahmudov and McKibben [21, 24] established existence and (optimal) controllability results for a delay variant of such equations under so-called Caratheodory growth conditions. The main purpose of the current investigation is to establish a generalization of the known existence and convergence results in [14, 15] for backward stochastic evolution equations to a class of so-called McKean-Vlasov equations.

From a theoretical standpoint, the results presented in the current manuscript constitute an extension and generalization of the theory presented in [1, 6, 7, 9, 14, 15, 17, 21, 27, 28] in that we allow for dependence of the nonlinearities on the probability distribution of the state process in (1.1). As such, the corresponding results in these papers can be viewed as corollaries of the main results of this manuscript. Further, our results elucidate the study of standard McKean-Vlasov equations in the form of BSDEs, which is useful from the viewpoint of optimal control. From a practical viewpoint, the results developed in this manuscript are applicable, in particular, to nonlinear diffusion equations and Sobolev-type equations (as outlined in Section 5).

The following is the outline of the paper. First, we make precise the necessary notation, function spaces, and definitions, and gather certain preliminary results in Section 2. We then state the main results concerning existence and convergence in Section 3, and provide the proofs in Section 4. We finally present two concrete examples in Section 5 in order to illustrate the applicability of the abstract theory.

2. PRELIMINARIES

For details of this section, we refer the reader to [1, 3, 6, 14, 16] and the references therein. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space equipped with filtration $\{\mathfrak{F}_t\}_{t\geq 0}$. Throughout this paper, H and K are given real separable Hilbert spaces with respective norms $\|\cdot\|$ and $\|\cdot\|_K$. The class of all bounded linear operators from H into H (equipped with the usual sup norm) is denoted by $BL(H)$, and the space of all Hilbert-Schmidt operators from K into H is denoted by $L_2(K;H)$ with norm denoted as $\|\cdot\|_{L_2(K;H)}$.

The following function spaces are adapted from those used in [1]; we recall them here for convenience. First, $\mathfrak{B}(H)$stands for the Borel class on H and $\mathfrak{P}(H)$ represents the space of all probability measures defined on $\mathfrak{B}(H)$ equipped with the weak convergence topology. Define $\lambda : H \times L_2(K;H) \to \mathbb{R}^+$ by $\lambda(x,y) = 1 + \|(x,y)\|_0$ (cf. (2.7)), and define the space

$$\mathfrak{C}_\rho(H) = \left\{ \varphi : H \times \mathfrak{L}^2(K;H) \to \mathbb{R} \middle| \, \varphi \text{ is continuous and } \|\varphi\|_{\mathfrak{C}_\rho} < \infty \right\},$$

where

$$\|\varphi\|_{\mathfrak{C}_\rho} = \sup_{(x,y)\in H\times L_2(K;H)} \frac{|\varphi(x,y)|}{\lambda^2(x,y)} + \sup_{(x,y)\neq(\bar{x},\bar{y})} \frac{|\varphi(x,y)-\varphi(\bar{x},\bar{y})|}{\|(x,y)-(\bar{x},\bar{y})\|_{H\times L_2(K;H)}} < \infty.$$

Next, let

$$\begin{aligned} \mathfrak{P}^s_{\lambda^2}(H) \;\; = \;\; & \{m : H \to \mathbb{R} \,|m \text{ is a signed measure on } H \text{ such that} \\ & \left. \|m\|_{\lambda^2} = \int_H \lambda^2(x,y)\,|m|\,(d(x,y)) < \infty \right\}, \end{aligned}$$

where $m = m^+ - m^-$is the Jordan decomposition of m, $|m| = m^+ + m^-$, and λ is defined above. Then, we can define the space

$$\mathfrak{P}_{\lambda^2}(H) = \mathfrak{P}^s_{\lambda^2}(H) \cap \mathfrak{P}(H)$$

equipped with the metric ρ given by

$$\rho(\nu_1,\nu_2) = \sup\left\{ \int_{H\times L_2(K;H)} \varphi(x,y)\,(\nu_1-\nu_2)\,(d(x,y)) : \|\varphi\|_{\mathfrak{C}_\rho} \le 1 \right\}. \tag{2.1}$$

It can be shown that $(\mathfrak{P}_{\lambda^2}(H),\rho)$ is a complete metric space. The space of all continuous $\mathfrak{P}_{\lambda^2}(H)$-valued measures defined on $[0,T]$, denoted by $\mathfrak{C}_{\lambda^2}$, is complete when equipped with the metric

$$D(\nu_1,\nu_2) = \sup_{0\le t\le T} \rho(\nu_1(t),\nu_2(t)), \quad \nu_1,\nu_2 \in \mathfrak{C}_{\lambda^2}. \tag{2.2}$$

For the remainder of this section, H_1 denotes any Hilbert space. The collection of all strongly measurable, square integrable H_1- valued random variables x is denoted by $L^2(\Omega;H_1)$ equipped with the norm

$$\|x(\cdot)\|_{L^2(\Omega;H_1)} = \left(E\,\|x(\cdot;\omega)\|^2_{H_1}\right)^{\frac{1}{2}}. \tag{2.3}$$

For any $0 \le t \le T$, the space of all L^2-continuous, H_1-valued stochastic processes $X : [t,T] \to H_1$, denoted by $C([t,T];H_1)$, is given by

$$C([t,T];H_1) = \left\{ X : [t,T] \to H_1 \mid \sup_{t\le s\le T} \|X(s,\cdot)\|_{L^2(\Omega;H_1)} < \infty \right\}. \tag{2.4}$$

Further, $L^2_{\mathfrak{F}}(0,T;L^2(\Omega;H_1))$ (written $L^2_{\mathfrak{F}}(0,T;H_1)$, for short) represents the function space

$$\left\{ z \mid z(t) \text{ is } \mathfrak{F}_t-\text{measurable, for all } 0 \le t \le T, \text{ and } \left(E\int_0^T \|z(t;\omega)\|^2_{H_1}\, dt\right)^{\frac{1}{2}} < \infty \right\}$$

endowed with the norm

$$\|z\|_{L^2_{\mathfrak{F}}(0,T;H_1)} = \left(E\int_0^T \|z(t;\omega)\|^2_{H_1}\, dt\right)^{\frac{1}{2}}. \tag{2.5}$$

Using these spaces, for any $0 \le t \le T$, we define the Banach space $M[t,T]$ by

$$M[t,T] = L^2_{\mathfrak{F}}(\Omega; C([t,T];H)) \times L^2_{\mathfrak{F}}([t,T]; L_2(K;H)) \tag{2.6}$$

equipped with the norm

$$\|(x,y)\|^2_t = E \sup_{t\le s\le T} \|x(s;\omega)\|^2 + E\int_t^T \|y(s;\omega)\|^2_{L_2(K;H)}\, ds. \tag{2.7}$$

Hereafter, for brevity, we suppress the dependence of all mappings on ω.

Next, we recall some properties of probability measures. The probability measure P induced by an H_1-valued random variable X, denoted P_X, is defined by $P \circ X^{-1} : \mathfrak{B}(H_1) \to [0,1]$. A sequence $\{P_n\} \subset \mathfrak{P}(H_1)$ is said to be weakly convergent to P if $\int_\Omega h dP_n \to \int_\Omega h dP$, for every bounded, continuous function $h : H_1 \to \mathbb{R}$; in such case, we write $P_n \xrightarrow{w} P$. A family $\{P_n\}$ is tight if for each $\varepsilon > 0$, there exists a compact set K_ε such that $P_n(K_\varepsilon) \ge 1-\varepsilon$, for all n. Prokhorov [3, 20] established the equivalence of tightness and relative compactness of a family of probability measures. Consequently, the Arzelá-Ascoli theorem can be used to establish tightness.

Definition 2.1. Let $P \in \mathfrak{P}(H_1)$, $0 \le t_1 < t_2 < \cdots < t_k \le T$, and $X \in C([0,T];H_1)$. Define $\pi_{t_1,\ldots,t_k} : C([0,T];H_1) \to H_1^k$ by $\pi_{t_1,\ldots,t_k}(X) = (X(t_1),\ldots,X(t_k))$. The probability measures induced by $\pi_{t_1,\ldots,t_k}$ are the finite dimensional joint distributions of P.

Proposition 2.2. *([20, pg. 37]) If a sequence $\{X_n\}$ of H_1-valued random variables converges weakly to an H_1-valued random variable X in $L^2(\Omega;H_1)$, then the sequence of finite dimensional joint distributions corresponding to $\{P_{X_n}\}$ converges weakly to the finite dimensional joint distribution of P_X.*

The next theorem, in conjunction with Proposition 2.2, is the main tool in establishing a convergence result in Section 4.

Theorem 2.3. *Let $\{P_n\} \subset \mathfrak{P}(H_1)$. If the sequence of finite dimensional joint distributions corresponding to $\{P_n\}$converges weakly to the finite dimensional joint distribution of P and $\{P_n\}$ is relatively compact, then $P_n \xrightarrow{w} P$.*

Finally, in addition to the familiar Young, Hölder, and Minkowski inequalities, the following inequality (which follows from the convexity of $x^m, m \geq 1$) is important:

$$\left(\sum_{i=1}^{n} a_i\right)^m \leq n^{m-1} \sum_{i=1}^{n} a_i^m,$$

where a_i is a nonnegative constant $(i = 1, \ldots, m)$.

3. MAIN RESULTS

Our examination of (1.1) begins by first establishing a result concerning the existence and uniqueness of an $\mathfrak{F}_t-$adapted solution in the spirit of those developed in [14, 15]. This result constitutes a direct extension of the corresponding results in these papers to a more general McKean-Vlasov type stochastic equation. We begin by considering the simpler backward stochastic evolution equation given by:

$$(3.1)\qquad \begin{gathered} dx(t) + Ax(t)dt = f(t, \mu(t))dt + [g(t) + y(t)]\, dW(t), 0 \leq t \leq T, \\ x(T) = \xi \end{gathered}$$

in a separable Hilbert space H, where $\mu(t)$ is the probability distribution of $(x(t), z(t))$, where $z(t) = \int_t^T y(s)dW(s)$ (this identification of z will be used throughout the manuscript), and $f : [0, T] \times \mathfrak{P}_{\lambda^2}(H) \to H$ and $g : [0, T] \to L_2(K; H)$ are given mappings.

Let $\mathfrak{M}$ represent the σ-algebra of $\mathfrak{F}_t-$measurable subsets of $\Omega \times [0, T]$. We consider (3.1) under the following conditions:

(A1): $A : D(A) \subset H \to H$ generates a strongly continuous semigroup $\left\{e^{At} : 0 \leq t \leq T\right\}$ on H with $\overline{M} = \sup_{0 \leq t \leq T} \left\|e^{At}\right\|_{BL(H)} < \infty$.

(A2): $f : [0, T] \times \mathfrak{P}_{\lambda^2}(H) \to H$ is an $\mathfrak{M} \otimes \mathfrak{B}\left(\mathfrak{P}_{\lambda^2}(H)\right)$-measurable mapping such that

- **(i)** for each $\mu \in \mathfrak{C}_{\lambda^2}$, $f(\cdot, \mu(\cdot)) \in L^2_{\mathfrak{F}}(0, T; H)$,
- **(ii)** there exists $M_f > 0$ such that $\|f(t, \mu) - f(t, \nu)\| \leq M_f \rho(\mu, \nu)$, for all $0 \leq t \leq T$, $\mu, \nu \in \mathfrak{C}_{\lambda^2}$.

(A3): $g : [0, T] \to L_2(K; H)$ is an $\mathfrak{M}$-measurable mapping such that $g(\cdot) \in L^2_{\mathfrak{F}}(0, T;\, L_2(K; H))$.

We seek to establish the existence and uniqueness of a solution to (3.1) in the following sense:

Definition 3.1. A solution to (3.1) is an $\mathfrak{F}_t-$adapted pair $\{(x(t), y(t)) : 0 \leq t \leq T\}$ in $L^2_{\mathfrak{F}}(0, T; H) \times L^2_{\mathfrak{F}}(0, T; L_2(K; H))$ that satisfies the variation of parameters formula

$$(3.2)\qquad \begin{aligned} x(t) = e^{A(T-t)}\xi &+ \int_t^T e^{A(s-t)} f(s, \mu(s))ds \\ &+ \int_t^T e^{A(s-t)}[g(s) + y(s)]dW(s), \quad 0 \leq t \leq T \quad P \text{ a.s} \end{aligned}$$

and $\mu(t)$ is the probability distribution of $(x(t), z(t))$, for all $0 \le t \le T$.

Lemma 3.2. *Let $\xi \in L^2(\Omega; H)$. If (A1)–(A3) are satisfied, then there exists a unique solution $\{(x(t), y(t)) : 0 \le t \le T\}$ of (3.1) satisfying Definition 3.1, as well as the following estimate, for all $0 \le t \le T$:*

$$\|(x,y)\|_t^2 \le 24\overline{M}^2 E\|\xi\|^2 + 24\overline{M}^2 (T-t) E\int_t^T \|f(s,\mu(s))\|^2\, ds + 2E\int_t^T \|g(s)\|_{L_2(K;H)}^2\, ds. \tag{3.3}$$

Using Lemma 3.2, we can establish the existence and uniqueness of a solution to the original problem (1.1). By a solution of (1.1), we mean a pair $\{(x(t), y(t)) : 0 \le t \le T\}$ satisfying Definition 3.1 with $f(s, \mu(s))$ and $g(s)+y(s)$ replaced by $f(s, \mu(s), x(s), y(s))$ and $g(s, x(s)) + y(s)$, respectively. Moreover, we replace the hypotheses **(A2)** and **(A3)** on f and g by the following slightly modified versions:

- **(A4):** $f : [0,T] \times \mathfrak{P}_{\lambda^2}(H) \times H \times L_2(K;H) \to H$ is an $\mathfrak{M} \otimes \mathfrak{B}(\mathfrak{P}_{\lambda^2}(H)) \otimes \mathfrak{B}(L_2(K;H)) / \mathfrak{B}(H) \otimes \mathfrak{B}(H)$-measurable mapping such that
  - **(i):** for each $\mu \in \mathfrak{C}_{\lambda^2}$, $f(\cdot, \mu(\cdot), 0, 0) \in L^2_{\mathfrak{F}}(0,T;H)$,
  - **(ii):** there exists $M_f > 0$ such that

$$\|f(t,\mu,x_1,y_1) - f(t,\nu,x_2,y_2)\| \le M_f \left[\rho(\mu,\nu) + \|y_1 - y_2\|_{L_2(K;H)} + \|x_1 - x_2\|\right],$$

for all $0 \le t \le T$, $\mu, \nu \in \mathfrak{C}_{\lambda^2}$, $y_1, y_2 \in L_2(K;H)$, and $x_1, x_2 \in H$.

- **(A5):** $g : [0,T] \times H \to L_2(K;H)$ is an $\mathfrak{M} \otimes \mathfrak{B}(H)$ −measurable mapping such that
  - **(i):** $g(\cdot, 0) \in L^2_{\mathfrak{F}}(0,T; L_2(K;H))$,
  - **(ii):** there exists $M_g > 0$ such that

$$\|g(t,x_1) - g(t,x_2)\|_{L_2(K;H)} \le M_g \|x_1 - x_2\|,$$

for all $0 \le t \le T$ and $x_1, x_2 \in H$.

Theorem 3.3. *Let $\xi \in L^2(\Omega; H)$. If (A1), (A4), and (A5) hold, then there exists a unique solution $\{(x(t), y(t)) : 0 \le t \le T\}$ of (1.1) such that $\mu(t)$ is the probability distribution of $(x(t), z(t))$, for each $0 \le t \le T$.*

The following continuous dependence result is useful in establishing the main convergence results. It follows from a standard application of Gronwall's lemma.

Proposition 3.4. *Assume that (A1), (A4), and (A5) hold. Then, for any $\xi_1, \xi_2 \in L^2(\Omega; H)$, there exist positive constants ς_1, ς_2 such that*

$$\|(x_1,y_1) - (x_2,y_2)\|_0^2 \le \varsigma_1 E\|\xi_1 - \xi_2\|^2 + \varsigma_2 D_T^2(\mu_1, \mu_2),$$

where (x_1, y_1) and (x_2, y_2) denote the solutions of (1.1) corresponding to ξ_1, ξ_2 with respective probability distributions μ_1, μ_2.

Next, for each $n \geq 1$, consider the Yosida approximation of (1.1) given by

$$\begin{aligned} dx_n(t) + Ax_n(t)dt &= nR(n;A)f(t,\mu_n(t),x_n(t),y_n(t))dt \\ &\quad + nR(n;A)(g(t,x_n(t)) + y_n(t))dW(t), \qquad 0 \leq t \leq T, \\ x_n(T) &= nR(n;A)\xi \end{aligned} \tag{3.4}$$

where $\mu_n(t)$ is the probability law of $(x_n(t), z_n(t))$, where $z_n(t) = \int_t^T y_n(s)dW(s)$, and $R(n;A) = (I - nA)^{-1}$ is the resolvent operator of A. Assuming that (A1), (A4), and (A5) hold, an application of Theorem 3.3 implies that (3.4) has a unique solution (x_n, y_n) in the sense of Definition 3.1. The following convergence result holds:

Proposition 3.5. *Let (x, y) denote the unique solution pair of (1.1) as guaranteed to exist by Theorem 3.3. Then, the sequence of solutions (x_n, y_n) of (3.4) converges to the solution (x, y) of (1.1) as $n \to \infty$ in the sense that $\|(x_n, y_n) - (x, y)\|_0^2 \to 0$.*

The following corollary is used to establish the weak convergence of probability measures (cf. Proposition 3.8). It follows immediately from the fact that for all $0 \leq t \leq T$,

$$D_T^2(\mu_n, \mu) = \sup_{0 \leq t \leq T} \rho^2\left(\mu_n(t), \mu(t)\right) \leq \|(x_n, y_n) - (x, y)\|_t^2 \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Corollary 3.6. The sequence of probability laws μ_n corresponding to the solutions (x_n, y_n) of (3.4) converges in $\mathfrak{C}_{\lambda^2}$ as $n \to \infty$ to the probability law μ of the solution (x, y) of (1.1).

Remark 3.7. We note for later purposes that Corollary 3.6 implies that

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq s \leq T} \|\mu_n(s)\|_{\mathfrak{C}_{\lambda^2}}^2 < \infty.$$

We now consider the weak convergence of the probability measures induced by the solutions of (3.4). Let $P_{(x,y)}$ denote the probability measure generated by the solution (x, y) of (1.1) and $P_{(x_n,y_n)}$ the probability measure generated by the solution (x_n, y_n) of (3.4). We have:

Proposition 3.8. *Assume that (A1), (A4), and (A5) hold, $E\|\xi\|_H^4 < \infty$, and*

- **(A6)** *There exists $\overline{M_f} > 0$ such that $\|f(t,\mu(t),0,0)\| \leq \overline{M_f}\left[1 + \|\mu\|_{\mathfrak{C}_{\lambda^2}}\right]$, for all $0 \leq t \leq T$, $\mu, \nu \in \mathfrak{C}_{\lambda^2}$.*
- **(A7)** *There exists $\overline{M_g} > 0$ such that $\sup_{0 \leq t \leq T} E\|g(t,0)\|_{L_2(K;H)}^2 < \overline{M_g}$. Then, $P_{(x_n,y_n)} \xrightarrow{w} P_{(x,y)}$ as $n \to \infty$.*

Finally, if A depends on t, then Eq. (1.1) becomes

$$\begin{aligned} dx(t) + A(t)x(t)dt = f(t,\mu(t),x(t),y(t))dt + (g(t,x(t)) + y(t))dW(t), 0 \leq t \leq T, \\ x(T) = \xi, \end{aligned} \tag{3.6}$$

where $\{A(t) : 0 \le t \le T\}$ is a family of linear operators on H with domains $D(A(t))$ such that $\overline{D(A(t))} = \overline{D}$ (independent of t) which generates an evolution operator $\{U(t,s) : 0 \le s \le t \le T\}$ of bounded linear operators on H satisfying the following properties:

1. $U(t,t) = I$,for all $0 \le t \le T$, (where I is the identity operator on H),
2. $U(t,r)U(r,s) = U(t,s)$, for all $0 \le s \le r \le t \le T$,
3. $U(t,s)$ is strongly continuous in s on $[0,T]$ and in t on $[s,T]$,
4. $\max\limits_{(t,s)\in\Delta} \|U(t,s)\|_{BL(H)} \le M_U$, for some positive constant M_U, where $\Delta = \{(t,s) : 0 \le s \le t \le T\}$.

Conditions that ensure $\{A(t) : 0 \le t \le T\}$ generates such an evolution operator are outlined in [35]. Similar arguments (which make use of the properties of the evolution system) can be used to establish results analogous to each of those presented in this section. Since only the natural modifications need to be made to the proofs in the following section, the technical details will be omitted.

4. PROOFS

Proof of Lemma 3.2 We employ a two-stage approach by combining the strategies used in [1] and [14, 15]. Precisely, arguing as in [14, 15], we first show that for a given $\mu \in \mathfrak{C}_{\lambda^2}$, (3.1) has a unique solution pair (x,y) in the sense of Definition 3.1 which satisfies estimate (3.3). Then, we argue as in [1] to show that $\mu(t)$ must be the probability distribution of $(x(t), z(t))$, for each $0 \le t \le T$, where $z(t) = \int_t^T y(s)dW(s)$. We proceed as follows.

Let $\mu \in \mathfrak{C}_{\lambda^2}$ be given. Then, (3.1) is a linear BSEE. Hence, by Theorem 1 in [14], (3.1) has a unique solution $(x,y) \in M[0,T]$ given by

$$x(t) = e^{A(T-t)} E\left(\xi \mid \mathfrak{F}_t\right) + \int_t^T e^{A(s-t)} E\left(f(s,\mu(s)) \mid \mathfrak{F}_t\right) ds \tag{4.1}$$

$$\tilde{y}(t) = e^{A(T-t)} L(t) - \int_t^T e^{A(s-t)} K_\mu(t,s)ds, \tag{4.2}$$

$$y(t) = \tilde{y}(t) - g(t), \tag{4.3}$$

where by the extended martingale representation theorem [7], $L \in L^2_{\mathfrak{F}}\left([0,T]; L_2(K;H)\right)$ and $K_\mu \in L^2_{\mathfrak{F}}\left([0,T]\times[0,T]; L_2(K;H)\right)$ satisfy the following, for each $0 \le t \le T$:

$$E\left(\xi \mid \mathfrak{F}_t\right) = E[\xi] + \int_0^t L(\theta)dW(\theta), \tag{4.4}$$

$$E\left(f(s,\mu(s)) \mid \mathfrak{F}_t\right) = E[f(s,\mu(s))] + \int_0^t K_\mu(s,\theta)dW(\theta). \tag{4.5}$$

Now, for $0 \le t \le T$ we estimate the solution (x, y) given by (4.1)–(4.3) in $M[t, T]$. Observe that (4.1) yields

$$\begin{aligned} E \sup_{t \le s \le T} \|x(s)\|^2 &\le 2\overline{M}^2 E \sup_{t \le s \le T} \|E(\xi \mid \mathfrak{F}_s)\|^2 \\ &\qquad + 2\overline{M}^2 E \sup_{t \le s \le T} \left(\int_t^T E(\|f(r, \mu(r))\| \mid \mathfrak{F}_r)\, dr \right)^2 \\ &\le 8\overline{M}^2 E\|\xi\|^2 + 8\overline{M}^2 (T-t) E \left(\int_t^T E\|f(r, \mu(r))\|^2\, dr \right). \end{aligned}$$

Similarly, using the inequalities

$$\|\tilde{y}(t)\|^2_{L_2(K;H)} \le 2\overline{M}^2 \left[\|L(t)\|^2_{L_2(K;H)} + (T-t) \int_t^T \|K_\mu(s, t)\|^2_{L_2(K;H)}\, ds \right],$$

$$E \int_t^s \|K_\mu(s, \theta)\|^2_{L_2(K;H)}\, d\theta \le 4E\|f(s, \mu(s))\|^2,$$

$$\int_t^T \|L(s)\|^2_{L_2(K;H)}\, ds \le 4E\|\xi\|^2,$$

we obtain the following estimate of $\tilde{y}$ in (4.2)

$$\begin{aligned} &E \int_t^T \|\tilde{y}(s)\|^2_{L_2(K;H)}\, ds \\ &\le 2\overline{M}^2 \left[E \int_t^T \|L(s)\|^2_{L_2(K;H)}\, ds + \int_t^T (T-s) \int_s^T \|K_\mu(r, s)\|^2_{L_2(K;H)}\, dr ds \right] \\ &\le 2\overline{M}^2 \left[E \int_t^T \|L(s)\|^2_{L_2(K;H)}\, ds + (T-t) \int_t^T \int_t^r \|K_\mu(r, s)\|^2_{L_2(K;H)}\, ds dr \right] \\ &\le 8\overline{M}^2 E\|\xi\|^2 + 8\overline{M}^2 (T-t) E \int_t^T \|f(s, \mu(s))\|^2\, ds, \end{aligned} \tag{4.6}$$

which, in turn, yields:

$$\begin{aligned} \|(x, y)\|_t^2 &= E \sup_{t \le s \le T} \|x(s)\|^2 + E \int_t^T \|y(s)\|^2_{L_2(K;H)}\, ds \\ &\le E \sup_{t \le s \le T} \|x(s)\|^2 + 2E \int_t^T \|\tilde{y}(s)\|^2_{L_2(K;H)}\, ds + 2E \int_t^T \|g(s)\|^2_{L_2(K;H)}\, ds \\ &\le 24\overline{M}^2 E\|\xi\|^2 + 24\overline{M}^2 (T-t) E \left(\int_t^T E\|f(r, \mu(r))\|^2\, dr \right) \\ &\qquad + 2E \int_t^T \|g(s)\|^2_{L_2(K;H)}\, ds, \end{aligned}$$

which is the desired estimate (3.3). This completes the first stage of the proof.

Next, we argue that $\mu(t)$ is, in fact, the probability distribution of $(x(t), z(t))$, for all $0 \le t \le T$. For the remainder of the argument, we shall write (x, z) as (x_μ, z_μ). Denote the probability law L of (x_μ, z_μ) by $L(x_\mu, z_\mu) = \{L(x_\mu(t), z_\mu(t)) : 0 \le t \le T\}$

and define the mapping $\Psi : \mathfrak{C}_{\lambda^2} \to \mathfrak{C}_{\lambda^2}$ by $\Psi(\mu) = L(x_\mu, z_\mu)$. We establish the following two claims.

Claim 1: Ψ is well-defined.

Proof. First, we argue that (x_μ, z_μ) is L^2-continuous on $[0,T]$. Let $0 \leq t_1 \leq T$ and $|h|$ be sufficiently small. Observe that

$$
\begin{aligned}
E\left\|x_\mu(t_1+h)-x_\mu(t_1)\right\|^2 &\leq 4\Big[E\left\|\left(e^{A(T-(t_1+h))}-e^{A(T-t_1)}\right)\xi\right\|^2 \\
&+E\left\|e^{A(T-(t_1+h))}\int_0^{t_1+h}L(\theta)dW(\theta)-e^{A(T-t_1)}\int_0^{t_1}L(\theta)dW(\theta)\right\|^2 \\
&+E\left\|\int_{t_1+h}^{T}e^{A(s-(t_1+h))}f(s,\mu(s))ds-\int_{t_1}^{T}e^{A(s-t_1)}f(s,\mu(s))ds\right\|^2 \\
+E\Big\|\int_{t_1+h}^{T}e^{A(s-(t_1+h))}\int_0^{t_1+h}K_\mu(s,\theta)dW(\theta)ds &-\int_{t_1}^{T}e^{A(s-t_1)}\int_0^{t_1}K_\mu(s,\theta)dW(\theta)ds\Big\|^2\Big] \\
&=4\sum_{i=1}^{4}E\left\|I_i(t_1+h)-I_i(t_1)\right\|^2 .
\end{aligned}
\tag{4.7}
$$

Certainly, $E\left\|I_1(t_1+h)-I_1(t_1)\right\|^2 \to 0$ as $|h| \to 0$. Next, note that

$$
\begin{aligned}
&E\left\|I_2(t_1+h)-I_2(t_1)\right\|^2 \\
&=E\left\|\left(e^{A(T-(t_1+h))}-e^{A(T-t_1)}\right)\int_0^{t_1}L(\theta)dW(\theta)-e^{A(T-(t_1+h))}\int_{t_1}^{t_1+h}L(\theta)dW(\theta)\right\|^2 \\
&\leq 2E\int_0^{t_1}\left\|\left(e^{A(T-(t_1+h))}-e^{A(T-t_1)}\right)L(\theta)\right\|^2 d\theta \;+2h\overline{M}^2E\int_{t_1}^{t_1+h}\left\|L(\theta)\right\|^2_{L_2(K;H)}d\theta.
\end{aligned}
\tag{4.8}
$$

An application of the Lebesgue dominated convergence theorem shows that the right-hand side of (4.8) goes to zero as $|h| \to 0$, due to the strong continuity of $\left\{e^{At} : 0 \leq t \leq T\right\}$. The same is true of $E\left\|I_3(t_1+h)-I_3(t_1)\right\|^2$ and $E\left\|I_4(t_1+h)-I_4(t_1)\right\|^2$ due to **(A2)**, **(A3)**, and the fact that $K_\mu \in L^2_{\mathfrak{F}}\left([0,T]\times[0,T];L_2(K;H)\right)$. Next, from (4.2)–(4.5), we have

$$
\begin{aligned}
E\left\|z_\mu(t_1+h)-z_\mu(t_1)\right\|^2 &= E\int_{t_1}^{t_1+h}\left\|y_\mu(s)\right\|^2_{L_2(K;H)}ds \\
&\leq\; 2\left(E\int_{t_1}^{t_1+h}\left\|\tilde{y}_\mu(s)\right\|^2_{L_2(K;H)}ds+E\int_{t_1}^{t_1+h}\left\|g(s)\right\|^2_{L_2(K;H)}ds\right),
\end{aligned}
\tag{4.9}
$$

where the right-hand side of (4.9) goes to 0 as $h \to 0$ using **(A3)**. As such, we have established the L^2-continuity of (x_μ, z_μ) on $[0,T]$. This, together with the fact that (x_μ, z_μ) is $\mathfrak{F}_t-$adapted, guarantees that $L(x_\mu(t), z_\mu(t)) \in \mathfrak{P}_{\lambda^2}(H)$, for all $0 \leq t \leq T$.

To complete the proof of Claim 1, it remains to verify the continuity of $t \mapsto L(x_\mu(t), z_\mu(t))$. To this end, let $0 \le c \le T$ and take $|h| > 0$ small enough to ensure that $0 \le c + h \le T$. From earlier computations, we know that

$$\lim_{h\to 0} \left(E \left\| x_\mu(c+h) - x_\mu(c) \right\|^2 + E \left\| z_\mu(c+h) - z_\mu(c) \right\|^2 \right) = 0, \text{ for all } 0 \le c \le T. \tag{4.10}$$

Hence, it follows that $\lim_{h\to 0} \|(x_\mu, y_\mu)(c+h) - (x_\mu, y_\mu)(c)\|_0^2 = 0$, for all $0 \le c \le T$ (cf. (2.7)).

Consequently, since for all $0 \le c \le T$ and $\varphi \in \mathfrak{C}_{\lambda^2}$, it is the case that

$$\begin{aligned} &\left| \int_H \varphi(x,y) \left(L(x_\mu(c+h), z_\mu(c+h)) - L(x_\mu(c), z_\mu(c)) \right) (d(x,y)) \right| \\ &\quad = \left| E \left[\varphi(x_\mu(c+h;\omega), z_\mu(c+h;\omega)) - \varphi(x_\mu(c;\omega), z_\mu(c;\omega)) \right] \right| \\ &\quad \le \|\varphi\|_{\mathfrak{C}_{\lambda^2}} \|(x_\mu, y_\mu)(c+h) - (x_\mu, y_\mu)(c)\|_0^2 \end{aligned}$$

we can conclude that

$$\begin{aligned} &\rho \left(L(x_\mu, z_\mu)(c+h), L(x_\mu, z_\mu)(c) \right) \\ = &\sup_{\|\varphi\|_{\mathfrak{C}_{\lambda^2}} \le 1} \int_{H \times L_2(K;H)} \varphi(x,y) \left(L(x_\mu, z_\mu)(c+h) - L(x_\mu, z_\mu)(c) \right) (d(x,y)), \end{aligned}$$

where the right-hand side goes to 0 as $|h| \to 0$, for any $0 \le c \le T$. Hence, $t \mapsto L\left(x_\mu(t), z_\mu(t)\right)$ is a continuous map, so that $L(x_\mu, z_\mu) \in \mathfrak{C}_{\lambda^2}$. Therefore, we conclude that Ψ is indeed well-defined.

Claim 2: Ψ has a unique fixed point in $\mathfrak{C}_{\lambda^2}\left([0,T]; (\mathfrak{P}_{\lambda^2}(H), \rho)\right)$.

Proof. Let $\mu, \nu \in \mathfrak{C}_{\lambda^2}$ and let $(x_\mu, y_\mu), (x_\nu, y_\nu)$ be the corresponding solution pairs of (3.1). Observe that

$$\begin{aligned} E \left\| x_\mu(s) - x_\nu(s) \right\|^2 &= E \left\| \int_s^T e^{A(\tau - s)} \left[f(\tau, \mu(\tau)) - f(\tau, \nu(\tau)) \right] d\tau \right\|^2 \\ &\le (T-s) \overline{M}^2 M_f^2 \int_s^T \rho^2(\mu(\tau), \nu(\tau)) d\tau \end{aligned} \tag{4.11}$$

and that

$$\begin{aligned} &E \int_t^T \left\| \tilde{y}_\mu(s) - \tilde{y}_\nu(s) \right\|_{L_2(K;H)}^2 ds \le 2\overline{M}^2 E \int_t^T \int_s^T \left\| K_\mu(\theta, s) - K_\nu(\theta, s) \right\|_{L_2(K;H)}^2 ds d\theta \\ &\le 8\overline{M}^2 E \int_t^T \left\| f(\theta, \mu(\theta)) - f(\theta, \nu(\theta)) \right\|^2 d\theta \\ &\le 8\overline{M}^2 M_f^2 E \int_t^T \rho^2(\mu(\theta), \nu(\theta)) d\theta. \end{aligned} \tag{4.12}$$

So, by (4.2)–(4.3), we see that (4.12) holds with y in place of $\tilde{y}$. Consequently, for all $0 \le t \le T$,

$$\|(x_\mu, y_\mu) - (x_\nu, y_\nu)\|_t^2 \le 9\overline{M}^2 M_f^2 (T-t). \tag{4.13}$$

Let $C(t) = 9\overline{M}^2 M_f^2(T-t)$. For $0 < \delta < T$, define $D_\delta^2(\mu,\nu) = \sup_{T-\delta \le s \le T} \rho^2(\mu(s),\nu(s))$. Then, taking the supremum over $[T-\delta, T]$ in (4.13) yields

$$\sup_{T-\delta \le t \le T} \|(x_\mu, y_\mu) - (x_\nu, y_\nu)\|_t^2 \le \sup_{T-\delta \le t \le T} C(t) \cdot D_\delta^2(\mu,\nu). \tag{4.14}$$

Choosing $0 < \delta_0 < T$ such that $\sup_{T-\delta_0 \le t \le T} C(t) \le \frac{1}{2}$ enables us to conclude from (4.14) that

$$\sup_{T-\delta_0 \le t \le T} \|(x_\mu, y_\mu) - (x_\nu, y_\nu)\|_t^2 < D_{\delta_0}^2(\mu,\nu). \tag{4.15}$$

By definition of ρ (cf. (2.1)), we know that

$$\rho\left(L(x_\mu(t), z_\mu(t)), L(x_\nu(t), z_\nu(t))\right) \le \|(x_\mu, y_\mu) - (x_\nu, y_\nu)\|_t^2,$$

and hence

$$D_{\delta_0}^2(\Psi(\mu), \Psi(\nu)) \le \sup_{T-\delta_0 \le t \le T} \|(x_\mu, y_\mu) - (x_\nu, y_\nu)\|_t^2 < D_{\delta_0}^2(\mu,\nu).$$

Thus, Ψ is a strict contraction on $\mathfrak{C}_{\lambda^2}\left([T-\delta_0, T]; (\mathfrak{P}_{\lambda^2}(H), \rho)\right)$ and hence, has a unique fixed point on this space. Performing the same argument on $[T-2\delta_0, T-\delta_0]$, $[T-3\delta_0, T-2\delta_0]$,and so on, we conclude after finitely many steps that, in fact, Ψ has a unique fixed point in $\mathfrak{C}_{\lambda^2}\left([0,T]; (\mathfrak{P}_{\lambda^2}(H), \rho)\right)$. This completes the proof of Claim 2.

Thus, we can conclude from the above argument that (3.1) has a unique solution pair $\{(x_\mu(t), y_\mu(t)) : 0 \le t \le T\}$ such that $\mu(t)$ is the probability distribution of $(x_\mu(t), z_\mu(t))$, for all such $0 \le t \le T$. This completes the proof of Lemma 3.2.

Proof of Theorem 3.3 First, note that for any fixed $\mu \in \mathfrak{C}_{\lambda^2}$ and $(\bar{x}, \bar{y}) \in M[0,T]$, it follows from **(A4)** and **(A5)** that

$$F(\cdot) = f(\cdot, \mu(\cdot), \bar{x}(\cdot), \bar{y}(\cdot)) \in L_{\mathfrak{F}}^2(0,T;H)$$
$$G(\cdot) = g(\cdot, \bar{x}(\cdot), \bar{y}(\cdot)) \in L_{\mathfrak{F}}^2(0,T;L_2(K,H))$$

Thus, by Lemma 3.2, the equation

$$\begin{aligned} x(t) = e^{A(T-t)}\xi &+ \int_t^T e^{A(s-t)} F(s, \mu(s), \bar{x}(s), \bar{y}(s)) ds \\ &+ \int_t^T e^{A(s-t)} (g(s, \bar{x}(s)) + y(s)) dW(s) \quad P \text{ a.s} \end{aligned} \tag{4.16}$$

has a unique solution $(x,y) \in M[0,T]$. Thus, the operator $\Phi : M[0,T] \to M[0,T]$ defined by $\Phi(\bar{x}, \bar{y}) = (x,y)$, where (x,y) is the solution to (4.16), is well-defined. As such, in order to verify Theorem 3.3 observe that from (3.3),

$$\begin{aligned}\|\Phi(\bar{x},\bar{y})-\Phi(\tilde{x},\tilde{y})\|_t^2 &\leq 12M_S^2(T-t)\int_t^T E\|f(s,\mu(s),\bar{x}(s),\bar{y}(s))\\ &\quad -f(s,\mu(s),\tilde{x}(s),\tilde{y}(s))\|^2 ds\\ &\quad +2M_S^2 E\int_t^T \|(g(s,\bar{x}(s))-g(s,\tilde{x}(s)))\|_{L_2(K;H)}^2\, ds\\ &\leq 24M_S^2M_f^2(T-t)E\int_t^T \left[\|\bar{x}(s)-\tilde{x}(s)\|^2+\|\bar{y}(s)-\tilde{y}(s)\|_{L_2(K;H)}^2\right] ds\\ &\quad +2M_S^2M_g^2(T-t)\sup_{t\leq s\leq T} E\left\|\bar{x}(s)-\tilde{x}(s)\right\|^2\\ &\leq 24M_S^2(T-t)(M_f^2+M_g^2)\sup_{t\leq s\leq T} E\left\|\bar{x}(s)-\tilde{x}(s)\right\|^2\\ &\quad +2M_S^2M_f^2(T-t)E\int_t^T \|\bar{y}(s)-\tilde{y}(s)\|_{L_2(K;H)}^2\, ds\\ &\leq (T-t)\max\{24M_S^2(M_f^2+M_g^2),2M_S^2M_f^2\}\left\|(\bar{x},\bar{y})-(\tilde{x},\tilde{y})\right\|_t^2. \end{aligned}\tag{4.17}$$

Thus, Φ is a contraction provided that

$$(T-t)\max\{24M_S^2(M_f^2+M_g^2),2M_S^2M_f^2\}<1$$

for sufficiently small $t=T-t_0$. In such case, Φ has a unique fixed point $(x_\mu,y_\mu)\in M[T-t_0,T]$. Performing the same argument on $[T-2t_0,T-t_0]$, $[T-3t_0,T-2t_0]$, and so on, we conclude after finitely many steps that, in fact, Φ has a unique fixed point in $M[0,T]$. In order to complete the proof, it remains to show that $\mu(t)$ is the probability distribution of $(x_\mu(t),z_\mu(t))$. The proof of this fact is similar to the corresponding portion of the proof of Lemma 3.2; the details are left to the reader.

Proof of Proposition 3.5 Observe that for $0\leq t\leq T$,

$$\begin{aligned}\|(x_n,y_n)-(x,y)\|_t^2 &\leq 24\overline{M}^2 E\left\|(nR(n;A)-I)\,e^{A(T-t)}\xi\right\|^2\\ &\quad +24\overline{M}^2(T-t)E\Big(\int_t^T E\|nR(n;A)f\left(s,\mu_n(s),x_n(s),y_n(s)\right)\\ &\qquad\qquad -f\left(s,\mu(s),x(s),y(s)\right)\|^2 ds\Big)\\ &\quad +2E\int_t^T \|nR(n;A)g(s,x_n(s))-g(s,x(s))\|_{L_2(K;H)}^2\, ds\\ &= J_1(t)+J_2(t)+J_3(t)\end{aligned}$$

The strong convergence of $nR(n;A)-I$ to 0 implies

$$\sup_{0\leq t\leq T} J_1(t)\to 0 \quad \text{as} \quad n\to\infty. \tag{4.18}$$

Standard computations lead to

$$J_2(t) \leqslant 48\overline{M}^2\,(T-t)\,E\int_t^T \left[\|(nR(n;A)-I)\,f\left(s,\mu_n(s),\,x_n(s),y_n(s)\right)\|^2\right.$$

$$+ M_f^2 \left(\rho^2 \left(\mu_n(s), \mu(s)\right) + \|x_n(s) - x(s)\|^2 + \|y_n(s) - y(s)\|_{L_2(K;H)}^2\right)\Big] \, ds$$

$$\leq 48\overline{M}^2 (T-t) \, E \int_t^T \|(nR(n;A) - I) \, f(s, \mu_n(s), x_n(s), y_n(s))\|^2 \, ds$$

$$+48\overline{M}^2 M_f^2 (T-t) \max((T-t), 1) \, \|(x_n, y_n) - (x,y)\|_t^2 + C_1(n) \tag{4.19}$$

where $C_1(n) \to 0$ as $n \to \infty$. Further,

$$E \int_t^T \|(nR(n;A) - I) \, f(s, \mu_n(s), x_n(s), y_n(s))\|^2 \, ds$$

(4.20)

$$\leqslant 2E \int_t^T \|nR(n;\, A) - I\|_{BL(H)}^2 \, \|f(s, \mu_n(s), x_n(s), y_n(s)) - f(s, \mu(s), x(s), y(s))\|^2 \, ds$$

$$+ 2E \int_t^T \|n\,R(n\,;\,A) \;-\; I\|_{BL(H)}^2 \, \|f(s,\, \mu(s),\, x(s),\, y(s))\|^2 ds.$$

Since $\|f(s,\, \mu(s),\, x(s),\, y(s))\|^2$ is bounded independently of n, the dominated convergence theorem, together with the strong convergence of $nR(n;\, A) - I$ to 0, enables us to conclude that the right-side of (4.20) converges to 0 as $n \to \infty$. Hence, we have

(4.21)

$$J_2(t) \leqslant \varsigma_1(n) + 48\overline{M}^2 M_f^2 (T-t) \max((T-t), 1) \, \|(x_n, y_n) - (x,y)\|_t^2, \quad 0 \leqslant t \leqslant T,$$

where $\varsigma_1(n) \to 0$ as $n \to \infty$. Similarly,

$$J_3(t) \leqslant \varsigma_2(n) + 4M_g^2 (T-t) \, \|(x_n,\, y_n) \;-\; (x,y)\|_t^2, \quad 0 \leqslant t \leqslant T, \tag{4.22}$$

where $\varsigma_2(n) \to 0$ as $n \to \infty$ Combining (4.17)–(4.22) yields

$$\|(x_n, y_n) - (x,y)\|_t^2 \leqslant (\varsigma_1(n) + \varsigma_2(n))$$

$$+ \left(48\overline{M}^2 M_f^2 \max((T-t), 1) + 4M_g^2\right) (T-t) \, \|(x_n, y_n) - (x,y)\|_t^2, \, 0 \leqslant t \leqslant T. \tag{4.23}$$

For sufficiently small $t_0 = T - t$

$$1 - \left(48\overline{M}^2 M_f^2 \max((T-t), 1) + 4M_g^2\right) (T-t) > 0.$$

So, $\|(x_n,\, y_n) \;-\; (x,y)\|_t^2 \;\to\; 0$ as $n \to \infty$. Performing the same argument on $[T - 2t_0, T - t_0]$, $[T - 3t_0, T - 2t_0]$, and so on, we conclude after finitely many steps that

$$\|(x_n,\, y_n) \;-\; (x,y)\|_0^2 \to 0 \quad \text{as} \quad n \to \infty,$$

which completes the proof of Proposition 3.5.

Proof of Proposition 3.8 We argue that $\left\{P_{(x_n,y_n)}\right\}_{n=1}^{\infty}$is relatively compact by using Arzelá-Ascoli, and then we invoke Theorem 2.3. Throughout the proof, $C_i$will denote a suitable positive constant independent of n. To begin, we show that $\{(x_n, y_n)\}$is uniformly bounded in the sense that $\sup_{n\in\mathbb{N}} \|(x_n, y_n)\|_0^2 < \infty$. Let $t \in [0, T]$. We estimate each term in the variation of parameters formula for $x_n(t)$ separately. Observe that

$$\sup{}_{n\in\mathbb{N}} E\,\|x_n(T)\|^2 = \sup{}_{n\in\mathbb{N}} E\left\|nR(n;A)e^{A(T-t)}\xi\right\|^2 < \infty, \tag{4.24}$$

since $nR(n;A)$ is contractive for all $n \in \mathbb{N}$. Next, using **(A4)** and **(A6)** yields

$$\begin{aligned} \|f(t,\,\mu_n(t),\,x_n(t),\,y_n(t))\| &\le \|f(t,\,\mu_n(t),\,x_n(t),\,y_n(t)) - f(t,\,\mu_n(t),\,0,\,0)\| \\ &\qquad + \|f(t,\,\mu_n(t),\,0,\,0)\| \\ &\le C_1\,\|\mu_n(t)\| + C_2\,\|x_n(t)\| \;+\; C_3\,\|y_n(t)\|_{L_2(K;H)}\,, \end{aligned} \tag{4.25}$$

for all $0 \leqslant t \leqslant T$. Using a routine argument involving (4.24) and Remark 3.7 (similar to the one used to establish (4.20)) yields

$$\begin{aligned} &E\left\|\int_t^T e^{A(s-t)} n\,R(n;A)\,f(s,\,\mu_n(s),\,x_n(s),\,y_n(s))\;ds\right\|^2 \\ &\qquad \leqslant (T-t)\,\overline{M}^2\left(C_4 + C_5\,\|(x_n,y_n)\|_t^2\right), \end{aligned} \tag{4.26}$$

for all $0 \leqslant t \leqslant T$. Similarly,

$$\begin{aligned} &E\left\|\int_t^T e^{A(s-t)} n\,R(n;A)\;(g(s,\,x_n(s)) +\; y_n(s))\,ds\right\|^2 \\ &\qquad \leqslant (T-t)\,\overline{M}^2\left(C_6 + C_7\,\|(x_n,y_n)\|_t^2\right), \end{aligned} \tag{4.27}$$

for all $0 \;\leqslant\; t \;\leqslant\; T$. Further, we have

$$E\int_t^T \|y_n(s) - y(s)\|_{L_2(K;H)}^2 ds \leqslant (T-t)\,\overline{M}^2\left(C_8 + C_9\,\|(x_n,y_n)\|_t^2\right), \tag{4.28}$$

for all $0 \;\leqslant\; t \;\leqslant\; T$. As such, since $1-(T-t)\,\overline{M}^2(C_5+C_7+C_9) > 0$ for sufficiently small $t_0 = T-t$, we note that there exists a constant C (independent of n) such that $\|(x_n,y_n)\|_{T-t}^2 \le C$, for all $n \in \mathbb{N}$. Repeating this argument on subsequent intervals $[T-2t_0, T-t_0]\,, [T-3t_0, T-2t_0]\,, \ldots$ we conclude after finitely many steps that, in fact, there exists a constant C (independent of n) such that $\|(x_n,y_n)\|_0^2 \le C$, for all $n \in \mathbb{N}$, thereby verifying the uniform boundedness of $\{(x_n,y_n)\}$.

Next, to verify the equicontinuity, we argue that for every $n \in \mathbb{N}$ and $0 \leqslant s \leqslant t \leqslant T$,$E\,\|x_n(t)-x_n(s)\|^4 \to 0$ as $t-s \to 0$, independently of n. Let $0 \leqslant s \leqslant t \leqslant T$. Since $\left\{e^{At} : 0 \le t \le T\right\}$ is a semigroup,

$$E\left\|(e^{At}-e^{As})nR(n;A)\xi\right\|^4 \leqslant E\left(\int_s^t \left\|e^{A\tau}AnR(n;A)\xi\right\|d\tau\right)^4 \leqslant \overline{M}^4 E\,\|\xi\|^4\,(t-s)^4. \tag{4.29}$$

Also,

$$E\left(\int_t^{T-t} \left\|[e^{A(T-t-\tau)} - e^{A(t-\tau)}]nR(n;A)f(\tau,\,\mu_n(\tau),\,x_n(\tau),\,y_n(\tau))\right\|d\tau\right.$$

$$
\begin{aligned}
&+ \int_{T-t}^{T} \left\| e^{A(T-t-\tau)} nR(n;A) f\left(\tau,\, \mu_n(\tau),\, x_n(\tau),\, y_n(\tau)\right) \right\| d\tau \Bigg)^4 \\
&\leqslant E\left(\int_t^T \int_{t-\tau}^{T-t-\tau} \left\| e^{Au} AnR(n;A) f\left(\tau,\, \mu_n(\tau),\, x_n(\tau),\, y_n(\tau)\right) \right\| du d\tau \right. \\
&\left. + \overline{M} M_f \left[1 + \|(x_n, y_n)\|_0 + \sup_{0 \leqslant t \leqslant T} \|\mu_n(t)\|_{\mathfrak{C}_{\lambda^2}} \right] (t-s) \right)^4 \\
&\leqslant C_{10}(t-s)^4. \qquad (4.30)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&E\left(\int_t^{T-t} \left\| [e^{A(T-t-\tau)} - e^{A(t-\tau)}] nR(n;A) \left(g(\tau,\, x_n(\tau)) +\, y_n(\tau)\right) \right\| d\tau + \right. \\
&\left. + \int_{T-t}^{T} \left\| e^{A(T-t-\tau)} nR(n;A) \left(g(\tau,\, x_n(\tau)) +\, y_n(\tau)\right) \right\| d\tau \right)^4 \\
&\leq C_{11}(t-s)^4. \qquad (4.31)
\end{aligned}
$$

The equicontinuity now follows directly from (4.29)–(4.31).

Therefore, the family $\left\{P_{(x_n,y_n)}\right\}_{n=1}^{\infty}$ is relatively compact by Arzelá-Ascoli, and therefore tight (cf. Section 2). Hence, by Proposition 2.2, the finite dimensional joint distributions of $P_{(x_n,y_n)}$ converge weakly to that of $P_{(x,y)}$and so, by Theorem 2.3, $P_{(x_n,y_n)} \xrightarrow{w} P_{(x,y)}$, as $n \to \infty$.

5. **APPLICATIONS**

Example 5.1 Let $\mathfrak{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\mathfrak{D}$. Consider the following initial boundary value problem:

$$
\begin{aligned}
\partial x(t,z) &= \left(\Delta_z x(t,z) + F_1(t,z,x(t,z),y(t,z)) + \int\limits_{L^2(\mathfrak{D})} F_2(t,z,w)\mu(t,z)(dw) \right) \partial t \\
&+ \left[G(t,z,x(t,z)) + y(t,z) \right] d\beta(t), \ \text{ a.e. on } (0,T)\times\mathfrak{D}
\end{aligned}
$$

(5.1)

$$
\begin{aligned}
x(t,z) &= 0, \ \text{ a.e. on } (0,T)\times\partial\mathfrak{D}, \\
x(T,z) &= \xi(T,z), \ \text{ a.e. on } \mathfrak{D},
\end{aligned}
$$

where $x : [0,T]\times\mathfrak{D} \to \mathbb{R}$, $y : [0,T]\times\mathfrak{D} \to L_2(\mathbb{R}^N; L^2(\mathfrak{D}))$, $F_1 : [0,T]\times\mathfrak{D}\times\mathbb{R}\times L_2(\mathbb{R}^N; L^2(\mathfrak{D})) \to \mathbb{R}$, $F_2 : [0,T]\times\mathfrak{D}\times L^2(\mathfrak{D}) \to L^2(\mathfrak{D})$, $\mu(t,\cdot) \in \mathfrak{P}_{\lambda^2}(L^2(\mathfrak{D}))$ is the probability law of $(x(t,\cdot), y(t,\cdot))$, $G : [0,T]\times\mathfrak{D}\times\mathbb{R} \to L_2(\mathbb{R}^N; L^2(\mathfrak{D}))$, β is a standard $N-$dimensional Brownian motion equipped
with filtration $\{\mathfrak{F}_t\}$, and $\xi : [0,T]\times\mathfrak{D} \to \mathbb{R}$ is an $\mathfrak{F}_0-$measurable random variable independent of β with finite second moment. We impose the following conditions:

(A8): F_1 satisfies the Caratheodory conditions (i.e., measurable in (t,z,x) and continuous in the fourth variable) and there exists $M_{F_1} > 0$ such that

$$|F_1(t,z,w_1,y_1) - F_1(t,z,w_2,y_2)| \le M_{F_1}\left[|w_1 - w_2| + \|y_1 - y_2\|_{L_2(\mathbb{R}^N;L^2(\mathfrak{D}))}\right],$$

for all $0 \le t \le T, z \in \mathfrak{D}, w_1, w_2 \in \mathbb{R}, y_1, y_2 \in L_2(\mathbb{R}^N; L^2(\mathfrak{D}))$.

(A9): F_2 satisfies the Caratheodory conditions and

(i) there exists $M_{F_2} > 0$ such that

$$|F_2(t,z,w_1) - F_2(t,z,w_2)| \le M_{F_2}\|w_1 - w_2\|_{L^2(\mathfrak{D})},$$

for all $0 \le t \le T, z \in \mathfrak{D}, w_1, w_2 \in L^2(\mathfrak{D})$.

(ii) there exists $\bar{M}_{F_2} > 0$ such that $\|F_2(t,z,w)\|_{L^2(\mathfrak{D})} \le \bar{M}_{F_2}\left[1 + \|w\|_{L^2(\mathfrak{D})}\right]$, for all $0 \le t \le T$, $z \in \mathfrak{D}, w \in L^2(\mathfrak{D})$,

(iii) $F_2(t,z,\cdot) : L^2(\mathfrak{D}) \to L^2(\mathfrak{D})$ is in $\mathfrak{C}_{\lambda^2}$, for each $0 \le t \le T, z \in \mathfrak{D}$.

(A10): G satisfies the Caratheodory conditions and there exists $M_G > 0$ such that

$$\|G(t,z,w_1) - G(t,z,w_2)\|_{L_2(\mathbb{R}^N;L^2(\mathfrak{D}))} \leqslant M_G\,|w_1 - w_2|,$$

for all $0 \leqslant t \leqslant T,\ z \in \mathfrak{D},\ w_1, w_2 \in \mathbb{R}$.

We have the following theorem:

Theorem 5.1. *If **(A8)-(A10)** are satisfied, then (5.1) has a unique solution*

$$(x,y) \in L^2_{\mathfrak{F}}\left(0,T;\ L^2(\Omega;\ L^2(\mathfrak{D}))\right) \times\ L^2_{\mathfrak{F}}\left(0,T;\ L^2(\mathbb{R}^N;\ L^2(\Omega;\ L^2(\mathfrak{D})))\right)$$

such that is the probability law of $(x(t,\cdot),\ \xi(t,\cdot))$, where $\xi(t,\cdot) = \int_t^T y(s,\cdot)d\beta(s)$.

Proof. Let $H = L^2(\mathfrak{D})$ and $K = \mathbb{R}^N$. Also, denote $\partial x/\partial t$ by $x'(t)$,and define the operator A by

$$Ax\,(t,\cdot) = \Delta_z x\,(t,\cdot)\,, \qquad x \in H^2(\mathfrak{D}) \cap H^1_0(\mathfrak{D}). \tag{5.2}$$

It is known that A generates a strongly continuous semigroup $\{S(t)\}$ on $L^2(\mathfrak{D})$ (see [32]). Define the maps $f\ :\ [0,T] \times \mathfrak{P}_{\lambda^2}(H) \times H \times L_2(K;H) \to H$ and $g :\ [0,T] \times H \to L_2(K;H)$ by

$$f(t,\mu(t),\,x(t),\,y(t))(z) =\ F_1(t,z,x(t,z),y(t,z)) + \int\limits_{L^2(\mathfrak{D})} F_2(t,z,w)\mu(t,z)(dw), \tag{5.3}$$

$$g(t,x(t))(z) = G(t,z,x(t,z)), \tag{5.4}$$

for all $0 \leqslant t \leqslant T,\ z \in \mathfrak{D}$. With these identifications, we observe that (5.1) can be written in the abstract form (1.1). As mentioned above, **(A1)** is satisfied. We now show that f and g as defined in (5.3) and (5.4) satisfy **(A4)** and **(A5)**, respectively. To this end, we first use **(A9)(ii)** and **(iii)**, together with the Hölder inequality, to observe that for any fixed $\mu \in \mathfrak{P}_{\lambda^2}(H)$,

$$\left\|\int_{L^2(\mathfrak{D})} F_2(t,\cdot,w)\mu(t,\cdot)(dw)\right\|_{L^2(\mathfrak{D})}$$

$$= \left[\int_{\mathfrak{D}} \left[\int_{L^2(\mathfrak{D})} F_2(t,z,w)\mu(t,z)(dw) \right]^2 dz \right]^{1/2}$$

$$\le \left[\int_{\mathfrak{D}} \|F_2(t,z,w)\|^2_{L^2(\mathfrak{D})} \mu(t,z)(dw) dz \right]^{1/2}$$

$$\le \overline{M}_{F_2} \left[\int_{\mathfrak{D}} \left(\int_{L^2(\mathfrak{D})} \left(1+\|w\|_{L^2(\mathfrak{D})}\right)^2 \mu(t,z)(dw) \right) dz \right]^{1/2} \tag{5.5}$$

$$\le \overline{M}_{F_2} \sqrt{m(\mathfrak{D})} \sqrt{\|\mu(t)\|_{\mathfrak{C}_{\lambda^2}}} \qquad \text{cf. (2.1)}$$

$$\le \overline{M}_{F_2} \sqrt{m(\mathfrak{D})} \left(1+\|\mu(t)\|_{\mathfrak{C}_{\lambda^2}}\right), \qquad \text{for all } 0 \le t \le T.$$

Also, from **(A8)(ii)**, we obtain

(5.6)

$$\|F_1(t,\cdot,x_1(\theta,\cdot),y_1(\theta,\cdot)) - F_1(t,\cdot,x_2(\theta,\cdot),y_2(\theta,\cdot))\|_{L^2(\mathfrak{D})}$$

$$\le 4M_{F_1} \left[\int_{\mathfrak{D}} \left(|x_1(\theta,z)-x_2(\theta,z)|^2 + \|y_1(\theta,z)-y_2(\theta,z)\|^2_{L_2(\mathbb{R}^N;L^2(\mathfrak{D}))} \right) dz \right]^{\frac{1}{2}}$$

$$\le 4M_{F_1} \max\left(1, \sqrt{m(D)}\right) \left[\|x_1-x_2\|_H + \|y_1-y_2\|_{L_2(\mathbb{R}^N;L^2(\mathfrak{D}))} \right].$$

(Here, m denotes Lebesgue measure in $\mathbb{R}^N$.) Hence, from (5.5) and (5.6), we deduce that f satisfies **(A4)(i)**.

Next, invoking **(A9)(i)** yields, for all μ, $\nu \in \mathfrak{P}_{\lambda^2}(H)$,

$$\left\| \int_{L^2(\mathfrak{D})} F_2(t,\cdot,w)\mu(t,\cdot)(dw) - \int_{L^2(\mathfrak{D})} F_2(t,\cdot,w)\nu(t,\cdot)(dw) \right\|_{L^2(\mathfrak{D})}$$

$$= \left\| \int_{L^2(\mathfrak{D})} F_2(t,\cdot,w)(\mu(t,\cdot)-\nu(t,\cdot))(dw) \right\|_{L^2(\mathfrak{D})}$$

$$\le \|\rho(\mu(t),\nu(t))\|_{L^2(\mathfrak{D})} \ \ \text{(cf. (2.1))} \tag{5.7}$$

$$= \sqrt{m(\mathfrak{D})}\rho(\mu(t),\nu(t)), \ \text{for all } 0 \le t \le T.$$

Combining (5.6) and (5.7), we see that f satisfies **(A4)(ii)** with

$$M_f = \max\left\{ 4M_{F_1},\ 4M_{F_1}\sqrt{m(D)},\ \sqrt{m(\mathfrak{D})} \right\}.$$

Also, it is easy to see that g satisfies **(A5)** with $M_g = M_G$. Thus, we can invoke Theorem 3.3 to conclude that (5.1) has a unique solution $(x,y) \in L^2_{\mathfrak{F}}(0,T;\ L^2(\Omega;\ L^2(\mathfrak{D}))) \times L^2_{\mathfrak{F}}\left(0,T;\ L^2(\mathbb{R}^N;\ L^2(\Omega;\ L^2(\mathfrak{D})))\right)$ such that is the probability law of $(x(t,\cdot),\ \xi(t,\cdot))$.

Example 5.3 Consider the following initial-boundary value problem of Sobolev type:

$$\begin{aligned} &\partial\left(x(t,z)-x_{zz}(t,z)\right)-x_{zz}(t,z)\partial t \\ &= \left(F_1(t,z,x(t,z),y(t,z))+\int_{L^2(0,\pi)} F_2(t,z,w)\mu(t,z)(dw)\right)\partial t \\ &\quad + \left(G\left(t,z,x(t,z)\right)+y(t,z)\right)dW(t), \quad 0\le z\le \pi,\ 0\le t\le T, \end{aligned}$$

$$x(t,0)=x(t,\pi)=0,\ \ 0\le t\le T, \tag{5.8}$$

$$x(T,z)=\xi(T,z),\ \ 0\le z\le \pi,$$

where $x:[0,T]\times[0,\pi]\to\mathbb{R}$, $F_1:[0,T]\times[0,\pi]\times\mathbb{R}\times L_2(\mathbb{R};L^2(0,\pi))\to\mathbb{R}$, $F_2:[0,T]\times[0,\pi]\times L^2(0,\pi)\to L^2(0,\pi)$, and $G:[0,T]\times[0,\pi]\times\mathbb{R}\to L_2(\mathbb{R};L^2(0,\pi))$ are given mappings satisfying **(A8)–(A10)** on the appropriate spaces; W is a standard $L^2(0,\pi)-$valued Wiener process equipped with filtration $\{\mathfrak{F}_t\}$; $\xi:[0,T]\times[0,\pi]\to\mathbb{R}$ is an $\mathfrak{F}_0-$measurable random variable independent of W with finite second moment; and $\mu(t,\cdot)\in\mathfrak{P}_{\lambda^2}(L^2(0,\pi))$ is the probability law of $(x(t,\cdot),y(t,\cdot))$. We have the following theorem.

Theorem 5.2. *Under the above assumptions, (5.8) has a unique solution*

$$(x,y)\in L^2_{\mathfrak{F}}\left(0,T;L^2(\Omega;L^2(0,\pi))\right)\times L^2_{\mathfrak{F}}\left(0,T;L^2(\mathbb{R}^N;L^2(\Omega;L^2(0,\pi)))\right),$$

such that $\{\mu(t,\cdot):0\le t\le T\}$ *is the probability law of* $(x(t,\cdot),\xi(t,\cdot))$*, where* $\xi(t,\cdot)=\int_t^T y(s,\cdot)dW(s)$.

Proof. Let $H=L^2(0,\pi), K=\mathbb{R}$, and define the operators $A:D(A)\subset H\to H$ and $B:D(B)\subset H\to H$, respectively, by

$$Ax(t,\cdot) \ = \ -x_{zz}(t,\cdot), \quad Bx(t,\cdot)=x(t,\cdot)-x_{zz}(t,\cdot),$$

with domains

$$\begin{aligned} D(A)&=D(B) \\ &=\Big\{x\in L^2(0,\pi): x, x_z \text{ are absolutely continuous}, \\ &\qquad x_{zz}\in L^2(0,\pi),\ x(0)=x(\pi)=0\Big\}. \end{aligned}$$

Define F_1,F_2,G as in Example 5.1 (with $L^2(0,\pi)$ in place of $L^2(\mathfrak{D})$). Then, (5.8) can be written in the abstract form

$$d\left(Bx(t)\right)+Ax(t)dt=f_1\left(t,\mu(t),x(t),y(t)\right)dt+\left[g\left(t,x(t)\right)+y(t)\right]dW(t), \quad 0\le t\le T,$$

$$x(T)=\xi \tag{5.9}$$

Upon making the substitution $v(t)=Bx(t)$ in (5.9), we arrive at the equivalent problem

$$dv(t)+AB^{-1}v(t)dt=f_1\left(t,\mu(t),B^{-1}v(t),y(t)\right)dt$$

$$+ \left[g\left(t, B^{-1}v(t)\right) + y(t)\right] dW(t), \quad 0 \le t \le T,$$

$$v(T) = B\xi \tag{5.10}$$

It is known that B is a bijective operator possessing a continuous inverse and that $-AB^{-1}$ is a bounded linear operator on $L^2(0,\pi)$ which generates a strongly continuous semigroup $\{T(t)\}$ on $L^2(0,\pi)$ satisfying **(A1)** with $M_T = \alpha = 1$ (see [27]). Further, f and g are shown to satisfy **(A4)** and **(A5)**, respectively, as in Example 5.1. Consequently, we can invoke Theorem 3.3 to conclude (5.10)) has a unique solution (v, y) which, in turn, yields the corresponding solution (x, y) (where $x = B^{-1}v$) to (5.9), and hence of (5.8).

Remark 5.5 This example provides a generalization of the work in [2, 8, 28, 33, 34] to the stochastic setting. Such initial-boundary value problems arise naturally in the mathematical modeling of various physical phenomena (e.g., thermodynamics [9], shear in second-order fluids [19, 34], fluid flow through fissured rocks [3], and consolidation of clay [32].

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