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UPPER AND LOWER SOLUTIONS METHOD AND A SUPERLINEAR SINGULAR DISCRETE BOUNDARY VALUE PROBLEM

DAQING JIANG, P. Y. H. PANG, AND RAVI P. AGARWAL

Department of Mathematics, Northeast Normal University, Changchun 130024, People's Republic of China (daqingjiang@163.com)

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543 (matpyh@nus.edu.sg)

Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901–6975, USA (agarwal@fit.edu)

ABSTRACT. In this paper, we study the singular discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + g(t, u(t)) = 0, & t \in \{1, 2, \ldots, T\}, \\ u(0) = u(T+1) = 0 \end{cases}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, and the function g is superlinear at infinity and may change sign or be singular at $u = 0$. Existence of solutions is obtained via an upper and lower solutions method.

Keywords and Phrases. Singular discrete boundary value problem, upper and lower solutions.

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1. INTRODUCTION

In this paper we study the existence of positive solutions for the singular discrete boundary value problem

$$(1.1) \qquad \begin{cases} \Delta[\phi(\Delta u(t-1))] + g(t, u(t)) = 0, & t \in Z[1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$; the function $g : Z[1, T] \times R_0^+ \to R$ ($R_0^+ = (0, \infty)$) is continuous in the second variable, is superlinear at infinity and may change sign.

Throughout this paper, for integers a, b with $a < b$, we shall use the notations $Z[a, b] = \{a, a + 1, \ldots, b\}$, $Z[a, b) = \{a, \ldots, b - 1\}$, $Z[a, \infty) = \{a, a + 1, \ldots\}$, etc.

Discrete boundary value problems have been the subject of many investigations. In particular, [1, 4–6, 8–10, 12, 14, 19–21, 27–30, 41–47], among others, have studied problems that are related to that of this paper. Several of these [4, 5, 10, 20, 21, 45] study the existence of positive solutions under the assumption that the nonlinear term is positive. In [5], g is allowed to be singular at $u = 0$ and superlinear at $u = \infty$.

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We note that (1.1) is a discrete model of the p-Laplace equation which occurs in the study of many diffusion phenomena such as non-Newtonian fluid flow and the turbulent flow of gas in porous media. We refer the readers to [2, 3, 11, 13, 15–18, 22, 26, 31–33, 35–40] for results in the continuous case as well as general results on singular boundary value problems. In particular, [2, 3, 33] deal with the superlinear problem; [11, 22] study the case where g is allowed to change sign, and [18] considers the possibility of singularities at $u = 0$, $t = 0$ or $t = 1$. More recently, in [7, 23–25, 34], the case where g may change sign and also be singular at $u = 0$, $t = 0$ or $t = 1$ is studied. Moreover, in [7, 23, 24, 34], $g(t, u)$ is allowed to be superlinear at $u = \infty$. The method of upper and lower solutions is used in these works.

The present work is inspired by [18, 24, 25]. In particular, we shall develop an upper and lower solutions method (Theorem 2.1) by extending that of [32, 38, 39] for the continuous case and [47] for the discrete case. Our main results (Theorems 3.1 and 3.2) extend those of [5, 37] as well as the discrete analogs of [2, 3, 7, 11, 18, 22–25, 37].

2. UPPER AND LOWER SOLUTIONS

Consider the discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + f(t, u(t)) = 0, & t \in Z[1, T], \\ u(0) = A, \quad u(T+1) = B, \end{cases} \tag{2.1}$$

where A and B are given real numbers, and $f(t, x) : Z[1, T] \times R \to R$ is continuous in x.

Definition 2.1. A function $\alpha(t) : Z[0, T+1] \to R$ is said to be a lower solution of (2.1) if

$$\begin{aligned} &\Delta[\phi(\Delta \alpha(t-1))] + f(t, \alpha(t)) \geq 0, \qquad t \in Z[1, T], \\ &\alpha(0) \leq A, \quad \alpha(T+1) \leq B. \end{aligned}$$

The definition of an upper solution β of (2.1) is given similarly by reversing all the above inequalities.

Theorem 2.1. *Let α, β be respectively a lower and an upper solution of* (2.1) *such that $\alpha(t) \leq \beta(t)$ for all $t \in Z[0, T+1]$. Then* (2.1) *has at least one solution $u(t)$ which satisfies*

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \forall\ t \in Z[0, T+1].$$

To prove Theorem 2.1, consider first the modified discrete boundary value problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + f^*(t, u(t)) = 0, & t \in Z[1, T], \\ u(0) = A, \quad u(T+1) = B, \end{cases} \tag{2.2}$$

where

$$f^*(t,x)=\begin{cases} f(t,\alpha(t))+\dfrac{\alpha(t)-x}{1+x^2}, & x<\alpha(t),\\ f(t,x), & \alpha(t)\le x\le\beta(t),\\ f(t,\beta(t))+\dfrac{\beta(t)-x}{1+x^2}, & x>\beta(t).\end{cases}$$

It is readily seen that $f^*(t,x):Z[1,T]\times R\to R$ is continuous in x. Moreover, there exists $H>0$ such that

$$|f^*(t,x)|\le H,\qquad \forall\ (t,x)\in Z[1,T]\times R. \tag{2.3}$$

Equip $E=\{u:Z[0,T+1]\to R\}$ with the norm $\|u\|=\max\{|u(t)|\,:\,t\in Z[0,T+1]\}$. Then E is the Banach space. Define the operator $\Phi:E\to E$ by

$$(\Phi u)(t)=\begin{cases} A, & t=0,\\ B+\displaystyle\sum_{s=t}^{T}\phi^{-1}\left(\tau+\sum_{r=1}^{s}f^*(r,u(r))\right), & t\in Z[1,T],\\ B, & t=T+1,\end{cases}$$

where τ is a solution of the equation

$$w(\tau):=\phi^{-1}(\tau)+\sum_{s=1}^{T}\phi^{-1}\left(\tau+\sum_{r=1}^{s}f^*(r,u(r))\right)=A-B. \tag{2.4}$$

In the next two lemmas, we will show that Φ is well-defined, bounded and continuous.

Lemma 2.1. *For each fixed $u\in E$, (2.4) has a unique solution τ, and $|\tau|\le C$, where C is a positive constant independent of u.*

Proof. Let $u\in E$ be fixed. Then we have, by the definition of w,

$$(T+1)\phi^{-1}(\tau-TH)\le w(\tau)\le(T+1)\phi^{-1}(\tau+TH), \tag{2.5}$$

for all $\tau\in R$, where H is as in (2.3). As ϕ^{-1} is a continuous, strictly increasing function on R with $\phi^{-1}(R)=R$, so is w (for each fixed $u\in E$). Thus, there exists a unique $\tau\in R$ satisfying (2.4). By (2.4) and (2.5), we have

$$\tau\le\phi\left(\frac{A-B}{T+1}\right)+TH,\qquad \tau\ge\phi\left(\frac{A-B}{T+1}\right)-TH,$$

and hence τ is bounded. This establishes Lemma 2.1. □

Lemma 2.2. *$\Phi:E\to E$ is bounded and continuous.*

Proof. Let $u\in E$ be fixed and $\tau\in R$ be the unique solution of (2.4) corresponding to u. Then by Lemma 2.1, we have

$$\|\Phi u\|\le M, \tag{2.6}$$

where M is a positive constant independent of u, showing that Φ is bounded.

Now let $\Big\{u_0, \{u_n\}\Big\} \subset E$ with $u_n \to u_0$. Then, by Lemma 2.1, $|\tau_n| \leq C$, $n = 0, 1, 2, \ldots$, where C is independent of u_n. Suppose that $\tau^* \in [-C, C]$ is an accumulation point of $\{\tau_n\}$. Then there is a subsequence of $\{\tau_n\}$ which converges to τ^*, and

$$\phi^{-1}(\tau^*) + \sum_{s=1}^{T} \phi^{-1}(\tau^* + \sum_{r=1}^{s} f^*(r, u_0(r))) = A - B.$$

It follows from the uniqueness of Lemma 2.1 that $\tau^* = \tau_0$, and hence $\tau_n \to \tau_0$. Thus,

$$\lim_{n\to\infty} (\Phi u_n)(t) = (\Phi u_0)(t)$$

and the proof is complete. □

By virtue of Lemma 2.2, the Brouwer fixed point theorem tells us that Φ has at least one fixed point in E. Let u be a fixed point of Φ. Then it is easy to see that

$$\Delta u(t) = \begin{cases} -\phi^{-1}(\tau), & t = 0, \\ -\phi^{-1}\left(\tau + \displaystyle\sum_{r=1}^{t} f^*(r, u(r))\right), & t \in Z[1, T], \end{cases}$$

and hence $u(t)$ is a solution to (2.2).

Proof of Theorem 2.1. To complete the proof of Theorem 2.1, we only need to show that the above solution $u(t)$ of (2.2) satisfies

$$\alpha(t) \leq u(t) \leq \beta(t)$$

for all $t \in Z[0, T+1]$.

To see that $u(t) \leq \beta(t)$ on $Z[0, T+1]$, let $x(t) = u(t) - \beta(t)$ and suppose that $u(t) > \beta(t)$ for some $t \in Z(0, T+1)$. Since $x(0) \leq 0$, $x(T+1) \leq 0$, there exists a point $t_0 \in Z(0, T+1)$ such that $x(t_0) = \max_{t\in Z[0,T+1]} x(t) > 0$, $\Delta x(t_0 - 1) \geq 0$, $\Delta x(t_0) \leq 0$, and

$$\begin{aligned} \Delta[\phi(\Delta u(t_0 - 1))] &= \phi[\Delta u(t_0)] - \phi[\Delta u(t_0 - 1)] \\ &\leq \phi[\Delta \beta(t_0)] - \phi[\Delta \beta(t_0 - 1)] \\ &= \Delta[\phi(\Delta \beta(t_0 - 1))]. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta[\phi(\Delta u(t_0 - 1))] &= -f^*(t_0, u(t_0)) \\ &= -\left[f(t_0, \beta(t_0)) + \frac{\beta(t_0) - u(t_0)}{1 + u^2(t_0)}\right] \\ &> \Delta[\phi(\Delta \beta(t_0 - 1))], \end{aligned}$$

which is a contradiction.

Similarly, we can prove $u(t) \geq \alpha(t)$ on $Z[0, T+1]$ and the proof of Theorem 2.1 is complete. □

Remark 2.1. It is an immediate corollary of Theorem 2.1 that if $f(t,x) : Z[1,T] \times R \to R$ is bounded and continuous in x, then (2.1) has at least one solution.

3. MAIN RESULTS

Motivated by the example $g(t,u) = \sigma(u^{-a} + u^b + \sin(8t/T))$, where $a > 0,\ b \geq 0,\ \sigma > 0$, we have the following main results for the singular discrete boundary value problem (1.1).

Theorem 3.1. *Assume that there exist constants $L > 0$ and $\varepsilon > 0$ such that*

$$g(t,x) > L, \qquad \forall (t,x) \in Z[1,T] \times (0,\varepsilon], \tag{3.1}$$

and that there exists a function $q : Z[1,T] \to (0,\infty)$ such that

$$|g(t,x)| \leq q(t)(F(x) + Q(x)), \qquad \forall (t,x) \in Z[1,T] \times R_0^+ \tag{3.2}$$

with $F > 0$ continuous and nonincreasing, $Q \geq 0$ continuous, and Q/F nondecreasing. Further, assume that

$$\sup_{c\in(0,\infty)} \frac{1}{\phi^{-1}\left(1 + \frac{Q(c)}{F(c)}\right)} \int_0^c \frac{du}{\phi^{-1}(F(u))} > b_0, \tag{3.3}$$

where

$$b_0 = \max_{t\in Z[1,T]} \max\left\{ \sum_{s=1}^{t} \phi^{-1}\left(\sum_{r=s}^{t} q(r)\right),\ \sum_{s=t}^{T} \phi^{-1}\left(\sum_{r=t}^{s} q(r)\right) \right\}.$$

Then (1.1) has at least one positive solution.

First, in view of (3.3), we may choose $0 < \mu < M$ such that

$$\frac{1}{\phi^{-1}\left(1 + \frac{Q(M)}{F(M)}\right)} \int_\mu^M \frac{du}{\phi^{-1}(F(u))} > b_0. \tag{3.4}$$

For μ above and ε in (3.1), choose a sequence $\{\varepsilon_n\}$ such that $\min\{\varepsilon,\mu\} > \varepsilon_n \downarrow 0$ as $n \to \infty$. Let $\lambda(t) = t(T+1-t)$, $t \in Z[0,T+1]$, and set

$$m = \min\left\{ 4\frac{\varepsilon - \varepsilon_1}{(T+1)^2},\ \left(\frac{L}{|\Delta[\phi(\Delta\lambda)]|_0 + 1}\right)^{1/(p-1)} \right\}, \tag{3.5}$$

where

$$|\Delta[\phi(\Delta\lambda)]|_0 = \max_{t\in Z[1,T]} |\Delta[\phi(\Delta\lambda(t-1))]|.$$

Now, we consider the sequence of boundary value problems

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + g(t,u(t)) = 0, & t \in Z[1,T], \\ u(0) = u(T+1) = \varepsilon_n. \end{cases} \tag{3.6$_n$}$$

It is clear that any solution $u_n(t)$ of $(3.6)_n$ is an upper solution for $(3.6)_{n+1}$.

Lemma 3.1. *The function $\alpha_n(t) = m\lambda(t) + \varepsilon_n$ is a lower solution of $(3.6)_n$.*

Proof. Note that

$$\alpha_n(t) \le m\lambda(t) + \varepsilon_1 \le m(T+1)^2/4 + \varepsilon_1 \le \varepsilon, \qquad \forall\, t \in z[0, T+1],$$

and thus by (3.1)

$$g(t, \alpha_n(t)) > L.$$

It follows that

$$\Delta[\phi(\Delta\alpha_n(t-1))] + g(t, \alpha_n(t)) > L - m^{p-1}(1 + |\Delta[\phi(\Delta\lambda)]|_0) \ge 0,$$

thus establishing the lemma. □

Lemma 3.2. *The problem* $(3.6)_1$ *has at least one solution.*

Proof. Consider the regular (nonsingular) boundary value problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + q(t)F(u(t))\left(1 + \frac{Q(M)}{F(M)}\right) = 0, & t \in Z[1, T], \\ u(0) = u(T+1) = \varepsilon_1. \end{cases} \tag{3.7}$$

It is easy to see $\alpha_0(t) \equiv \varepsilon_1$ is a lower solution of (3.7). By Remark 2.1, one can see that the boundary value problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + q(t)F(\varepsilon_1)\left(1 + \frac{Q(M)}{F(M)}\right) = 0, & t \in Z[1, T], \\ u(0) = u(T+1) = \varepsilon_1, \end{cases} \tag{3.8}$$

has a solution $\beta_0(t)$. Since $\Delta[\phi(\Delta\beta_0(t-1))] \le 0$ on $Z[1, T]$, $\beta_0(t)$ is concave on $Z[0, T+1]$, and hence $\beta_0(t) \ge \varepsilon_1$. Further,

$$\begin{aligned} \Delta[\phi(\Delta\beta_0(t-1))] &= -q(t)F(\varepsilon_1)\left(1 + \frac{Q(M)}{F(M)}\right) \\ &\le -q(t)F(\beta_0(t))\left(1 + \frac{Q(M)}{F(M)}\right), \end{aligned}$$

so that β_0 is an upper solution of (3.7). Thus by Theorem 2.1, (3.7) has a solution $u(t)$ such that $\varepsilon_1 \le u(t) \le \beta_0(t)$.

Since $\Delta[\phi(\Delta u(t-1))] \le 0$, we note that the solution $u(t)$ of (3.7) is concave on $Z[0, T+1]$, and there exists $t_0 \in Z(0, T+1)$ with $u(t_0) = \|u\|$, $\Delta u(t) \ge 0$ on $Z[0, t_0)$ and $\Delta u(t) \le 0$ on $Z[t_0, T+1)$.

For $0 \le s < t_0$, sum (3.7) from s to t_0 to obtain

$$\phi[\Delta u(t_0)] = \phi[\Delta u(s)] - \left(1 + \frac{Q(M)}{F(M)}\right) \sum_{r=s}^{t_0-1} F(u(r+1))q(r+1).$$

Since $\Delta u(t_0) \le 0$, we have

$$\begin{aligned}\phi[\Delta u(s)] &= \phi[\Delta u(t_0)] + \left(1 + \frac{Q(M)}{F(M)}\right) \sum_{r=s}^{t_0-1} F(u(r+1))q(r+1)\\ &\le F(u(s+1))\left(1 + \frac{Q(M)}{F(M)}\right) \sum_{r=s+1}^{t_0} q(r).\end{aligned}$$

It follows that

$$\frac{\Delta u(s)}{\phi^{-1}(F(u(s+1)))} \le \phi^{-1}\left(1 + \frac{Q(M)}{F(M)}\right)\phi^{-1}\left(\sum_{r=s+1}^{t_0} q(r)\right). \tag{3.9}$$

Since $F(u(s+1)) \le F(u) \le F(u(s))$ as $u(s) \le u \le u(s+1)$, we have

$$\int_{u(s)}^{u(s+1)} \frac{du}{\phi^{-1}(F(u))} \le \frac{\Delta u(s)}{\phi^{-1}(F(u(s+1)))}. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\int_{u(s)}^{u(s+1)} \frac{du}{\phi^{-1}(F(u))} \le \phi^{-1}\left(1 + \frac{Q(M)}{F(M)}\right)\phi^{-1}\left(\sum_{r=s+1}^{t_0} q(r)\right).$$

Summing from 0 to $t_0 - 1$, we obtain

$$\begin{aligned}\int_{\varepsilon_1}^{u(t_0)} \frac{du}{\phi^{-1}(F(u))} &\le \phi^{-1}\left(1 + \tfrac{Q(M)}{F(M)}\right)\sum_{s=0}^{t_0-1} \phi^{-1}\left(\sum_{r=s+1}^{t_0} q(r)\right)\\ &= \phi^{-1}\left(1 + \tfrac{Q(M)}{F(M)}\right)\sum_{s=1}^{t_0} \phi^{-1}\left(\sum_{r=s}^{t_0} q(r)\right).\end{aligned} \tag{3.11}$$

Similarly, for $s \ge t_0$,

$$\phi[\Delta u(s)] = \phi[\Delta u(t_0 - 1)] - \left(1 + \frac{Q(M)}{F(M)}\right) \sum_{r=t_0-1}^{s-1} F(u(r+1))q(r+1),$$

and, making use of $\Delta u(t_0 - 1) \ge 0$, we have

$$\int_{\varepsilon_1}^{u(t_0)} \frac{du}{\phi^{-1}(F(u))} \le \phi^{-1}\left(1 + \frac{Q(M)}{F(M)}\right)\sum_{s=t_0}^{T} \phi^{-1}\left(\sum_{r=t_0}^{s} q(r)\right). \tag{3.12}$$

Now (3.11) and (3.12) imply that

$$\int_{\varepsilon_1}^{u(t_0)} \frac{du}{\phi^{-1}(F(u))} \le b_0\phi^{-1}\left(1 + \frac{Q(M)}{F(M)}\right).$$

Together with (3.4), this implies $u(t_0) = \|u\| \le M$.

Finally, by (3.2),

$$\begin{aligned}\Delta[\phi(\Delta u(t-1))] + g(t, u(t)) &\le -q(t)F(u(t))\left(1 + \tfrac{Q(M)}{F(M)}\right) + |g(t, u(t))|\\ &\le q(t)F(u(t))\left(\tfrac{Q(u(t))}{F(u(t))} - \tfrac{Q(M)}{F(M)}\right)\\ &\le 0,\end{aligned}$$

so that $\beta = u$ is an upper solution of $(3.6)_1$. Together with the lower solution $\alpha \equiv \varepsilon_1$, we conclude by Theorem 2.1 that there is a solution $u_1(t)$ of $(3.6)_1$ such that

$$\varepsilon_1 = \alpha(t) \leq u_1(t) \leq \beta, \qquad \forall\, t \in Z[0, T+1].$$

The proof of Lemma 3.2 is complete. □

Proof of Theorem 3.1. By Lemmas 3.1 and 3.2, and also the fact that any solution $u_n(t)$ of $(3.6)_n$ is an upper solution of $(3.6)_{n+1}$, we obtain, by Theorem 2.1, a sequence of solutions $\{u_n(t)\}$ to $(3.6)_n$ such that $m\lambda(t) + \varepsilon_n = \alpha_n(t) \leq u_n(t) \leq u_{n-1}(t)$ and $u_n(0) = u_n(T+1) = \varepsilon_n$.

Consider now the pointwise limit $z(t) \lim_{n\to\infty} u_n(t)$. As $|g(t,x)|$ is bounded for $t \in Z[1,T]$ and $m\lambda(t) \leq x \leq u_1(t)$, it is easy to see by the Arzela-Ascoli theorem that $z(t)$ is a positive solution of (1.1). This completes the proof of Theorem 3.1. □

By similar arguments as above and [18, Theorem 2], we can prove:

Theorem 3.2. *Suppose* (3.1) *holds for some positive constants* L *and* ε *and that for any* $r > 0$ *there exists a function* $h_r(t) : Z[1,T] \to R^+$ *such that*

$$|g(t,x)| \leq h_r(t), \qquad \forall\, t \in Z[1,T],\; x \geq r.$$

Then (1.1) *has at least one positive solution. If, moreover,* $g(t,x)$ *is strictly decreasing in* x*, then the solution is unique.*

Proof. We only give the proof of the uniqueness. Let $u_1(t)$ and $u_2(t)$ be two solutions, and write $x(t) = u_1(t) - u_2(t)$. Suppose that $|x(t)| > 0$ for some $t \in Z(0, T+1)$. Without loss of generality, we may assume that there exists a point $t_0 \in Z(0, T+1)$ such that $x(t_0) = \max_{t\in Z[0,T+1]} x(t) > 0$. Then, $\Delta x(t_0 - 1) \geq 0$, $\Delta x(t_0) \leq 0$, and

$$\begin{aligned}\Delta[\phi(\Delta u_1(t_0-1))] &= \phi[\Delta u_1(t_0)] - \phi[\Delta u_1(t_0-1)] \\ &\leq \phi[\Delta u_2(t_0)] - \phi[\Delta u_2(t_0-1)] \\ &= \Delta[\phi(\Delta u_2(t_0-1))].\end{aligned}$$

This implies that

$$\begin{aligned}\Delta[\phi(\Delta u_1(t_0-1))] &= -f(t_0, u_1(t_0)) \\ &> -f(t_0, u_2(t_0)) = \Delta[\phi(\Delta u_2(t_0-1))],\end{aligned}$$

which is a contradiction. □

The following corollary of Theorem 3.2 extends the result of [37] as well as the discrete analogs of those of [18, 25].

Corollary 3.1. *Let* $A(t) : Z[1,T] \to R_0^+$, $B(t) : Z[1,T] \to R$ *and* $f : R_0^+ \to R_0^+$ *be given continuous function such that* f *is strictly decreasing and* $\lim_{u\to 0^+} f(u) = \infty$.

Then the problem

$$\begin{cases} \Delta[\phi(\Delta u(t-1))] + A(t)f(u(t)) = B(t), \qquad t \in Z[1,T], \\ u(0) = u(T+1) = 0 \end{cases}$$

has a unique positive solution.

Finally, we give an example. Consider the singular discrete boundary value problem (1.1) with $p = 2$ and $g(t,u) = \sigma q(t)(u^{-a} + u^b + \sin(8t/T))$, where $a > 0$, $b \geq 0$ and $\sigma > 0$ are given constants. Using $F(u) = \sigma u^{-a}$ and $Q(u) = \sigma(u^b + 1)$, we can see from Theorem 3.1 that this problem has at least one positive solutions if

$$\sigma < \sup_{x \in (0,\infty)} \frac{x^{a+1}}{b_0(a+1)(1 + x^a + x^{a+b})}.$$

In particular, if $0 \leq b < 1$, then the problem has at least one positive solution for all $\sigma > 0$.

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