MULTIPLE SOLUTIONS FOR A DIRICHLET PROBLEM INVOLVING THE P-LAPLACIAN

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ABSTRACT. In this paper, we establish some multiplicity results for a Dirichlet problem related to a parametric equation involving the p-Laplacian operator. To this aim we make use of a recent local minima result of B. Ricceri.

1. INTRODUCTION

Here and in the sequel \( \Omega \) is a non-empty bounded open subset of \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \) and \( p > N \). We are interested in the multiplicity of weak solutions of the following Dirichlet problem

\[
(\text{D}_{\lambda, \mu}) \quad \begin{cases}
-\Delta_p u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( f, g : \Omega \times \mathbb{R} \to \mathbb{R} \) are two Carathéodory functions, \( \lambda, \mu \) are two positive parameters and \( \Delta_p = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the p-Laplacian.

In [2], making use of a three critical points theorem of Ricceri ([6], Theorem 1), the authors established the existence of three weak solutions for the problem \( (\text{D}_{\lambda, \mu}) \) in the case \( \mu = 0 \). Still in the case \( \mu = 0 \) but with \( f \) depending only on \( u \) and having discontinuous nonlinearities, a multiplicity result for \( (\text{D}_{\lambda, \mu}) \) is obtained in [1]. Recently, in [3], the autonomous case of the problem \( (\text{D}_{\lambda, \mu}) \) when \( p = 2 \) and \( N = 1 \) has been studied. Here, thanks to a recent result of Ricceri, we will obtain two (or three) solutions of \( (\text{D}_{\lambda, \mu}) \) when \( \mu \neq 0 \).

Now, we recall the Ricceri’s results that will be used in our arguments.

**Proposition 1.1** ([5], Proposition 3.1). Let \( X \) be a non-empty set, and \( \Phi, J \) two real functions on \( X \). Assume that there are \( \sigma > 0 \), \( x_0, x_1 \in X \), such that

\[ \Phi(x_0) = J(x_0) = 0, \quad \Phi(x_1) > \sigma, \]

Because of a surprising coincidence of names within the same Department, we have to point out that the first author was born on August 4, 1968.

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Then, for each $\rho$ satisfying
\[
\sup_{x \in \Phi^{-1}([-\infty, \sigma])} J(x) \leq \sigma \frac{J(x_1)}{\Phi(x_1)}
\]
one has
\[
\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))).
\]

**Theorem 1.1** ([7], Theorem 4). Let $X$ be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $\Psi : X \times I \to \mathbb{R}$ a function such that $\Psi(x, \cdot)$ is concave in $I$ for all $x \in X$, $\Psi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in $X$ for all $\lambda \in I$. Further, assume that
\[
\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).
\]
Then, for each $\alpha > \sup_I \inf_X \Psi$ there exists a non-empty open set $A_\alpha \subseteq I$ with the following property: for every $\lambda \in A_\alpha$ and every sequentially weakly lower semicontinuous functional $H : X \to \mathbb{R}$, there exists $\delta_{\lambda, H} > 0$ such that, for each $\mu \in ]0, \delta_{\lambda, H}[$, the functional $\Psi(\cdot, \lambda) + \mu H(\cdot)$ has at least two local minima lying in the set \{ $x \in X : \Psi(x, \lambda) < \alpha$ \}.

Before introducing our results, we precise some notation. On the Sobolev space $W^{1,p}_0(\Omega)$ we consider the norm
\[
\|u\| = \left( \int_{\Omega} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.
\]
We denote by $k$ the constant
\[
k := \sup \left\{ \frac{\max_{x \in \Omega} |u(x)|}{\|u\|} : u \in W^{1,p}_0(\Omega), u \neq 0 \right\}.
\]
The weak solutions of $(D_{\lambda, \mu})$ are the functions $u \in W^{1,p}_0$ such that
\[
\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx = \lambda \int_{\Omega} f(x, u(x)) v(x) dx + \mu \int_{\Omega} g(x, u(x)) v(x) dx
\]
for each $v \in W^{1,p}_0$. We put
\[
F(x, t) = \int_0^t f(x, \xi) d\xi
\]
for each $(x, t) \in \Omega \times \mathbb{R}$.

Fix $x_0 \in \Omega$ and $D > 0$ such that $B(x_0, D) \subseteq \Omega$ where $B(x_0, D)$ denotes the open ball of $\mathbb{R}^N$ centered on $x_0$ and having radius $D$. Moreover, we put
\[
m := \left[ \frac{\omega}{D^{p-N}} \left( 1 - \frac{1}{2^N} \right) \right]^\frac{1}{p}
\]
2. MAIN RESULTS

Theorem 2.1. Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function with \( \sup_{|s| \leq s} |f(\cdot, \xi)| \in L^1(\Omega) \) for each \( s > 0 \). Assume that there exist two positive numbers \( c, h \) with \( c < 2hmk \) such that

(i) \( F(x, \xi) \geq 0 \) for each \( (x, \xi) \in B(x_0, D) \times [0, h] \);

(ii) \( \int_{\Omega} \sup_{t \in [-c, c]} F(x, t)dx < \left( \frac{c}{2hmk} \right)^p \int_{B(x_0, \frac{D}{2})} F(x, h)dx \);

(iii) \( \limsup_{|\xi| \to +\infty} \sup_{x \in \Omega} \frac{F(x, \xi)}{|\xi|^p} \leq 0. \)

Then, there exist a number \( r \in \mathbb{R} \) and an open interval \( A \subseteq [0, +\infty[ \) with the following property: for every \( \lambda \in A \) and for every Carathéodory function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) with \( \sup_{|s| \leq s} |g(\cdot, \xi)| \in L^1(\Omega) \) for each \( s > 0 \), there exists \( \delta > 0 \) such that, for each \( \mu \in ]0, \delta[ \), the problem \((D_{\lambda, \mu})\) has at least two weak solutions whose norms are less than \( r \).

Proof. We put \( X = W^{1,p}_0(\Omega) \) and we define the functionals \( \Phi \) and \( J \) as follows

\[
\Phi(u) = \frac{1}{p} \|u\|^p \quad \text{and} \quad J(u) = \int_{\Omega} F(x, u(x)) \, dx
\]

for each \( u \in X \). Let \( \bar{u} \in X \) defined by

\[
(1) \quad \bar{u}(x) = \begin{cases} 
0 & x \in \Omega \setminus B(x_0, D) \\
h & x \in B(x_0, \frac{D}{2}) \\
\frac{2h}{D} (D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2})
\end{cases}
\]

where \( |\cdot| \) denotes the euclidean norm on \( \mathbb{R}^N \). We have

\[
\Phi(\bar{u}) = \frac{1}{p} \int_{\Omega} |\nabla \bar{u}(x)|^p \, dx = \frac{1}{p} \int_{B(x_0, D) \setminus B(x_0, \frac{D}{2})} \frac{2h^p}{D^p} \, dx = \frac{1}{p} (2hm)^p
\]

and, by (i),

\[
J(\bar{u}) = \int_{\Omega} F(x, \bar{u}(x)) \, dx \geq \int_{B(x_0, \frac{D}{2})} F(x, h) \, dx.
\]

Now, taking into account that, for every \( u \in X \), one has

\[
\max_{x \in \Omega} |u(x)| \leq k\|u\|,
\]

and put \( \sigma = \frac{1}{p} \left( \frac{c}{k} \right)^p \), condition (ii) assures that

\[
\sup_{u \in \Phi^{-1}([0, \sigma])} (J(u)) \leq \int_{\Omega} \sup_{t \in [-c, c]} F(x, t)dx < \sigma \frac{J(\bar{u})}{\Phi(\bar{u})}.
\]
At this point, chosen
\[\sup_{x \in \Phi^{-1}([-\infty, \sigma])} J(u) < \rho < \sigma \frac{J(\bar{u})}{\Phi(\bar{u})},\]

Proposition 1.1 assures that
\[\sup_{\lambda \geq 0} \inf_{u \in X} \Psi(u, \lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} \Psi(u, \lambda)\]
where
\[\Psi(u, \lambda) = \Phi(u) - \lambda J(u) + \lambda \rho\]
for each \((u, \lambda) \in X \times [0, +\infty[.\) We apply Theorem 1.1 to the functional \(\Phi\) by choosing \(I = [0, +\infty[.\)

Easily, we can observe that \(\Psi(u, \cdot)\) is concave in \(I\) for each \(u \in X\) while classical arguments provide the sequential weak lower semicontinuity and the continuity of \(\Psi(\cdot, \lambda)\) for each \(\lambda \in I\).

Now, we want to prove that the functional \(\Psi(\cdot, \lambda)\) is coercive for each \(\lambda \in I\). It is obvious that \(\Psi(\cdot, 0)\) is coercive. Fixed \(\lambda \in [0, +\infty[\) and \(0 < \epsilon < \frac{1}{p \lambda}\), condition (iii) implies that there exists \(b \in L^1(\Omega)\) such that
\[F(x, \xi) \leq \epsilon |\xi|^p + b(x)\]
for all \(x \in X\) and \(\xi \in \mathbb{R}\). Then, for each \(u \in X\), we have that
\[\Psi(u, \lambda) \geq \left( \frac{1}{p} - \lambda \epsilon \right) \|u\|^p - \lambda \int_{\Omega} b(x) dx + \lambda \rho\]
i.e. \(\Psi(\cdot, \lambda)\) is coercive. Fixed \(\alpha > \sup_{\lambda \geq 0} \inf_{u \in X} \Psi(u, \lambda)\), Theorem 1.1 ensures in particular that there exists an open interval \([a, b] \subseteq I\) with the following property: for every \(\lambda \in [a, b]\) and every Carathéodory function \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) with \(\sup_{|\xi| \leq s}|g(\cdot, \xi)| \in L^1(\Omega)\) for each \(s > 0\), there exists \(\delta > 0\) such that, for each \(\mu \in [0, \delta]\), the functional \(E(u) = \Psi(u, \lambda) - \mu H_g(u)\) has at least two local minima lying in the set \(\{u \in X : \Psi(u, \lambda) < \alpha\}\), where \(H_g\) is the weakly sequentially lower semicontinuous functional defined by
\[H_g(u) = \int_{\Omega} \left( \int_0^{u(x)} g(x, \xi) d\xi \right) dx\]
for each \(u \in X\). These two local minima are critical points of \(E\) and then are weak solutions of the problem \((D_{\lambda, u})\).

Now we observe that
\[\bigcup_{\lambda \in [a, b]} \{u \in X : \Psi(u, \lambda) < \alpha\} \subseteq \{u \in X : \Psi(u, a) \leq \alpha\} \cup \{u \in X : \Psi(u, b) \leq \alpha\}\]
and so
\[S := \bigcup_{\lambda \in [a, b]} \{u \in X : \Psi(u, \lambda) < \alpha\}\]
is bounded. The conclusion follows taking \(A = ]a, b[\) and \(r = \sup_{u \in S} \|u\|\). \(\square\)
To obtain three solutions of \((D_{\lambda, \mu})\) instead of two, we add another hypothesis on the function \(g\).

**Theorem 2.2.** Let assume the same hypotheses of Theorem 2.1. Then, there exists an open interval \(A \subseteq [0, +\infty[\) such that, for every \(\lambda \in A\) and every Carathéodory function \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) with \(\sup_{|\xi| \leq s}|g(\cdot, \xi)| \in L^1(\Omega)\) for each \(s > 0\), and

\[
(iv) \quad \limsup_{|\xi| \to +\infty} \sup_{x \in \Omega} \frac{\int_0^\xi g(x, t) dt}{|\xi|^p} < +\infty,
\]

there exists \(\delta > 0\) such that, for each \(\mu \in ]0, \delta[\), the problem \((D_{\lambda, \mu})\) has at least three weak solutions.

**Proof.** Let \(A \subseteq [0, +\infty[\) be an open interval as in the conclusion of Theorem 2.1. In particular, fixed a Carathéodory function \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) with \(\sup_{|\xi| \leq s}|g(\cdot, \xi)| \in L^1(\Omega)\) for each \(s > 0\) and satisfying \((iv)\), for each \(\lambda \in A\), there exists \(\delta > 0\) such that for every \(\mu \in ]0, \delta[\) the problem \((D_{\lambda, \mu})\) has at least two solutions which are critical points of the functional

\[
E(u) = \Psi(u, \lambda) - \mu H_g(u).
\]

In order to obtain a third solution of \((D_{\lambda, \mu})\), we prove the coercivity of \(E\). From \((iii)\) there exist \(a > 0\) and \(l \in L^1(\Omega)\) such that

\[
\int_0^\xi g(x, t) dt \leq a|\xi|^p + l(x)
\]

for all \(x \in \Omega\) and \(\xi \in \mathbb{R}\). Then, for each \(u \in X\), we have

\[
H_g(u) = \int_\Omega \left( \int_0^{u(x)} g(x, \xi) d\xi \right) dx \leq a\|u\|^p + \int_\Omega l(x) dx.
\]

Fix \(\lambda \in A\) and \(0 < \tilde{\delta} < \min\{\delta, \frac{1}{ap}\}\). Then, for each \(\mu \in ]0, \tilde{\delta}[\), choosen \(0 < \epsilon < \frac{\lambda}{\lambda + 1}(\frac{1}{p} - \mu a)\), condition \((iii)\) implies that there exists \(b_\epsilon \in L^1(\Omega)\) such that the inequality

\[
\Psi(u, \lambda) \geq \left( \frac{1}{p} - \lambda \epsilon \right) \|u\|^p - \lambda \int_\Omega b_\epsilon(x) dx + \lambda \rho
\]

holds for each \(u \in X\). Then we have

\[
E(u) \geq \left( \frac{1}{p} - \lambda \epsilon - \mu a \right) \|u\|^p - \lambda \int_\Omega b_\epsilon(x) dx + \lambda \rho - \mu \int_\Omega l(x) dx
\]

for each \(u \in X\). The last condition provides the coercivity of \(E\).

Standard arguments assure that the functional \(\Phi'\) admits a continuous inverse on \(X^*\) while \(J'\) and \(H'_g\) are compact. Then, by Example 38.25 of [8] we deduce that the functional \(E\) has the Palais-Smale property. Finally, by using Corollary 1 of [4] and taking into account that the functional \(E\) is \(C^1\) on \(X\), there exists a third critical point of \(E\) which is a third solution of problem \((D_{\lambda, \mu})\). \(\square\)

Now, we present an example in which Theorem 2.2 is applied.
Example 2.1. Let $N = 3$, $p = 4$ and $\Omega = \{ x \in \mathbb{R}^n : |x| < 1 \}$. In this case $k = (\frac{a}{4\pi})^{\frac{1}{2}}$.

Choose $x_0 = 0$ and $D = 1$. Hence $m = (\frac{\pi}{6})^\frac{1}{2}$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ and $a : \Omega \to \mathbb{R}$ be defined by setting

$$\alpha(\xi) = \begin{cases} \xi^{2\beta} & \text{if } \xi \leq 1 \\ \xi^{\alpha} & \text{if } \xi > 1 \end{cases}$$

for each $\xi \in \mathbb{R}$, with $\beta > \frac{3}{2}$ and $0 < \alpha < 3$ and

$$a(x) = \begin{cases} 1 & \text{if } x \in B(0, \frac{1}{2}) \\ 2(1 - |x|) & \text{if } x \in B(0, 1) \setminus B(0, \frac{1}{2}) \end{cases}$$

for each $x \in \Omega$.

Then, there exists an open interval $A \subseteq [0, +\infty[$ with the following property: for every $\lambda \in A$, for every continuous function $b : \Omega \to \mathbb{R}$ and every $\gamma \in ]0, 3]$, there exists $\delta > 0$ such that, for each $\mu \in ]0, \delta[$, the problem

$$\begin{cases} -\Delta u = \lambda a(x)\alpha(u) + \mu b(x)|u|^{\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

has at least two nontrivial weak solutions.

Let $f(x, t) = a(x)\alpha(t)$ and $c, h \in ]0, 1[$ such that $(\frac{15}{8}(2mk)^4)^{\frac{1}{2\beta}} < \frac{c}{h} < 1$.

With such choices one has

$$F(x, \xi) = \int_{0}^{\xi} a(x)\alpha(t)dt = \begin{cases} a(x)\frac{\xi^{2\beta + 1}}{2\beta + 1} & \text{if } \xi \leq 1 \\ a(x)\left(\frac{1}{2\beta + 1} + \frac{1}{\alpha + 1} (\xi^{\alpha + 1} + 1)\right) & \text{if } \xi > 1 \end{cases}$$

for each $(x, \xi) \in \Omega \times \mathbb{R}$ and so conditions (i) and (iii) follows obviously. On the other hand it results that

$$\int_{B(0,1)} \sup_{t \in [-c,c]} F(x, t) \, dx = \frac{c^{2\beta + 1}}{2\beta + 1} \int_{B(0,1)} a(x) \, dx = \frac{5}{8}\pi \frac{c^{2\beta + 1}}{2\beta + 1}$$

and condition (ii) follows taking into account that

$$\frac{1}{(2mk)^4} \left(\frac{c}{h}\right)^{\frac{1}{2}} \int_{B(0,\frac{1}{2})} F(x, h) \, dx = \frac{1}{(2mk)^4} \left(\frac{c}{h}\right)^{\frac{1}{2}} \int_{B(0,\frac{1}{2})} a(x) \, dx = \frac{1}{(2mk)^4} \left(\frac{c}{h}\right)^{\frac{1}{2}} \int_{B(0,\frac{1}{2})} \frac{h^{2\beta + 1}}{2\beta + 1} \, dx = \frac{1}{(2mk)^4} \left(\frac{c}{h}\right)^{\frac{1}{2}} \frac{h^{2\beta + 1}}{2\beta + 1} \frac{\pi}{6} = \frac{1}{(2mk)^4} \left(\frac{c}{h}\right)^{\frac{1}{2}} \frac{c^{2\beta + 1}}{2\beta + 1} \frac{\pi}{6} > \frac{5}{8}\pi \frac{c^{2\beta + 1}}{2\beta + 1}.$$

Finally, condition (iv) is satisfied with $g(x, t) = b(x)|t|^{\gamma}$ for each $(t, x) \in \Omega \times \mathbb{R}$.

Applying Theorem 2.1 instead of Theorem 2.2, with $f$ defined as in Example 2.1, we obtain the following conclusion: there exist a number $r \in \mathbb{R}$ and an open interval $A \subseteq [0, +\infty[$ with the following property: for every $\lambda \in A$, for every continuous
function $b : \Omega \to \mathbb{R}$ and every $\gamma > 0$, there exists $\delta > 0$ such that, for each $\mu \in ]0, \delta[$, the problem $(D)$ has at least a non trivial weak solution whose norm is less than $r$.

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