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NONLOCAL CAUCHY PROBLEM FOR SECOND ORDER STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACES

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ABSTRACT. Existence of mild solutions of second order nonlinear stochastic evolution equations with nonlocal conditions in Hilbert spaces is established. The results are obtained by using the Schaefer fixed point theorem. Application for the beam equation is also discussed to illustrate the theory.

Key Words. Existence of solutions, Second order, Stochastic evolution equation, Nonlocal condition, Schaefer's theorem

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1. INTRODUCTION

Byszewski [13] introduced nonlocal initial conditions into the initial-value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single (possibly erroneous) initial measurement. In the deterministic cases Byszewski [10] has studied the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\frac{dx(t)}{dt} + Ax(t) = f(t, x(t)), \qquad t \in (0, a]$$
$$x(t_0) + g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0$$

where $0 \leq t_0 < t_1 < ... < t_p \leq a, a > 0, -A$ is the infinitesimal generator of a C_0-semigroup in a Banach space $X, x_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X \to X$ are given functions. He has also investigated the same type of problem for different kinds of evolution equations in Banach spaces [11, 12]. Since then, many authors have continued this work in several directions including nonlocal condition and established existence theories for various functional differential equations. The elaborate discussions of Cauchy problems with nonlocal conditions the reader may refer the latest papers, for the deterministic cases [8] and for the stochastic evolution equations [2, 19] and the references contained therein. Concrete nonlocal parabolic and

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elliptic partial (integro-) differential equations arising in the mathematical modeling of various physical, biological, and ecological phenomena as well as a discussion of the advantages of replacing the classical initial condition with a more general functional can be found in [10, 11].

In many cases it is advantageous to treat the second order abstract differential equations directly rather than converting them into first order systems. The deterministic version of second order systems have been thoroughly investigated by several authors (see [6, 30, 39, 40]) while the stochastic version has been in growing state. In fact, abstract second-order stochastic evolution equations have been discussed recently in [21, 23, 24] also second order stochastic inclusions has been studied in [27].

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems (see [14, 17, 41]). Stochastic differential equations are important from the viewpoint of applications since they incorporate (natural) randomness into the mathematical description of the phenomena, and, therefore, provide a more accurate description of it. Numerous papers and books devoted to the formulation of theory of such equations have been written during the past two decades (for example see [22, 36]).

The physical motivation for the study of second order stochastic equation is given. Fitzgibbon [16] used the second order abstract differential equations for establishing the boundedness of solutions of the following partial differential equations governing the dynamical buckling of a hinged extensible beam which is stretched or compressed by an axial force

$$\frac{\partial^2 z}{\partial t^2}+\kappa\frac{\partial^4 z}{\partial x^4}-\left(\alpha+\beta\int_0^L\left|\frac{\partial z(x,s)}{\partial s}\right|^2 ds\right)\frac{\partial^2 z}{\partial x^2}+f(\frac{\partial z)}{\partial t})=0,$$

where $z(x,t)$ is the deflection of the beam at point x at time t, f is a nondecreasing numerical function, L is the length of the beam and $\alpha,\beta,\kappa>0$ are given parameters. The nonlinear friction force $f(\frac{\partial z}{\partial t})$ is the dissipative term. When $f=0$, this equation reduced to the equation introduced in [42] as a model for the transverse motion of an extensible beam whose ends are held a fixed distance apart. Several authors (see [15, 25, 28, 31, 32, 33]) used various approaches to study the estimate of weak and classical solutions of the above equation as well as the asymptotic behaviour of these solutions. These equations take the abstract form as

$$z''+A^2z+M(\|A^{\frac{1}{2}}z\|_H^2)Az+f(z')=0$$

where A is a linear operator in a Hilbert space H, M and f are real functions. Existence of solutions of this kind and more general equations are discussed in [4, 5, 6, 7, 16, 29, 33].

Recently Balachandran et. al [1] studied another type of abstract equation modelled as

$$z'' = Az + f(t, z, z')$$

in which the nonlinear damping term $f(t, z, z')$ accounts for the affects of axial force taking into account the fact that, during vibration, the elements of a beam perform not only a translatory motion, but also rotate [26, 38] which generalizes the model discussed in [33].

All results in the aforementioned papers were established for the deterministic case (without accounting for noise). As pointed out in Kannan and Bharucha-Reid [18], if experimentally there is variance in measurements, then it is advantageous to study a stochastic version of the model to better understand the effects of so-called noise on the behavior of the phenomenon.

Hence for the more realistic abstract model of the above equation it is necessary to represent certain interval structural damping behaviour in the damping term $f(t, z, z')$, tension of the beam. This can be treated by introducing noise (one more term) that represents the change in tension of the beam. This is precisely the principal goal of the present manuscript to consider the abstract form as

$$dz' = [Az + f(t, z, z')]dt + g(t, z, z')dw(t).$$

The results presented in the manuscript constitute a continuation and generalization of existence results from [2, 3, 9, 19, 37] to the second order semilinear stochastic delay evolution in Hilbert spaces with nonlocal conditions in two ways. For one, we study the second order nonlinear stochastic evolution equation using fundamental solution developed by Kozak [20] instead of cosine and sine family of operators discussed mostly in all the literatures; to the authors knowledge, this approach has not yet been treated in the study of such second order stochastic problems. And two, our result contribute a stochastic variant of the results concerning the existence of mild solutions in [1] this enables one to introduce noise into the concrete models that one subsumed as a special cases of the abstract evolution system being studied thereby allowing for a more accurate description of the phenomenon also coupling the existence results with a nonlocal initial condition strengthens the model even further.

The main aim of this paper is to establish the existence of solutions of the following second order nonlinear stochastic evolution equation with nonlocal condition

$$\begin{aligned} dx'(t) &= [A(t)x(t) + f(t, x(t), x'(t))]dt + g(t, x(t), x'(t))dw(t), \ t \in J = [0, T], \\ x(0) &+ h(x) = x_0, \qquad x'(0) = y_0, \end{aligned} \tag{1}$$

where $A(t)$ is a closed densely defined operator defined on a separable Hilbert space H with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let K be another separable Hilbert space with inner product $(\cdot, \cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t)\}_{t\geq 0}$ is a given K–valued

Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We are also employing the same notation $\|\cdot\|$ for the norm $BL(K,H)$, where $BL(K,H)$ denotes the space of all bounded linear operators from K into H. Further, $f : J \times H \times H \to H$ and $g : J \times H \times H \to L_Q(K,H)$ are measurable mappings in H-norm and $L_Q(K,H)$-norm respectively. Here $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which is going to be defined below and $h : C(J,H) \to H$ is a given continuous function.

2. PRELIMINARIES

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t \in J\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. An *$H$-valued random variable* is an $\mathfrak{F}$-measurable function $x(t) : \Omega \to H$ and a collection of random variables $S = \{x(t,w) : \Omega \to H | t \in J\}$ is called a *stochastic process*. Usually we suppress the dependence on $w \in \Omega$ and write $x(t)$ instead of $x(t,w)$ and $x(t) : J \to H$ in the place of S. Let $\beta_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathfrak{F}, P)$. Set $w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\zeta_n,\ t \geq 0$, where $\lambda_n \geq 0$, $(n = 1, 2, \ldots)$ are nonnegative real numbers and $\{\zeta_n\}$ $(n = 1, 2, \ldots)$ is complete orthonormal basis in K. Let $Q \in BL(K)$ be an operator defined by $Q\zeta_n = \lambda_n\zeta_n$ with finite $Tr\ Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, (Tr denotes the trace of the operator). Then the above K-valued stochastic process $w(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t = \sigma(w(s) : 0 \leq s \leq t)$ is the σ-algebra generated by w and $\mathfrak{F}_T = \mathfrak{F}$. Let $\varphi \in L(K,H)$ and define

$$\|\varphi\|_Q^2 = Tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\zeta_n\|^2.$$

If $\|\varphi\|_Q < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\varphi : K \to H$. The completion $L_Q(K,H)$ of $BL(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q^2 = \langle\langle \varphi, \varphi \rangle\rangle$ is a Hilbert space with the above norm topology.

The collection of all strongly-measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P; H) \equiv L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = \left(E\|x(\cdot; w)\|_H^2\right)^{\frac{1}{2}}$, where the expectation, E is defined by $E(h) = \int_\Omega h(w)dP$. Let $Z = C^1(J, L_2(\Omega; H))$ be the Banach space of all continuously differentiable maps from J into $L_2(\Omega; H)$ satisfying the condition $E\|x(t)\|^2 < \infty$ and $E\|x'(t)\|^2 < \infty$ and let $\|\cdot\|_Z$ be a norm in Z defined by

$$\|x\|_Z = \max\{\|x\|_0, \|x\|_1\},$$

where $\|x\|_0 = \left(\sup_{t \in J} \|x(t)\|_{L_2}^2\right)^{\frac{1}{2}}$ and $\|x\|_1 = \left(\sup_{t \in J} \|x'(t)\|_{L_2}^2\right)^{\frac{1}{2}}$. It is easy to verify that Z furnished with the norm topology as defined above, is a Banach space. An important subspace is given by $L_2^0(\Omega, H) = \{\xi \in L_2(\Omega, H) : \xi \text{ is } \mathfrak{F}_0\text{-measurable}\}$.

Let H be a real separable Hilbert space and, for each $t \in J$, let $A(t) : H \to H$ be a closed densely defined operator. The fundamental solution for the second order evolution equation

$$x''(t) = A(t)x(t) \tag{2}$$

developed by Kozak [20] is as follows. Let us assume that the domain of $A(t)$ does not depend on $t \in J$ and denote it by $D(A)$ (for each $t \in J,\ D(A(t)) = D(A)$).

Definition 2.1. [20] A family $\mathcal{S}$ of bounded linear operators $S(t,s) : H \to H$, $t,s \in J$, is called a fundamental solution of a second order equation if:

$[Z_1]$: For each $x \in H$ the mapping $J \times J \ni (t,s) \to S(t,s)x \in H$ is of class C^1 and

(i) for each $t \in J$, $S(t,t) = 0$,

(ii) for all $t,s \in J$, and for each $x \in H$,

$$\frac{\partial}{\partial t}S(t,s)\bigg|_{t=s} x = x, \qquad \frac{\partial}{\partial s}S(t,s)\bigg|_{t=s} x = -x.$$

$[Z_2]$: For all $t,s \in J$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $J \times J \ni (t,s) \to S(t,s)x \in H$ is of class C^2 and

(i) $\dfrac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$,

(ii) $\dfrac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$,

(iii) $\dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t}S(t,s)\bigg|_{t=s} x = 0.$

$[Z_3]$: For all $t,s \in J$, if $x \in D(A)$, then $\dfrac{\partial}{\partial s}S(t,s)x \in D(A)$, there exist $\dfrac{\partial^2}{\partial t^2}\dfrac{\partial}{\partial s}S(t,s)x$, $\dfrac{\partial^2}{\partial s^2}\dfrac{\partial}{\partial t}S(t,s)x$ and

(i) $\dfrac{\partial^2}{\partial t^2}\dfrac{\partial}{\partial s}S(t,s)x = A(t)\dfrac{\partial}{\partial s}S(t,s)x$,

(ii) $\dfrac{\partial^2}{\partial s^2}\dfrac{\partial}{\partial t}S(t,s)x = \dfrac{\partial}{\partial t}S(t,s)A(s)x$,

and the mapping $J \times J \ni (t,s) \to A(t)\dfrac{\partial}{\partial s}S(t,s)x$ is continuous.

Similar to Pazy [34] we shall define the following solution.

Definition 2.2. A $\mathfrak{F}_t$-adapted continuous stochastic process $x(t) : J \to H$ is called a mild solution of the problem (1) if the following hold:

(i) x_0, y_0 satisfying $\|x_0\|^2 < \infty$, $\|y_0\|^2 < \infty$ such that $x_0, y_0 \in L_2^0(\Omega, H)$,

(ii) the following equation satisfied for all continuous function $h : C(J,H) \to L_2^0(\Omega, H)$ and a. e. $t \in J$

$$x(t) = -\frac{\partial}{\partial s}S(t,s)\bigg|_{s=0}[x_0 - h(x)] + S(t,0)y_0 + \int_0^t S(t,s)f(s,x(s),x'(s))ds$$

$$+\int_0^t S(t,s)g(s,x(s),x'(s))dw(s). \tag{3}$$

To establish our main theorem we need the following assumptions.

(H_1): $x(t) \in D(A(t))$, for each $t \in J$.

(H_2): There exists a fundamental solution $S(t,s)$ of (2).

(H_3): $S(t,s)$ is compact for each $t,s \in J$ and there exist positive constants M, M^* and N, N^* such that

$$M = \sup\{\|S(t,s)\|^2 : t,s \in J\}, \quad M^* = \sup\{\|\tfrac{\partial}{\partial s}S(t,s)\|^2 : t,s \in J\},$$

and $N = \sup\{\|\frac{\partial}{\partial t}S(t,s)\|^2 : t,s \in J\}$, $N^* = \sup\{\|\frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\|^2 : t,s \in J\}$ respectively.

(H_4): $h : C(J,H) \to L_2^0(\Omega,H)$ is continuous and satisfying the following Lipschitz condition

$$E\|h(x)-h(y)\|^2 \le M_h E\|x-y\|^2, \quad \text{for} \;\; x,y \in C(J,H)$$

and the set $\{(x_0 - h(x)) : x \in Z, \|x\|_Z \le k\}$ is precompact in $L_2^0(\Omega,H)$ where M_h is a positive constant satisfying $5(M^*+N^*)M_h < 1$.

(H_5): $f(t,\cdot,\cdot) : H \times H \to H$ and $g(t,\cdot,\cdot) : H \times H \to BL(K,H)$ are continuous for each $t \in J$ and the functions $f(\cdot,x,y) : J \to H$, $g(\cdot,x,y) : J \to BL(K,H)$ are strongly measurable functions for each $(x,y) \in H \times H$.

(H_6): For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\ \|y\| \le k} E\|f(t,x,y)\|^2 \bigvee \sup_{\|x\|,\ \|y\| \le k} E\|g(t,x,y)\|_Q^2 \le \alpha_k(t) \quad \text{for a.a } t \in J.$$

(H_7): $f : J \times H \times H \to H$, $g : J \times H \times H \to BL(K,H)$ are continuous and there exists an integrable function $m : J \to [0,\infty)$ such that

$$E\|f(t,x,y)\|^2 \bigvee E\|g(t,x,y)\|_Q^2 \le m(t)\mho(E\|x\|^2 + E\|y\|^2), \quad t \in J, \;\; x,y \in H,$$

where $\mho : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function and

$$5\big(T + TrQ\big)\Big(\frac{M}{k_1} + \frac{N}{k_2}\Big)\int_0^T m(s)ds < \int_c^\infty \frac{ds}{\mho(s)},$$

where $c = 5(\frac{M^*}{k_1} + \frac{N^*}{k_2})\|x_0\|_Z^2 + 5(\frac{M}{k_1} + \frac{N}{k_2})\|y_0\|_Z^2$.

Lemma 2.3 (Schaefer's Theorem [35])**.** *Let X be a normed linear space. Let $\Phi : X \to X$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(\Phi) = \{x \in X : x = \lambda\Phi x \;\; \textit{for some}\; 0 < \lambda < 1\}.$$

Then either $\zeta(\Phi)$ is unbounded or Φ has a fixed point.

3. MAIN RESULT

Theorem 3.1. *If the assumptions* $(H_1)-(H_7)$ *hold, then the problem (1) has a mild solution on* J.

Proof. In order to establish the existence of a mild solution to the problem (1), we have to apply the Lemma 2.3. First we obtain *a priori* bounds for the mild solution of the following equation

$$\begin{aligned} dx'(t) &= [A(t)x(t) + \lambda f(t, x(t), x'(t))]dt + \lambda g(t, x(t), x'(t))dw(t), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J, \quad \lambda \in (0,1) \\ x(0) &+ \lambda h(x) = \lambda x_0, \qquad x'(0) = \lambda y_0, \end{aligned} \tag{4}$$

Let x be a mild solution of the problem (4). Then from

$$\begin{aligned} x(t) &= -\lambda \frac{\partial}{\partial s} S(t,s)\Big|_{s=0} [x_0 - h(x)] + \lambda S(t,0)y_0 + \lambda \int_0^t S(t,s) f(s, x(s), x'(s))ds \\ & \quad + \lambda \int_0^t S(t,s) g(s, x(s), x'(s))dw(s), \quad t \in J, \end{aligned} \tag{5}$$

we have

$$\begin{aligned} E\|x(t)\|^2 &\leq 5\big\{M^*[E\|x_0\|^2 + E\|h(x)\|^2] + ME\|y_0\|^2 + MT\int_0^t m(s)\mho(E\|x(s)\|^2 \\ & \quad + E\|x'(s)\|^2)ds + MTrQ\int_0^t m(s)\mho(E\|x(s)\|^2 + E\|x'(s)\|^2)ds\big\}. \end{aligned}$$

Thus we have

$$\begin{aligned} \|x(t)\|_Z^2 &\leq 5\big\{M^*[\|x_0\|_Z^2 + M_h\|x\|_Z^2] + M\|y_0\|_Z^2 + MT\int_0^t m(s)\mho(\|x(s)\|_Z^2 \\ & \quad + \|x'(s)\|_Z^2)ds + MTrQ\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds\big\} \\ &\leq \frac{5}{k_1}\big\{M^*\|x_0\|_Z^2 + M\|y_0\|_Z^2 + MT\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds \\ & \quad + MTrQ\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds\big\}, \end{aligned}$$

where $k_1 = 1 - 5M^*M_h$. Denoting by $v(t)$ the right-hand side of the above inequality we have

$$\begin{aligned} \|x(t)\|_Z^2 &\leq v(t), \quad t \in J, \\ v(0) &= \frac{5}{k_1}[M^*\|x_0\|_Z^2 + M\|y_0\|_Z^2], \\ v'(t) &= \frac{5}{k_1}\big(T + TrQ\big)Mm(t)\mho(\|x(t)\|_Z^2 + \|x'(t)\|_Z^2), \qquad t \in J. \end{aligned}$$

For $t \in J$, from (5)

$$\begin{aligned} x'(t) &= -\lambda \frac{\partial}{\partial t}\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[x_0 - h(x)] + \lambda\frac{\partial}{\partial t}S(t,0)y_0 \\ &\quad +\lambda\int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x(s),x'(s))ds + \lambda\int_0^t \frac{\partial}{\partial t}S(t,s)g(s,x(s),x'(s))dw(s) \end{aligned}$$

and we have

$$\begin{aligned} E\|x'(t)\|^2 &\le 5\big\{N^*[E\|x_0\|^2 + E\|h(x)\|^2] + NE\|y_0\|^2 + NT\int_0^t m(s)\mho(E\|x(s)\|^2 \\ &\quad +E\|x'(s)\|^2)ds + NTrQ\int_0^t m(s)\mho(E\|x(s)\|^2 + E\|x'(s)\|^2)ds\big\}. \end{aligned}$$

Thus

$$\begin{aligned} \|x'(t)\|_Z^2 &\le 5\big\{N^*[\|x_0\|_Z^2 + M_h\|x\|_Z^2] + N\|y_0\|_Z^2 + NT\int_0^t m(s)\mho(\|x(s)\|_Z^2 \\ &\quad +\|x'(s)\|_Z^2)ds + NTrQ\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds\big\} \\ &\le \frac{5}{k_2}\big\{N^*\|x_0\|_Z^2 + N\|y_0\|_Z^2 + NT\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds \\ &\quad +NTrQ\int_0^t m(s)\mho(\|x(s)\|_Z^2 + \|x'(s)\|_Z^2)ds\big\}, \end{aligned}$$

where $k_2 = 1 - 5N^*M_h$. Denoting by $r(t)$ the right-hand side of the above inequality we have

$$\begin{aligned} \|x'(t)\|_Z^2 &\le r(t), \quad t \in J, \\ r(0) &= \frac{5}{k_2}[N^*\|x_0\|_Z^2 + N\|y_0\|_Z^2], \\ r'(t) &= \frac{5}{k_2}\big(T + TrQ\big)Nm(t)\mho(\|x(t)\|_Z^2 + \|x'(t)\|_Z^2), \qquad t \in J. \end{aligned}$$

Let
$$w(t) = v(t) + r(t), \qquad t \in J.$$

Then
$$w(0) = v(0) + r(0) = c, \text{ and}$$

$$\begin{aligned} w'(t) &= v'(t) + r'(t) \\ &\le 5\big(T + TrQ\big)\Big(\frac{M}{k_1} + \frac{N}{k_2}\Big)m(t)\mho(w(t)). \end{aligned}$$

This gives

$$\begin{aligned} \int_{w(0)}^{w(t)} \frac{ds}{\mho(s)} &\le 5\big(T + TrQ\big)\Big(\frac{M}{k_1} + \frac{N}{k_2}\Big)\int_0^t m(s)ds \\ &\le 5\big(T + TrQ\big)\Big(\frac{M}{k_1} + \frac{N}{k_2}\Big)\int_0^T m(s)ds < \int_c^\infty \frac{ds}{\mho(s)}. \end{aligned}$$

This inequality implies that there is a constant k such that

$$w(t) = v(t) + r(t) \leq k, \qquad t \in J.$$

Thus $\|x(t)\|^2 \leq k, \ \ \|x'(t)\|^2 \leq k, \ \ t \in J$, and hence

$$\|x\|_Z \leq k,$$

where k depends only on T and on the functions m and $\mho$.

Now we shall prove that the operator $\Phi : Z \to Z$ defined by

$$\begin{aligned}(\Phi x)(t) \ \ &= \ \ -\frac{\partial}{\partial s} S(t,s)\Big|_{s=0} [x_0 - h(x)] + S(t,0)y_0 + \int_0^t S(t,s) f(s, x(s), x'(s)) ds \\ &\qquad + \int_0^t S(t,s) g(s, x(s), x'(s)) dw(s), \quad t \in J\end{aligned}$$

is a completely continuous operator.

Let $B_q = \{x \in Z : \|x\|_Z \leq q\}$ for some $q \geq 1$. We first show that Φ maps B_q into an equicontinuous family. Let $x \in B_q$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned}
&E\|(\Phi x)(t_1) - (\Phi x)(t_2)\|^2 \\
&\leq \ 6\Big\{E\left\|\frac{\partial}{\partial s}[S(t_1,s) - S(t_2,s)]\Big|_{s=0}[x_0 - h(x)]\right\|^2 + E\|[S(t_1,0) - S(t_2,0)]y_0\|^2 \\
&\quad + T\int_0^{t_1} \|S(t_1,s) - S(t_2,s)\|^2 \alpha_k(s) ds + (t_2 - t_1)\int_{t_1}^{t_2} \|S(t_2,s)\|^2 \alpha_k(s) ds \\
&\quad + TrQ \int_0^{t_1} \|S(t_1,s) - S(t_2,s)\|^2 \alpha_k(s) ds + TrQ\int_{t_1}^{t_2} \|S(t_2,s)\|^2 \alpha_k(s) ds\Big\} \\
&\to \ 0 \quad \text{as} \quad t_1 \to t_2
\end{aligned}$$

and similarly

$$\begin{aligned}
&E\|(\Phi x)'(t_1) - (\Phi x)'(t_2)\| \\
&\leq \ 6\Big\{E\left\|\frac{\partial}{\partial s}[\frac{\partial}{\partial t_1}S(t_1,s) - \frac{\partial}{\partial t_2}S(t_2,s)]\Big|_{s=0}[x_0 - h(x)]\right\|^2 \\
&\quad + \|[\frac{\partial}{\partial t_1}S(t_1,0) - \frac{\partial}{\partial t_2}S(t_2,0)]\|^2 E\|y_0\|^2 \\
&\quad + T\int_0^{t_1} \|[\frac{\partial}{\partial t_1}S(t_1,s) - \frac{\partial}{\partial t_2}S(t_2,s)]\|^2 \alpha_k(s) ds \\
&\quad + (t_2 - t_1)\int_{t_1}^{t_2} \|\frac{\partial}{\partial t_2}S(t_2,s)\|^2 \alpha_k(s) ds \\
&\quad + TrQ\int_0^{t_1} \|[\frac{\partial}{\partial t_1}S(t_1,s) - \frac{\partial}{\partial t_2}S(t_2,s)]\|^2 \alpha_k(s) ds
\end{aligned}$$

$$+ TrQ \int_{t_1}^{t_2} \|\frac{\partial}{\partial t_2} S(t_2, s)\|^2 \alpha_k(s) ds\Big\} \quad \to 0 \quad \text{as} \quad t_1 \to t_2.$$

Thus Φ maps B_q into an equicontinuous family of functions. It is easy to see that the family ΦB_q is uniformly bounded.

Next we show that $\overline{\Phi B_q}$ is compact. Since we have shown ΦB_q is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that Φ maps B_q into a precompact set in H.

Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_q$ we define

$$(\Phi_\epsilon x)(t) = -\frac{\partial}{\partial s} S(t, s)\Big|_{s=0} [x_0 - h(x)] + S(t, 0)y_0 + \int_0^{t-\epsilon} S(t, s) f(s, x(s), x'(s)) ds$$
$$+ \int_0^{t-\epsilon} S(t, s) g(s, x(s), x'(s)) dw(s), \ t \in J.$$

Since $S(t, s)$ is a compact operator, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_q\}$ is precompact in H for every ϵ, $0 < \epsilon < t$. Moreover for every $x \in B_q$ we have

$$\begin{aligned} E\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\|^2 &\leq 2\epsilon \int_{t-\epsilon}^{t} \|S(t, s)\|^2 E\|f(s, x(s), x'(s))\|^2 ds \\ &\quad +2TrQ \int_{t-\epsilon}^{t} \|S(t, s)\|^2 E\|g(s, x(s), x'(s))\|_Q^2 ds \\ &\leq 2\epsilon \int_{t-\epsilon}^{t} \|S(t, s)\|^2 \alpha_k(s) ds + 2TrQ \int_{t-\epsilon}^{t} \|S(t, s)\|^2 \alpha_k(s) ds \\ &\to 0 \quad \text{as} \quad \epsilon \to 0 \end{aligned}$$

and

$$\begin{aligned} E\|(\Phi x)'(t) - (\Phi_\epsilon x)'(t)\|^2 &\leq 2\epsilon \int_{t-\epsilon}^{t} E\|\frac{\partial}{\partial t} S(t, s) f(s, x(s), x'(s))\|^2 ds \\ &\quad +2TrQ \int_{t-\epsilon}^{t} E\|\frac{\partial}{\partial t} S(t, s) g(s, x(s), x'(s))\|_Q^2 ds \\ &\leq 2\epsilon \int_{t-\epsilon}^{t} \|\frac{\partial}{\partial t} S(t, s)\|^2 \alpha_k(s) ds \\ &\quad +2TrQ \int_{t-\epsilon}^{t} \|\frac{\partial}{\partial t} S(t, s)\|^2 \alpha_k(s) ds \ \to \ 0 \ \ as \ \ \epsilon \to 0. \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_q\}$. Hence the set $\{(\Phi x)(t) : x \in B_q\}$ is precompact in H.

It remains to show that $\Phi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer ν such that $E\|x_n(t)\|^2 \leq \nu, E\|x_n'(t)\|^2 \leq \nu$ for all n and $t \in J$, so $E\|x(t)\|^2 \leq \nu, E\|x'(t)\|^2 \leq \nu$ and $x, x' \in B_\nu$. By (H_5)

$$f(s, x_n(s), {x_n}'(s)) \to f(s, x(s), x'(s))$$

and

$$g(s, x_n(s), {x_n}'(s)) \to g(s, x(s), x'(s))$$

for each $t \in J$. Further

$$E\|f(s, x_n(s), {x_n}'(s)) - f(s, x(s), x'(s))\|^2 \leq 2\alpha_\nu(s),$$

and

$$E\|g(s, x_n(s), {x_n}'(s)) - g(s, x(s), x'(s))\|^2 \leq 2\alpha_\nu(s),$$

we have by dominated convergence theorem

$$\begin{aligned}
&E\|\Phi x_n - \Phi x\|^2 \\
&= 3\sup_{t\in J}\Big\{E\left\|\frac{\partial}{\partial s}S(t,s)|_{s=0}h(x_n) - \frac{\partial}{\partial s}S(t,s)|_{s=0}h(x)\right\|^2 \\
&\quad +E\left\|\int_0^t S(t,s)f(s,x_n(s),x_n'(s))ds - \int_0^t S(t,s)f(s,x(s),x'(s))ds\right\|^2 \\
&\quad +E\left\|\int_0^t S(t,s)g(s,x_n(s),x_n'(s))dw(s) - \int_0^t S(t,s)g(s,x(s),x'(s))dw(s)\right\|^2\Big\} \\
&\leq 3\big\{M^*E\|h(x_n)-h(x)\|^2 + T\int_0^t E\|S(t,s)[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))]\|^2 ds \\
&\quad +TrQ\int_0^t E\|S(t,s)[g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))]\|_Q^2 ds\big\} \to 0 \text{ as } n\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&E\|(\Phi x_n)' - (\Phi x)'\|^2 \\
&= 3\sup_{t\in J}\Big\{E\left\|\frac{\partial}{\partial s}[\frac{\partial}{\partial t}S(t,s)]|_{s=0}h(x_n) - \frac{\partial}{\partial s}[\frac{\partial}{\partial t}S(t,s)]|_{s=0}h(x)\right\|^2 \\
&\quad +E\left\|\int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x_n(s),x_n'(s))ds - \int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x(s),x'(s))ds\right\|^2 \\
&\quad +E\left\|\int_0^t \frac{\partial}{\partial t}S(t,s)g(s,x_n(s),x_n'(s))dw(s) - \int_0^t \frac{\partial}{\partial t}S(t,s)g(s,x(s),x'(s))dw(s)\right\|^2\Big\} \\
&\leq 3\big\{N^*E\|h(x_n)-h(x)\|^2 \\
&\quad +T\int_0^t E\|\frac{\partial}{\partial t}S(t,s)[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))]\|^2 ds \\
&\quad +TrQ\int_0^t E\|\frac{\partial}{\partial t}S(t,s)[g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))]\|_Q^2 ds\big\} \\
&\quad \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Thus Φ is continuous. This completes the proof that Φ is completely continuous.

We have already proved that the set $\zeta(\Phi) = \{x \in Z : x = \lambda\Phi x, \lambda \in (0,1)\}$ is bounded. Hence by the Schaefer fixed point theorem the operator Φ has a fixed point in Z. This means that any fixed point of Φ is a mild solution of (1) on J satisfying $(\Phi x)(t) = x(t)$. Thus the initial value problem (1) has at least one mild solution on J.

4. STOCHASTIC INTEGRODIFFERENTIAL EQUATION

The derived theory is also easily can be applied to the following general second-order stochastic integrodifferential equation with nonlocal condition of the form

$$\begin{aligned} dx'(t) &= \big[A(t)x(t) + f\big(t, x(t), \textstyle\int_0^t \eta_1(t,s)\mu(s,x(s),x'(s))ds, x'(t)\big)\big]dt \\ &\quad + g\big(t, x(t), \textstyle\int_0^t \eta_2(t,s)\mu(s,x(s),x'(s))ds, x'(t)\big)dw(t), \quad t \in J \\ x(0) + h(x) &= x_0, \quad x'(0) = y_0 \end{aligned} \tag{6}$$

where $A(t), h$ are as in the previous section and $f : J \times H \times H \times H \to H$, $g : J \times H \times H \times H \to BL(K,H)$, $\eta_i : J \times J \to R$, for $i = 1, 2$, $\mu : J \times H \times H \to H$ are given functions. If $x(t)$ is a solution of the problem (6) then for $t \in J$

$$\begin{aligned} x(t) &= -\frac{\partial}{\partial s} S(t,s)\Big|_{s=0} [x_0 - h(x)] + S(t,0)y_0 \\ &\quad + \int_0^t S(t,s) f(s, x(s), \int_0^s \eta_1(s,\tau)\mu(\tau, x(\tau), x'(\tau))d\tau, x'(s))ds \\ &\quad + \int_0^t S(t,s) g(s, x(s), \int_0^s \eta_2(s,\tau)\mu(\tau, x(\tau), x'(\tau))d\tau, x'(s))dw(s). \end{aligned} \tag{7}$$

The above equation (7) is more general than equation (6) and every solution of this is called mild solution of (6).

Assume the following conditions:

(C_1)**:** $f(t,\cdot,\cdot,\cdot) : H \times H \times H \to H$ and $g(t,\cdot,\cdot,\cdot) : H \times H \times H \to BL(K,H)$ are continuous for each $t \in J$ and the functions $f(\cdot,x,y,z) : J \to H$, $g(\cdot,x,y,z) : J \to BL(K,H)$ are strongly measurable functions for each $(x,y,z) \in H \times H \times H$.

(C_2)**:** For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|,\|z\| \le k} E\|f(t,x,y,z)\|^2 \bigvee E\|g(t,x,y,z)\|^2 \le \alpha_k(t) \quad \text{for a.a} \quad t \in J.$$

(C_3)**:** $\mu : J \times H \times H \to H$ is continuous and there exists an integrable function $m : J \to [0,\infty)$ such that

$$E\|\mu(t,x,y)\|^2 \le m(t)\mho(E\|x\|^2 + E\|y\|^2), \quad t \in J, \quad x, y \in H,$$

where $\mho : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(C_4): $f : J \times H \times H \times H \to H,\ g : J \times H \times H \times H \to BL(K, H)$ are continuous and there exists an integrable function $p : J \to [0, \infty)$ such that $t \in J,\ x, y, z \in H$,

$$E\|f(t, x, y, z)\|^2 \bigvee E\|g(t, x, y, z)\|^2 \leq p(t)\mho_0(E\|x\|^2 + E\|y\|^2 + E\|z\|^2),$$

where $\mho_0 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(C_5): $\eta_i : J \times J \to R$, (i = 1, 2) are measurable and there exists a constant L such that

$$|\eta_i(t, s)|^2 \leq L, \text{ for } \ t \geq s \geq 0.$$

Theorem 4.1. *If the assumptions $(H_1) - (H_4)$ and $(C_1) - (C_5)$ hold and if*

$$\int_0^T \tilde{m}(s)ds < \int_c^\infty \frac{ds}{\mho(s) + \mho_0(s)},$$

where $\tilde{m}(t) = \max\{5(T + TrQ)(\frac{M}{k_1} + \frac{N}{k_2})p(t), Lm(t)\}$, then the problem (6) has at least one mild solution on J.

Moreover using the same method as in Theorem 3.1 we can easily prove that the problem (6) has at least one mild solution on J.

5. EXAMPLE

Let $\mathfrak{D}$ be a bounded domain in R^N with smooth boundary $\partial\mathfrak{D}$ and consider the stochastic partial differential equation

$$\begin{aligned} \partial\Big(\frac{\partial x(t,z)}{\partial t}\Big) &= [\frac{\partial^2 x(t,z)}{\partial z^2} + f\big(t, x(t,z), \frac{\partial x(t,z)}{\partial t}\big)]\partial t \\ &\quad + g\big(t, x(t,z), \frac{\partial x(t,z)}{\partial t}\big)d\beta(t), \ a.e. \ (0,T) \times \mathfrak{D} \\ x(0,z) &= \sum_{i=1}^n \xi_i(z)x(t_i, z), \quad a.e. \ on \ \mathfrak{D} \\ \frac{\partial}{\partial t}x(0,z) &= \xi_0(z), \quad a.e. \ on \ \mathfrak{D} \\ x(t,z) &= 0, \quad a.e. \ on \ (0,T) \times \partial\mathfrak{D} \end{aligned} \tag{8}$$

where $0 \leq t_1 < t_2 < \cdots < t_n \leq T$ are given and $\beta(t)$ is a $L_2(\mathfrak{D})$-valued Brownian motion (see [14]). We consider (8) under the following conditions:

(C_6): $f(t, \cdot, \cdot) : R \times R \to R$ and $g(t, \cdot, \cdot) : R \times R \to BL(L_2(\mathfrak{D}))$ are continuous for each $t \in J$ and the functions $f(\cdot, x, y) : J \to H,\ g(\cdot, x, y) : J \to BL(L_2(\mathfrak{D}))$ are strongly measurable functions for each $(x, y) \in R \times R$.

(C_7): For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\ \|y\| \leq k} E\|f(t, x, y)\|^2 \bigvee \sup_{\|x\|,\ \|y\| \leq k} E\|g(t, x, y)\|_Q^2 \leq \alpha_k(t) \quad \text{for a.a} \quad t \in J.$$

(C_8): $f : J \times R \times R \to R$, $g : J \times R \times R \to BL(L_2(\mathfrak{D}))$ are continuous and there exists an integrable function $m : J \to [0, \infty)$ such that

$$E\|f(t,x,y)\|^2 \bigvee E\|g(t,x,y)\|_Q^2 \le m(t)\mho(E\|x\|^2 + E\|y\|^2), \quad t \in J, \;\; x, y \in R,$$

(C_9): $\xi_i \in L_2^0(\Omega, L_2(\mathfrak{D}))$, $i = 0, 1, 2 \cdots, n$.

The third right hand side term in the first equation of (8) represents certain change in tension of the beam. This example introduces the noise in the model governing the displacement of the beam. Let $H = K = L_2(\mathfrak{D})$ and let $A : H \to H$ be defined by

$$Ay = y'', \qquad y \in D(A) = H_2(\mathfrak{D}) \cup H_1^0(\mathfrak{D}).$$

It is easily shown that A generates a fundamental solution $S(t)$, $t \in R$. Also the nonlocal condition satisfies the Lipschitz condition as

$$\|h(x)(\cdot) - h(y)(\cdot)\|^2_{L^2(\mathfrak{D})} \le \sum_{i=1}^{n} \|\xi_i(\cdot)\|^2 \|x(t_i, \cdot) - y(t_i, \cdot)\|^2$$

where $M_h = \sum_{i=1}^{n} \|\xi_i(\cdot)\|^2$. Then, (8) is the abstract formulation of (1), existence solution follows immediately from Theorem 3.1.

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