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ON THE WAVE EQUATION WITH A TEMPORAL NON-LOCAL TERM

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ABSTRACT. This paper is concerned with the asymptotic behavior for an integro-differential equation which appears in viscoelasticity. It is proved that the energy of the system decays exponentially to zero as time goes to infinity provided that the kernel in the memory term is also exponentially decaying. New assumptions are discussed.

Key words and phrases: Exponential decay, memory term, relaxation function, viscoelasticity

AMS subject classifications: 35L20, 35B40, 45K05

1. INTRODUCTION

We shall consider the following wave equation with a temporal non-local term and a weak internal damping

$$(1)\qquad \begin{cases} u_{tt} + au_t = \Delta u - \int_0^t h(t-s)\Delta u(s)ds, & \text{in } \Omega \times \mathbf{R}_+ \\ u = 0, \quad \text{on } \Gamma \times \mathbf{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given initial data and the relaxation function $h(t)$ will be specified later on.

This problem models some phenomena in viscoelasticity, see [16,4] for a discussion on how these models arise. Similar linear problems as well as some nonlinear versions have been studied by many authors [6,16,3,4,17,1] see also the references in [16] and [4]. For similar problems with singular integrable and non-integrable kernels we refer the reader to [10,14,5,7,8,9,15,11] for instance. In [1], this problem was considered on star-shaped domains and with a nonlinear dissipation $g(u_t)$. The case where a localized viscoelastic dissipation acting on a certain subdomain is complemented by a weak internal dissipation on the other part of the domain is considered in [2]. We obtain results of the same nature considering different approaches. The problem in [2] is nonlinear and therefore requires a more delicate treatement. We note here that,

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in all the previous works, the following restriction on the kernel

$$h'(t) \leq -\eta h(t), \quad \forall t \geq 0$$

was imposed.

It is the purpose of this paper to give an explicit rate of convergence for solutions of problem (1). It will be shown that the solution is exponentially asymptotically stable provided that the kernel appearing in the memory term is also exponentially decaying to zero. Moreover, we replace the above frequently used assumption by the conditions $h'(t) \leq 0$ and $e^{\alpha t} h(t) \in L^1(0, \infty)$ for some $\alpha > 0$. No other condition on the derivative of $h(t)$ is imposed. Our argument has two features: it is simple (no heavy machinery is needed) and it covers some kernels which were not treated previously. An interesting family of examples may be constructed by taking non-increasing regular functions below $e^{-\alpha t}$ for some $\alpha > 0$ and which are constants on some small intervals. On these intervals where the function is constant the condition $h'(t) \leq -\eta h(t)$, $\forall t \geq 0$ is not satisfied for any $\eta > 0$. One can avoid this situation by starting the argument of the exponential decay from a point t_0 on the right hand side of these intervals but in this case we may consider an infinite (but countable) sequence of intervals as in the following second family of examples. Another type of examples consists in choosing (almost everywhere regular) functions of the form

$$h(t) = \begin{cases} (1 + n2^n)e^{-n} - 2^n e^{-n} t, \text{ on } [n, n + \frac{1}{2^n}] \\ e^{-1} + (n+1)2^{n+6} e^{-n} - 2^{n+6} e^{-n} t, \text{ on } [n+1, n+1+\frac{1}{2^{n+3}}) \\ e^{-t}, \text{ everywhere else} \end{cases}$$

for $n \geq 3$. We can easily check that $\int_t^{+\infty} h(s)ds \leq Ce^{-t}$ which implies that $\int_0^{+\infty} h(s)e^{\alpha s} ds$ for $0 < \alpha < 1$. On the other hand, $\limsup_{t\to\infty} h(t) = 1/e$ and hence $\limsup_{t\to\infty} h(t)e^{\delta t} = +\infty$ for any $\delta > 0$. Consequently, the condition $h'(t) \leq -\eta h(t)$, $\forall t \geq 0$ cannot be satisfied.

To this end we establish a new "Lyapunov functional." Indeed, we modify the energy associated to the system by an additional, suitably chosen, term which will cancel out some undesirable terms.

We also discuss the case $a = 0$, that is without the internal dissipation. In this case we will show that the integral term induces a weak damping which alone is capable of driving the system to rest also in an exponential manner.

For well posedness and regularity we refer the reader to the aforementioned works and to [13].

Theorem 1. *Assume that h is a continuous function and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then there exists a unique solution to problem (1) such that*

$$u \in L^\infty(0, \infty; H_0^1(\Omega)), \quad u_t \in L^\infty(0, \infty; L^2(\Omega)), \quad u_{tt} \in L^2(0, \infty; L^2(\Omega)).$$

The assumption on $h(t)$ we replaced is not needed to prove the existence, uniqueness or continuous dependence results. In this paper, we will concentrate only on the asymptotic behavior question.

The plan of the paper is as follows: in the next section we consider the case of an internal damping ($a > 0$) and in Section 3, we discuss the case $a = 0$.

2. ASYMPTOTIC BEHAVIOR

In this section we state and prove our result. First, we suppose that the kernel $h(t)$ is a $C^1(\mathbf{R}_+, \mathbf{R}_+)$ function satisfying

(A1) $h^{'}(t) \leq 0$ for all $t \in \mathbf{R}_+$,

(A2) $1 - \int_0^\infty h(s)ds = l > 0$,

(A3) $e^{\alpha t}h(t) \in L^1(\mathbf{R}_+)$ for some $\alpha > 0$.

Next, we define the energy of (1) by

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dx.$$

In this section we treat the case $a > 0$. Without loss of generality we may suppose that $a = 1$.

Theorem 2. *Assume that the hypotheses (A1)–(A3) hold. Then the energy of (1) decays to zero exponentially, that is, there exist positive constants C and $\beta > 0$ such that*

$$E(t) \leq Ce^{-\beta t}, \quad t \geq 0.$$

Proof: A differentiation of $E(t)$ with respect to t yields

$$\frac{dE(t)}{dt} = -\int_\Omega |u_t|^2 dx + \int_\Omega \nabla u_t \int_0^t h(t-s)\nabla u(s)ds\, dx. \tag{2}$$

Setting,

$$(h * u)(t) = \int_\Omega \int_0^t h(t-s)\left|\nabla u(t) - \nabla u(s)\right|^2 ds\, dx,$$

it is easy to see that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}(h * \nabla u)(t) = -\textstyle\int_\Omega \nabla u_t \int_0^t h(t-s)\nabla u(s)ds\, dx + \tfrac{1}{2}(h^{'} * \nabla u)(t) \\ +\tfrac{1}{2}\tfrac{d}{dt}\left\{\left(\textstyle\int_0^t h(s)ds\right)\int_\Omega |\nabla u|^2 dx\right\} - \tfrac{1}{2}h(t)\textstyle\int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{3}$$

Then, defining

$$e(t) := \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\left(1 - \int_0^t h(s)ds\right)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}(h * \nabla u)(t),$$

we obtain from (2) and (3),

$$e^{'}(t) = -\int_\Omega |u_t|^2 dx - \frac{1}{2}h(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}(h^{'} * \nabla u)(t). \tag{4}$$

Observe that by assumption (A1) we have $e^{'}(t) \leq 0,\ t \geq 0$. Moreover, from the definitions of $e(t)$, $(h * \nabla u)(t)$ and (A2), there exists $M > 0$ such that

$$E(t) \leq Me(t),\ t \geq 0. \tag{5}$$

Next, we introduce two functionals

$$\Phi(t) = \int_\Omega u_t u\, dx$$

and

$$\Psi(t) = \int_\Omega \int_0^t H_\alpha(t-s) \left|\nabla u(s)\right|^2 ds\, dx$$

where

$$H_\alpha(t) = e^{-\alpha t} \int_t^{+\infty} h(s) e^{\alpha s} ds$$

and α is as in (A3).

Using equation $(1)_1$ of our problem, we obtain

$$\begin{aligned}
\frac{d\Phi(t)}{dt} &= \int_\Omega |u_t|^2\, dx + \int_\Omega u_{tt} u\, dx \\
&= \int_\Omega |u_t|^2\, dx - \int_\Omega u_t u\, dx - \int_\Omega |\nabla u|^2\, dx + \int_\Omega \nabla u \int_0^t h(t-s) \nabla u(s) ds\, dx.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\int_\Omega \nabla u \int_0^t h(t-s)\nabla u(s) ds\, dx &\leq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega \left(\int_0^t h(t-s)\nabla u(s) ds\right)^2 dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_0^t h(t-s) ds \int_\Omega \int_0^t h(t-s) \left|\nabla u(s)\right|^2 ds\, dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1-l}{2}\int_\Omega \int_0^t h(t-s) \left|\nabla u(s)\right|^2 ds\, dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d\Phi(t)}{dt} \leq & \int_\Omega |u_t|^2\, dx - \int_\Omega u_t u\, dx - \int_\Omega |\nabla u|^2\, dx \\
& + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1-l}{2}\int_\Omega \int_0^t h(t-s) \left|\nabla u(s)\right|^2 ds\, dx.
\end{aligned} \tag{6}$$

A differentiation of $\Psi(t)$ with respect to t using Leibnitz rule yields

$$\frac{d\Psi(t)}{dt} = \left(\int_0^{+\infty} h(s) e^{\alpha s} ds\right) \int_\Omega |\nabla u|^2\, dx - \int_\Omega \int_0^t h(t-s) \left|\nabla u(s)\right|^2 ds\, dx - \alpha\Psi(t). \tag{7}$$

Let us introduce the functional

$$V(t) = e(t) + \varepsilon\Phi(t) + \eta\Psi(t)$$

with $0 < \varepsilon < 1$ and $\eta > 0$. From the above relations (4), (6) and (7) we infer that

$$V^{'}(t) = e^{'}(t) + \varepsilon\Phi^{'}(t) + \eta\Psi^{'}(t)$$

$$
\begin{aligned}
&\leq -\int_\Omega |u_t|^2\,dx - \frac{1}{2}h(t)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}(h' * \nabla u)(t) + \varepsilon \int_\Omega |u_t|^2\,dx \\
&\quad - \varepsilon \int_\Omega u_t u\,dx - \varepsilon \int_\Omega |\nabla u|^2\,dx \\
&\quad + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\varepsilon(1-l)}{2}\int_\Omega \int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx \\
&\quad + \eta \left(\int_0^{+\infty} h(s)e^{\alpha s}ds\right)\int_\Omega |\nabla u|^2\,dx \\
&\quad - \eta \int_\Omega \int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx - \alpha\eta\Psi(t)
\end{aligned}
$$

or

$$
\begin{aligned}
V'(t) &\leq -(1-\varepsilon)\int_\Omega |u_t|^2\,dx - \left[\tfrac{\varepsilon}{2} - \eta\left(\int_0^{+\infty} h(s)e^{\alpha s}ds\right)\right]\int_\Omega |\nabla u|^2\,dx \\
&+\tfrac{1}{2}(h' * \nabla u)(t) - \varepsilon\Phi(t) - \alpha\eta\Psi(t) - \left(\eta - \tfrac{\varepsilon(1-l)}{2}\right)\int_\Omega\int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx.
\end{aligned} \tag{8}
$$

If we choose α such that $\int\limits_0^{+\infty} h(s)e^{\alpha s}ds < \frac{1}{1-l}$, then we can select η such that

$$
\frac{\varepsilon(1-l)}{2} < \eta < \frac{\varepsilon}{2}\left(\int_0^{+\infty} h(s)e^{\alpha s}ds\right)^{-1}.
$$

Consequently, the coefficients of $\int_\Omega |\nabla u|^2\,dx$ and $\int_\Omega \int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx$ in (8) are negative.

Let us add and subtract $\mu(h * \nabla u)(t)$ to the right hand side of (8), then using the estimation

$$
\begin{aligned}
(h * \nabla u)(t) &= \int_\Omega\int_0^t h(t-s)\,|\nabla u(t) - \nabla u(s)|^2\,ds\,dx \\
&\leq 2(1-l)\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega\int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx
\end{aligned}
$$

we get

$$
\begin{aligned}
V'(t) \leq &-\left[\tfrac{\varepsilon}{2} - \eta\left(\int_0^{+\infty} h(s)e^{\alpha s}ds\right) - 2(1-l)\mu\right]\int_\Omega |\nabla u|^2\,dx \\
&-(1-\varepsilon)\int_\Omega |u_t|^2\,dx - \varepsilon\Phi(t) - \alpha\eta\Psi(t) - \mu(h * \nabla u)(t) \\
&-\left(\eta - \tfrac{\varepsilon(1-l)}{2} - 2\mu\right)\int_\Omega\int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx.
\end{aligned}
$$

Finally, we choose μ small enough so that the coefficients are positive. Therefore, there exists a positive constant $\beta > 0$ such that

$$
V'(t) \leq -\beta V(t),\ t \geq 0.
$$

We deduce that

$$
V(t) \leq V(0)e^{-\beta t},\ t \geq 0.
$$

From the definitions of $V(t)$ and $e(t)$ we conclude the assertion of our theorem. The proof is complete. $\square$

3. THE UNDAMPED CASE $a = 0$

If h satisfies the additional condition that $\left|h'(t)\right| e^{\sigma t} \in L^1(0,\infty)$ for some $\sigma > 0$ then it is possible to prove an exponential decay result without the internal dissipation. That is for the case $a = 0$. This situation is less favorable than the first one since we are left only with the memory term. We can show that this memory term alone is enough to produce a weak dissipation which is able to drive the system to rest in an exponential manner. The proof is essentially similar to the proof of the previous theorem. We need only to make clear how we compensate the term $-\int_\Omega |u_t|^2\,dx$ in (2) which we will lose by taking $a = 0$. To this end we introduce first the functional

$$\Lambda_1(t) := \int_\Omega u_t \int_0^t h(t-s)\,(u(t)-u(s))\,ds\,dx$$

with an appropriate coefficient. Notice that by the inequality $ab \le \delta a^2 + \frac{1}{4\delta}b^2$, we have

$$\begin{aligned}\Lambda_1(t) \le \delta_1 C_p(1-l)\textstyle\int_\Omega |\nabla u|^2\,dx + \frac{1-l}{4}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)\int_\Omega |u_t|^2\,dx \\ +(1-l)C_p\delta_2 \textstyle\int_\Omega\int_0^t h(t-s)\,|\nabla u(s)|^2\,ds\,dx,\end{aligned}$$

where C_p is the Poincaré constant. This implies that, adding this term to $V(t)$, we obtain a new functional which is again bounded below by a positive constant times $e(t)$. Therefore the exponential decay of this new functional will imply the exponential decay of $e(t)$. On the other hand the derivative of $\Lambda_1(t)$ with respect to t is equal to

$$\begin{aligned}\frac{d\Lambda_1(t)}{dt} &= -\textstyle\int_\Omega \nabla u \int_0^t h(t-s)\,(\nabla u(s)-\nabla u(t))\,ds\,dx \\ &+\textstyle\int_\Omega\left(\int_0^t h(t-s)\nabla u(s)ds\right)\left(\int_0^t h(t-s)\,(\nabla u(s)-\nabla u(t))\,ds\right)dx \\ &+\textstyle\int_\Omega u_t\int_0^t h'(t-s)\,(u(s)-u(t))\,ds\,dx - \left(\int_0^t h(s)ds\right)\int_\Omega |u_t|^2\,dx.\end{aligned} \tag{9}$$

If $t \ge t_0 > 0$, then this derivative will provide us with a $-\left(\int_0^t h(s)ds\right)\int_\Omega |u_t|^2\,dx$. The only term in the right hand side of the relation (9) which cannot be readily controlled is the third one. We have

$$\begin{aligned}&\textstyle\int_\Omega u_t\int_0^t h'(t-s)\,(u(s)-u(t))\,ds\,dx \\ &= \textstyle\int_\Omega u_t\int_0^t h'(t-s)u(s)ds\,dx - \int_0^t h'(t-s)ds\int_\Omega u_t u\,dx\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_\Omega u_t\int_0^t h'(t-s)u(s)ds\,dx \\ &\le \delta_3\textstyle\int_\Omega |u_t|^2\,dx + \frac{C_p}{4\delta_3}\int_0^\infty \left|h'(s)\right|ds\int_\Omega\int_0^t \left|h'(t-s)\right|\,|\nabla u(s)|^2\,ds\,dx.\end{aligned} \tag{10}$$

For the second term in the right hand side of (10) we need to introduce the functional

$$\Lambda_2(t) = \int_\Omega\int_0^t \tilde{H}_\sigma(t-s)\,|\nabla u(s)|^2\,ds\,dx$$

where

$$\tilde{H}_\sigma(t) = e^{-\sigma t}\int_t^{+\infty}\left|h'(s)\right| e^{\sigma s}ds.$$

The derivative of $\Lambda_2(t)$ will provide us with a $-\int_\Omega \int_0^t \left|h^{'}(t-s)\right| \left|\nabla u(s)\right|^2 ds\, dx$, see (7).

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