Dynamic Systems and Applications 16 (2007) 755-770

POSITIVE SOLUTIONS OF SINGULAR BOUNDARY VALUE PROBLEMS FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Existence criteria are established for positive solutions of singular boundary value problems for nonlinear second order ordinary and delay differential equations. Here the nonlinearities may be singular at phase variables and positive solutions 'pass through' the singularities.
Keywords and Phrases. Singular boundary value problem, existence, positive solution, Vitali's convergence theorem.

AMS Subject Classification. 34K10, 34B16, 34B18.

1. INTRODUCTION

Let a, b, c and T be positive numbers, $b < a$, $c < a$ and $J = [0, T]$. Let $\alpha_i \in C^1(J)$, $i = 1, 2$, satisfy the following conditions:

(j) $\alpha_i(t) < t$ and $\alpha_i'(t) \geq \Delta_i > 0$ for $t \in J$,

(jj) there exist $\tau_i \in (0, T)$ such that $\alpha_i(\tau_i) = 0$.

Set $r = \min\{\alpha_1(0), \alpha_2(0)\}$ (< 0). Let $\varphi \in C^0([r, 0])$, $\varphi(0) = a$, $0 < \varphi(t) < a$ for $t \in [r, 0)$ and $\varphi(\alpha_2(t)) > c$ for $t \in [0, \tau_2]$. Consider the singular boundary value problem (BVP)

$$x''(t) = \mu q(t)\Big(f_1(t, x(t), x(\alpha_1(t))) + f_2(t, x(t), x(\alpha_2(t)))\Big), \tag{1}$$

$$x(t) = \varphi(t) \text{ for } t \in [r, 0], \quad x(T) = 0, \tag{2}$$

where $\mu \geq 0$ is a constant, $q(t) > 0$ for $t \in (0, T)$, $f_1(t, x, y) \geq 0$ for $(t, x, y) \in J \times (0, b) \cup (b, a) \times (0, a)$ and $f_2(t, x, y) \geq 0$ for $(t, x, y) \in J \times (0, a) \times [0, c) \cup (c, a)$. The function f_1 (resp. f_2) may be singular at the points $x = 0$, $x = b$ and $x = a$ (resp. $x = 0$ and $x = a$) of the phase variable x and at the points $y = 0$ and $y = a$ (resp. $y = c$ and $y = a$) of the phase variable y.

Set

$$\mathcal{E} = \{x : x \in C^0([r, T]) \cap C^1(J),\ x \text{ is decreasing on } J\}.$$

Staněk is supported by grant no. 201/01/1451 of the Grant Agency of Czech Republic and by the Council of Czech Government J14/98:153100011.

Received September 19, 2005 1056-2176 $15.00

We say that x is *a solution of BVP* (1), (2) *in the set* $\mathcal{E}$ if $x \in \mathcal{E} \cap C^2(J_1)$, where J_1 denotes the interval $(0,T)$ with the exception of at most two point if f_1 is singular at the point $x = b$ of the phase variable x and f_2 is singular at the point $y = c$ of the phase variable y, x fulfills the boundary conditions (2) and (1) is satisfied on J_1.

If (1) is independent of the delays α_1 and α_2, we obtain a singular BVP of the type

$$x''(t) = \mu q(t) f_*(t, x(t)), \tag{3}$$

$$x(0) = a, \ x(T) = 0 \tag{4}$$

where $f_* \geq 0$ may be singular at $x = 0$, $x = b$ and $x = a$ of the phase variable x. This problem has been considered for instance in [3], [6] and [22] but here f_* may be singular only at $x = 0$.

Many existence results have been established for boundary value problems with second–order functional differential equations with delay which have no singularities in the phase variables; see, e.g., [2–5,8,11,13,14,18–21] and their references. Boundary value problems for second order functional differential equations with singularities in the phase variable having positive solutions have been considered in [1], [7] and [16]. For example in [1] is discussed BVP

$$x''(t) + q(t)\tilde{f}(t, x(t-r)) = 0, \quad t \in (0,1) \setminus \{r\},$$

$$x(t) = \mu(t) \text{ for } t \in [-r, 0], \quad x(1) = 0$$

where $0 < r < 1$ is a constant, $\mu(0) = 0$, $\mu > 0$ on $[-r, 0)$ and $\tilde{f} \geq 0$ on $(0,1) \times (0,\infty)$ may by singular at $x = 0$ of the phase variable x. Here the singularity of $\tilde{f}$ at $x = 0$ 'appears' in positive solutions only at the fixed point $t = 0$ and $t = 1$ where solutions vanishing. In our paper the singularities of f_1 and f_2 'appear' in positive solutions of BVP (1), (2) not only at the fixed point $t = 0$ and $t = T$, but positive solutions 'pass through' the singularities of f_1 and f_2 in inner points of $(0,T)$ (if f_1 is singular at $x = b$ and f_2 is singular at $y = c$).

The aim of this paper is to give conditions for the existence of a solution of BVP (1), (2) in the set $\mathcal{E}$. Our results are proved by the regularity and sequential techniques. First, we construct 2–parameter family of auxiliary regular BVPs $(8)_{n\lambda}$, $(9)_n$ depending on parameters $(\lambda, n) \in [0,1] \times \mathbb{N}$ and obtain *a priori* bounds for their solutions (Lemma 1). Then applying the topological transversality theorem (see, e.g., [9], [10]), we prove the existence of positive solutions of the auxiliary regular BVPs (Lemma 2). The main result for the original BVP (1), (2) (Theorem 1) follows from Arzelà–Ascoli's theorem and a modification of Vitali's convergence theorem (see, e.g., [12], [17]) given in Lemma 3. Finally, Corollary 1 generalized results obtained in [3], [6] and [22].

Throughout the paper we will use the following assumptions:

(H_1) $q \in C^0((0,T))$, $q > 0$ on $(0,T)$ and $Q = \sup\{q(t) : t \in J\} < \infty$;

(H_2) $f_1 : J \times D_1 \times (0,a) \to [0,\infty)$ is continuous with $D_1 = (0,b) \cup (b,a)$, $f_1(t,x,y) \le g_1(x)+h_1(y)$, $(t,x,y) \in J \times D_1 \times (0,a)$, where $g_1 \ge 0$ is continuous on D_1, $h_1 \ge 0$ is continuous on $(0,a)$ and $\int_0^a (g_1(s)+h_1(s))\,ds < \infty$, $\int_0^{\tau_1} h_1(\varphi(\alpha_1(t)))\,dt < \infty$;

(H_3) $f_2 : J \times (0,a) \times D_2 \to [0,\infty)$ is continuous with $D_2 = [0,c) \cup (c,a)$, $f_2(t,x,y) \le g_2(x)+h_2(y)$, $(t,x,y) \in J \times (0,a) \times D_2$, where $g_2 \ge 0$ is continuous on $(0,a)$, $h_2 \ge 0$ is continuous on D_2 and $\int_0^a (g_2(s)+h_2(s))\,ds < \infty$, $\int_0^{\tau_2} h_2(\varphi(\alpha_2(t)))\,dt < \infty$;

(H_4) h_1 and h_2 are nondecreasing on $(a-\varepsilon_0, a)$ for an $\varepsilon_0 > 0$;

(H_5) $\lim_{v\to 0} v g_1(b+v) = 0$, $\lim_{v \to 0} v h_2(c+v) = 0$.

2. NOTATION AND LEMMAS

Let assumptions (H_1)–(H_5) be satisfied. Let $n_* \in \mathbb{N}$,

$$n_* > 2\max\left\{\frac{1}{b}, \frac{1}{c}, \frac{1}{a-b}, \frac{1}{a-c}, \frac{1}{2\varepsilon_0}\right\}$$

and $\mathbb{N}_* = \{n : n \in \mathbb{N}, n \ge n_*\}$. By Urysohn's lemma, for each $n \in \mathbb{N}_*$, there exists $p_n \in C^0(J \times \mathbb{R})$ such that $0 \le p_n(t,x) \le 1$ for $t \in J \times \mathbb{R}$, $p_n(t,x) = 0$ for $(t,x) \in J \times (-\infty, 1/(2n)]$ and $p_n(t,x) = 1$ for $(t,x) \in J \times [1/n, \infty)$. Let $\varphi_n \in C^0([r,0])$, $\xi_n \in C^0(\mathbb{R})$, f_{1n}, $\tilde{f}_{2n} \in C^0(J \times \mathbb{R}^2)$, $g_{1n} \in C^0((0,a))$, $h_{2n} \in C^0([0,a))$ and l_{1n}, $l_{2n} \in \mathbb{R}$ be defined for $n \in \mathbb{N}_*$ by the formulas

$$\varphi_n(t) = \begin{cases} a-(1/n) & \text{if } \varphi(t) \ge a-(1/n) \\ \varphi(t) & \text{if } \varphi(t) < a-(1/n), \end{cases}$$

$$\xi_n(x) = \begin{cases} a-(1/n) & \text{for } x \ge a-(1/n) \\ x & \text{for } 1/(2n) \le x < a-(1/n) \\ 1/(2n) & \text{for } x < 1/(2n), \end{cases}$$

$$f_{1n}(t,x,y) = \begin{cases} f_1(t, a-(1/n), \xi_n(y)) & \text{for } (t,x,y) \in J \times [a-(1/n), \infty) \times \mathbb{R} \\ f_1(t,x,\xi_n(y)) & \text{for } (t,x,y) \in J \times (b+(1/n), a-(1/n)) \times \mathbb{R} \\ (n/2)\Big(f_1(t, b+(1/n), \xi_n(y))(x-b+(1/n)) & \\ \qquad - f_1(t, b-(1/n), \xi_n(y))(x-b-(1/n))\Big) & \\ & \text{for } (t,x,y) \in J \times [b-(1/n), b+(1/n)] \times \mathbb{R} \\ p_n(t,x) f_1(t,x,\xi_n(y)) & \text{for } (t,x,y) \in J \times [1/(2n), b-(1/n)) \times \mathbb{R} \\ x-(1/2n) & \text{for } (t,x,y) \in J \times (-\infty, 1/(2n)) \times \mathbb{R}, \end{cases}$$

$$f_{2n}(t,x,y)=\begin{cases} f_2(t,\xi_n(x),a-(1/n)) & \text{for } (t,x,y)\in J\times\mathbb{R}\times[a-(1/n),\infty)\\ f_2(t,\xi_n(x),y) & \text{for } (t,x,y)\in J\times\mathbb{R}\times(c+(1/n),a-(1/n))\\ (n/2)\Big(f_2(t,\xi_n(x),c+(1/n))(y-c+(1/n)) & \\ \qquad -f_2(t,\xi_n(x),c-(1/n))(y-c-(1/n))\Big) & \\ \qquad\qquad \text{for } (t,x,y)\in J\times\mathbb{R}\times[c-(1/n),c+(1/n)] & \\ p_n(t,x)p_n(t,y)f_2(t,\xi_n(x),y) & \text{for } (t,x,y)\in J\times\mathbb{R}\times(-\infty,c-(1/n)), \end{cases}$$

$$g_{1n}(x)=\begin{cases} g_1(x) \quad \text{for } x\in(0,b-(1/n))\cup(b+(1/n),a)\\ (n/2)\Big(g_1(b+(1/n))(x-b+(1/n))+g_1(b-(1/n))(b+(1/n)-x)\Big)\\ \qquad \text{for } \ x\in[b-(1/n),b+(1/n)], \end{cases}$$

$$h_{2n}(y)=\begin{cases} h_2(y) \quad \text{for } y\in(0,c-(1/n))\cup(c+(1/n),a)\\ (n/2)\Big(h_2(c+(1/n))(y-c+(1/n))+h_2(c-(1/n))(c+(1/n)-y)\Big)\\ \qquad \text{for } \ y\in[c-(1/n),c+(1/n)], \end{cases}$$

$$l_{1n}=\frac{1}{n}\left(g_1\left(b-\frac{1}{n}\right)+g_1\left(b+\frac{1}{n}\right)\right),$$
$$l_{2n}=\frac{1}{n}\left(h_2\left(c-\frac{1}{n}\right)+h_2\left(c+\frac{1}{n}\right)\right).$$

By (H_2), (H_3) and (H_5), $\lim_{n\to\infty} l_{1n}=\lim_{n\to\infty} l_{2n}=0$,

$$f_{1n}(t,x,y)\le g_{1n}(x)+h_1(y),\quad f_{2n}(t,x,y)\le g_2(x)+h_{2n}(y) \tag{5}$$

for $(t,x,y)\in J\times[1/(2n),a-(1/n)]\times[1/(2n),a-(1/n)]$. In addition,

$$\begin{aligned} &\int_u^v g_{1n}(s)\,ds\le\int_u^v g_1(s)\,ds+nl_{1n}\min\left\{\frac{2}{n},v-u\right\},\\ &\int_u^v h_{2n}(s)\,ds\le\int_u^v h_2(s)\,ds+nl_{2n}\min\left\{\frac{2}{n},v-u\right\} \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\int_{\mathcal{S}} g_{1n}(s)\,ds\le\int_{\mathcal{S}} g_1(s)\,ds+nl_{1n}\min\left\{\frac{2}{n},m(\mathcal{S})\right\},\\ &\int_{\mathcal{S}} h_{2n}(s)\,ds\le\int_{\mathcal{S}} h_2(s)\,ds+nl_{2n}\min\left\{\frac{2}{n},m(\mathcal{S})\right\} \end{aligned} \tag{7}$$

for $n\in\mathbb{N}_*$, $0\le u\le v\le a$ and any measurable $\mathcal{S}\subset[0,a]$, where $m(\mathcal{S})$ stands for the Lebesgue measure of $\mathcal{S}$.

Consider the family of the auxiliary regular BVPs

$$x''(t)=\lambda\mu q(t)\Big(f_{1n}(t,x(t),x(\alpha_1(t)))+f_{2n}(t,x(t),x(\alpha_2(t)))\Big), \tag*{$(8)_{n\lambda}$}$$

$$x(t)=\varphi_n(t)\quad \text{for } t\in[r,0],\quad x(T)=\frac{1}{2n} \tag*{$(9)_n$}$$

depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}_*$.

We say that $x \in C^0([r,T]) \cap C^1(J) \cap C^2((0,T))$ is a *solution of BVP* $(8)_{n\lambda}$, $(9)_n$ if x fulfills the boundary conditions $(9)_n$ and $(8)_{n\lambda}$ is satisfied for $t \in (0,T)$.

Set

$$A_1 = \mu Q \left(\int_0^{\tau_1} h_1(\varphi(\alpha_1(t)))\, dt + \int_0^{\tau_2} h_2(\varphi(\alpha_2(t)))\, dt \right), \tag{10}$$

$$B_1 = 2\mu Q \Bigg(\int_0^a (g_1(s)+g_2(s))ds + \frac{1}{\Delta_1}\int_0^a h_1(s)ds + \frac{1}{\Delta_2}\int_0^a h_2(s)ds + 2\sup\left\{ l_{1n} + \frac{l_{2n}}{\Delta_2} : n \in \mathbb{N}_* \right\} \Bigg) \tag{11}$$

and

$$K = A_1 + \sqrt{A_1^2 + B_1 + \left(\frac{a}{T}\right)^2}. \tag{12}$$

Lemma 1. *Let assumptions* (H_1)–(H_5) *be satisfied,* $(\lambda, n) \in [0,1] \times \mathbb{N}_*$. *Let* x *be a solution of BVP* $(8)_{n\lambda}$, $(9)_n$. *Then*

$$\frac{1}{2n} \le x(t) \le a - \frac{1}{n}, \tag{13}$$

$$0 \le -x'(t) \le K \tag{14}$$

for $t \in J$, *where* K *is defined by* (12).

Proof. If $\lambda\mu = 0$ then

$$x(t) = \begin{cases} \varphi_n(t) & \text{for } t \in [r,0) \\ \dfrac{3-2na}{2nT}t + a - \dfrac{1}{n} & \text{for } t \in J \end{cases}$$

and x satisfies (13) and (14). Let $\lambda\mu > 0$. Since $x'' \ge 0$ on $(0,T)$, x' is nondecreasing on J and $x(0) = a - (1/n) > 0$, $x(T) = 1/(2n) < x(0)$ imply $x'(0) < 0$. Hence $x(t) \le a - (1/n)$ for $t \in J$. If $\min\{x(t) : t \in J\} = x(t_0) < 1/(2n)$ then $t_0 \in (0,T)$, $x'(t_0) = 0$ and

$$\begin{aligned} x''(t_0) &= \lambda\mu q(t_0)\Big(f_{1n}(t_0, x(t_0), x(\alpha_1(t_0))) + f_{2n}(t_0, x(t_0), x(\alpha_2(t_0)))\Big) \\ &= \lambda\mu q(t_0)\left(x(t_0) - \frac{1}{2n}\right) < 0, \end{aligned}$$

which is impossible. Hence $x(t) \ge 1/(2n)$ for $t \in J$. In addition, $x' \le 0$ on J since from $x'(\nu) > 0$ for some $\nu \in (0,T)$ we have $x' > 0$ on $[\nu, T]$ which implies $x(T) > 1/(2n)$, a contradiction. We have proved that x satisfies inequalities (13) and $x' \le 0$ on J.

We are going to show that $-x'(t) \le K$ for $t \in J$. Since $x'' \ge 0$ on $(0,T)$ and $x' \le 0$ on J, we have $x'(0) \le x'(t) \le x'(T)$ for $t \in J$ and from $x(0) = a - (1/n)$,

$x(T) = 1/(2n)$ it follows that $x'(T) \geq (1/T)(3/(2n) - a) > -a/T$. By (5), (13) and (H_3),

$$\begin{aligned} x''(t) &= \lambda\mu q(t)\Big(f_{1n}(t, x(t), x(\alpha_1(t))) + f_{2n}(t, x(t), x(\alpha_2(t)))\Big) \\ &\leq \mu Q\Big(g_{1n}(x(t)) + h_1(x(\alpha_1(t))) + g_2(x(t)) + h_{2n}(x(\alpha_2(t)))\Big) \end{aligned}$$

for $t \in (0, T)$. Whence

$$(15) \quad 2x''(t)x'(t) \geq 2\mu Q\Big(g_{1n}(x(t)) + h_1(x(\alpha_1(t))) + g_2(x(t)) + h_{2n}(x(\alpha_2(t)))\Big)x'(t)$$

for $t \in (0, T)$. By condition (j), $1/\alpha_i'(t) \leq 1/\Delta_i,\ t \in J,\ i = 1, 2$ and from (j), (jj) and (H_4) we obtain $x'(\alpha_i(t)) \leq x'(t),\ t \in [\tau_i, T]$, and $h_i(\varphi_n(\alpha_i(t))) \leq h_i(\varphi(\alpha_i(t))),\ t \in [0, \tau_i]$ for $i = 1$, 2 and $n \in \mathbb{N}_*$. Consequently (cf. (6)),

$$\begin{aligned} \int_0^T g_{1n}(x(t))x'(t)\,dt &= \int_{x(0)}^{x(T)} g_{1n}(s)\,ds = \int_{a-(1/n)}^{1/(2n)} g_{1n}(s)\,ds \\ &\geq -\int_0^a g_{1n}(s)\,ds \geq -\int_0^a g_1(s)\,ds - 2l_{1n}, \end{aligned}$$

$$\begin{aligned} \int_0^T h_1(x(\alpha_1(t)))x'(t)\,dt &= \int_0^{\tau_1} h_1(x(\alpha_1(t)))x'(t)\,dt + \int_{\tau_1}^T h_1(x(\alpha_1(t)))x'(t)\,dt \\ &\geq x'(0)\int_0^{\tau_1} h_1(\varphi_n(\alpha_1(t)))\,dt + \int_{\tau_1}^T \frac{h_1(x(\alpha_1(t)))(x(\alpha_1(t)))'}{\alpha_1'(t)}\,dt \\ &\geq x'(0)\int_0^{\tau_1} h_1(\varphi(\alpha_1(t)))\,dt + \frac{1}{\Delta_1}\int_{x(0)}^{x(\alpha_1(T))} h_1(s)\,ds \\ &\geq x'(0)\int_0^{\tau_1} h_1(\varphi(\alpha_1(t)))\,dt - \frac{1}{\Delta_1}\int_0^a h_1(s)\,ds, \end{aligned}$$

$$\int_0^T g_2(x(t))x'(t)\,dt = \int_{x(0)}^{x(T)} g_2(s)\,ds \geq -\int_0^a g_2(s)\,ds$$

and

$$\begin{aligned} \int_0^T h_{2n}(x(\alpha_2(t)))x'(t)\,dt &= \int_0^{\tau_2} h_{2n}(x(\alpha_2(t)))x'(t)\,dt + \int_{\tau_2}^T h_{2n}(x(\alpha_2(t)))x'(t)\,dt \\ &\geq x'(0)\int_0^{\tau_2} h_2(\varphi_n(\alpha_2(t)))\,dt + \int_{\tau_2}^T \frac{h_{2n}(x(\alpha_2(t)))(x(\alpha_2(t)))'}{\alpha_2'(t)}\,dt \\ &\geq x'(0)\int_0^{\tau_2} h_2(\varphi(\alpha_2(t)))\,dt + \frac{1}{\Delta_2}\int_{x(0)}^{x(\alpha_2(T))} h_{2n}(s)\,ds \\ &\geq x'(0)\int_0^{\tau_2} h_2(\varphi(\alpha_2(t)))\,dt - \frac{1}{\Delta_2}\Big(\int_0^a h_2(s)\,ds + 2l_{2n}\Big). \end{aligned}$$

Integrating (15) from 0 to T and using the above inequalities, we see that

$$(x'(T))^2 - (x'(0))^2 \geq 2A_1x'(0) - B_1$$

where A_1 and B_1 are defined by (10) and (11) respectively. Then

$$-x'(0) \le A_1 + \sqrt{A_1^2 + B_1 + (x'(T))^2} \le A_1 + \sqrt{A_1^2 + B_1 + \left(\frac{a}{T}\right)^2} = K$$

since $(x'(T))^2 \le (a/T)^2$. □

Lemma 2. *Let assumptions* (H_1)–(H_5) *be satisfied and* $n \in I\!N_*$. *Then there exists a solution* x *of BVP* $(8)_{n1}$, $(9)_n$ *satisfying inequalities* (13) *and* (14) *for* $t \in J$, *where* K *is defined by* (12).

Proof. Set

$$\mathcal{U} = \{x : x \in C^0([r,T]),\ x(t) = \varphi_n(t) \text{ for } t \in [r,0]\}$$

and

$$\mathcal{K} = \{x : x \in \mathcal{U},\ 0 < x(t) < a \text{ for } t \in J\}.$$

Then $\mathcal{U}$ is a convex subset of the Banach space $C^0([r,T])$ equipped with the sup–norm and $\mathcal{K}$ is open in $\mathcal{U}$. Let $\overline{\mathcal{K}}$ and $\partial\mathcal{K}$ denote, respectively, the closure and the boundary of $\mathcal{K}$ in $\mathcal{U}$. Define the operator

$$\Lambda : [0,1] \times \overline{\mathcal{K}} \to C^0([r,T])$$

by

$$\Lambda(\lambda, x)(t) = \varphi_n(t) \quad \text{for } t \in [r,0),$$

$$\begin{aligned}\Lambda(\lambda, x)(t) = a &- \frac{1}{n} + \frac{3-2na}{2nT}t \\ &- \frac{\lambda\mu t}{T}\int_0^T (T-s)q(s)\Big(f_{1n}(s,x(s),x(\alpha_1(s))) + f_{2n}(s,x(s),x(\alpha_2(s)))\Big)ds \\ &+ \lambda\mu\int_0^t (t-s)q(s)\Big(f_{1n}(s,x(s),x(\alpha_1(s))) + f_{2n}(s,x(s),x(\alpha_2(s)))\Big)\,ds\end{aligned}$$

for $t \in [0,T]$. Obviously, Λ is a compact operator. Suppose that $\Lambda(\lambda_0, x_0) = x_0$ for some $(\lambda_0, x_0) \in [0,1] \times \partial\mathcal{K}$. Then x_0 is a solution of BVP $(8)_{n\lambda_0}$, $(9)_n$, and so $1/(2n) \le x_0(t) \le a - (1/n)$ for $t \in J$ by Lemma 1. Hence $x_0 \notin \partial\mathcal{K}$, a contradiction. Therefore $\Lambda(\lambda, x) \ne x$ for $(\lambda, x) \in [0,1] \times \partial\mathcal{K}$. Since for $x \in \overline{\mathcal{K}}$ we have

$$\Lambda(0,x)(t) = \begin{cases} \varphi_n(t) & \text{for } t \in [r,0) \\ a - \dfrac{1}{n} + \dfrac{3-2na}{2nT}t & \text{for } t \in J, \end{cases}$$

$\Lambda(0,\cdot)$ is a constant operator and $\Lambda(0,\cdot) \in \mathcal{K}$. By the topological transversality theorem, there exists a fixed point x of the operator $\Lambda(1,\cdot)$. Clearly, x is a solution of BVP $(8)_{n1}$, $(9)_n$ and, by Lemma 1, (13) and (14) hold. □

Lemma 3. *Let $\{w_n(t)\} \subset L_1([\alpha,\beta])$ be a sequence of nonnegative functions on $[\alpha,\beta]$ converging in measure on $[\alpha,\beta]$ to $w(t)$. Let $\{p_n(t)\} \subset L_1([\alpha,\beta])$ and $w_n(t) \le p_n(t)$ for a.e. $t \in [\alpha,\beta]$ and each $n \in \mathbb{N}$. Suppose that for every $\varepsilon > 0$, there is a $\delta > 0$ such that for each at most countable set pairwise disjoint intervals $(u_i, v_i) \subset [\alpha,\beta]$, $i \in I$ with $\sum_{i\in I}(v_i - u_i) < \delta$ we have $\sum_{i\in I}\int_{u_i}^{v_i} p_n(s)\,ds < \varepsilon$ for $n \in \mathbb{N}$. Then $w \in L_1([\alpha,\beta])$ and*

$$\lim_{n\to\infty}\int_\alpha^\beta w_n(t)\,dt = \int_\alpha^\beta w(t)\,dt. \tag{16}$$

Proof. To prove the assertion of the lemma we use Vitali's convergence theorem. Fix $\varepsilon > 0$ and let $\delta > 0$ be from the assumption. Let $\mathcal{M} \subset [\alpha,\beta]$ be a measurable set, $m(\mathcal{M}) < \delta/2$. Then there exists an open set $\mathcal{M}_1 \subset [\alpha,\beta]$, $\mathcal{M} \cap (\alpha,\beta) \subset \mathcal{M}_1$ such that $m(\mathcal{M}_1) < \delta$. As $\mathcal{M}_1$ is open bounded, $\mathcal{M}_1$ is a union of at most countable set of intervals (α_i,β_i), $i \in I_*$ without common elements, $\mathcal{M}_1 = \bigcup_{i\in I_*}(\alpha_i,\beta_i)$. Then $\sum_{i\in I_*}\int_{\alpha_i}^{\beta_i} p_n(t)\,dt < \varepsilon$ and

$$\int_{\mathcal{M}} w_n(t)\,dt \le \int_{\mathcal{M}} p_n(t)\,dt \le \int_{\mathcal{M}_1} p_n(t)\,dt = \sum_{i\in I_*}\int_{\alpha_i}^{\beta_i} p_n(t)\,dt < \varepsilon$$

for $n \in \mathbb{N}$. We have proved that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 \le \int_{\mathcal{M}} w_n(t)\,dt < \varepsilon$ for each $n \in \mathbb{N}$ and any measurable $\mathcal{M} \subset [\alpha,\beta]$, $m(\mathcal{M}) < \delta/2$. Hence $\{w_n(t)\}$ has uniformly absolutely continuous integrals on $[\alpha,\beta]$ and, by Vitali's convergence theorem, $w \in L_1([\alpha,\beta])$ and (16) holds. □

3. EXISTENCE RESULTS AND AN EXAMPLE

Theorem 1. *Let assumptions (H_1)–(H_5) be satisfied. Set*

$$\mu_T = \frac{a^2}{(2aA + BT)T} \tag{17}$$

where

$$A = Q\left(\int_0^{\tau_1} h_1(\varphi(\alpha_1(t)))\,dt + \int_0^{\tau_2} h_2(\varphi(\alpha_2(t)))\,dt\right) \tag{18}$$

and

$$B = 2Q\left(\int_0^a (g_1(s) + g_2(s))\,ds + \frac{1}{\Delta_1}\int_0^a h_1(s)\,ds + \frac{1}{\Delta_2}\int_0^a h_2(s)\,ds\right). \tag{19}$$

If $0 \le \mu \le \mu_T$ then BVP (1), (2) has a solution in the set $\mathcal{E}$.

Proof. Fix $\mu \in [0,\mu_T]$. By Lemma 2, for each $n \in \mathbb{N}_*$, there exists a solution x_n of BVP $(8)_{n1}$, $(9)_n$ such that $x_n(t) = \varphi_n(t)$ for $t \in [r,0]$ and

$$1/2n \le x_n(t) \le a - (1/n), \quad 0 \le -x_n'(t) \le K \tag{20}$$

for $t \in J$, where $K > 0$ is defined by (12). Since $|\varphi_n(t) - \varphi(t)| \le 1/n$ for $t \in [r,0]$ and $n \in \mathbb{N}_*$, there is no loss of generality in assuming $\{x_n(t)\}_{n\in\mathbb{N}_*}$ is uniformly convergent on $[r,T]$, and let $\lim_{n\to\infty} x_n(t) = x(t)$. Then $x \in C^0([r,T])$, $x(t) = \varphi(t)$ for $t \in [r,0]$, $x(0) = a$, $x(T) = 0$ and x is nondecreasing on J. Hence there exists $\gamma \in (0,T]$ such that $a \ge x(t) > 0$ for $t \in [0,\gamma)$ and $x(t) = 0$ for $t \in [\gamma,T]$. We now show that x is decreasing on $[0,\gamma]$. If not, there exist $0 \le t_1 < t_2 < \gamma$ such that $x(t) = x(t_1)$ for $t \in [t_1,t_2]$. From $x_n'' \ge 0$ on $(0,T)$ and the equalities

$$x_n(t_2) - x_n(t_1) = x_n'(\varrho_n)(t_2 - t_1),$$

$$x_n(t_2) - 1/(2n) = x_n(t_2) - x_n(T) = x_n'(\tau_n)(t_2 - T),$$

where $t_1 < \varrho_n < t_2 < \tau_n < T$, we obtain

$$x_n'(\varrho_n) \le x_n'(\tau_n) = \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}.$$

Then

$$x_n(t_2) - x_n(t_1) \le \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}(t_2 - t_1), \quad n \in \mathbb{N}_*$$

and

$$\lim_{n\to\infty}(x_n(t_2) - x_n(t_1)) \le \lim_{n\to\infty} \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}(t_2 - t_1) = \frac{x(t_2)}{t_2 - T}(t_2 - t_1) < 0,$$

contrary to $\lim_{n\to\infty}(x_n(t_2) - x_n(t_1)) = x(t_2) - x(t_1) = 0$. Hence

$$\begin{aligned} &\lim_{n\to\infty}\Big(f_{1n}(t,x_n(t),x_n(\alpha_1(t))) + f_{2n}(t,x_n(t),x_n(\alpha_2(t)))\Big) \\ &\qquad = f_1(t,x(t),x(\alpha_1(t))) + f_2(t,x(t),x(\alpha_2(t))) \end{aligned} \tag{21}$$

for a.e. $t \in [0,\gamma]$.

Fix $t_* \in (0,\gamma)$. We are going to show that there exist $\Phi < 0$ and $n_1 \in \mathbb{N}_*$ such that

$$x_n'(t) \le \Phi \quad \text{for } t \in [0,t_*], \quad n \ge n_1. \tag{22}$$

Assume, on the contrary, that there exists a subsequence $\{k_n\}$ of $\mathbb{N}_*$ such that $\lim_{n\to\infty} x_{k_n}'(t_*) = 0$. From

$$x_{k_n}(t_*) - 1/(2k_n) = x_{k_n}(t_*) - x_{k_n}(T) = x_{k_n}'(\delta_n)(t_* - T) \le x_{k_n}'(t_*)(t_* - T),$$

where $\delta_n \in (t_*,T)$, it follows that

$$\lim_{n\to\infty}(x_{k_n}(t_*) - 1/(2k_n)) \le (t_* - T)\lim_{n\to\infty} x_{k_n}'(t_*) = 0,$$

contrary to $\lim_{n\to\infty}(x_{k_n}(t_*) - 1/(2k_n)) = x(t_*) > 0$. In order to prove that

$$f_1(t,x(t),x(\alpha_1(t))) + f_2(t,x(t),x(\alpha_2(t))) \in L_1([0,t_*]) \tag{23}$$

and

$$\begin{aligned} &\lim_{n\to\infty}\int_0^{t_*}\Big(f_{1n}(s,x_n(s),x_n(\alpha_1(s))) + f_{2n}(s,x_n(s),x_n(\alpha_2(s)))\Big)\,ds \\ &\quad = \int_0^{t_*}\Big(f_1(s,x(s),x(\alpha_1(s))) + f_2(s,x(s),x(\alpha_2(s)))\Big)\,ds \end{aligned} \tag{24}$$

we use Lemma 3. Let $\varepsilon > 0$. Then there exists $n_2 \in \mathbb{N}$, $n_2 \geq n_1$, such that

$$l_{1n} < \frac{\varepsilon|\Phi|}{14}, \quad l_{2n} < \frac{\varepsilon\Delta_2|\Phi|}{14}$$

for $n \geq n_2$ and there exists $\delta_0 > 0$ such that (cf. (H_2) and (H_3))

$$\begin{gathered} \int_{\mathcal{D}_0}(g_1(s)+g_2(s))\,ds < \frac{\varepsilon|\Phi|}{7}, \quad \int_{\mathcal{D}_1} h_1(s)\,ds < \frac{\varepsilon\Delta_1|\Phi|}{7}, \\ \int_{\mathcal{D}_2} h_2(s)\,ds < \frac{\varepsilon\Delta_2|\Phi|}{7}, \quad \int_{\mathcal{D}\cap[0,\tau_1]} h_1(\varphi(\alpha_1(t)))\,dt < \frac{\varepsilon|\Phi|}{7K} \end{gathered} \tag{25}$$

and

$$\int_{\mathcal{D}\cap[0,\tau_2]} h_2(\varphi(\alpha_2(t)))\,dt < \frac{\varepsilon|\Phi|}{7K}. \tag{26}$$

for any measurable subsets $\mathcal{D}_j$ of $[0,a]$ $(j=0,1,2)$ and measurable $\mathcal{D} \subset J$ such that $m(\mathcal{D}_0) < \delta_0$, $m(\mathcal{D}_j) < \delta_0 \max\{\alpha_j'(t) : t \in J\}$ $(j=1,2)$ and $m(\mathcal{D}) < \delta_0/K$. We recall that τ_1 and τ_2 are numbers from condition (jj). Set

$$\delta = \frac{1}{K}\min\left\{\delta_0, \frac{\varepsilon|\Phi|}{7\max\{nl_{1n} : n_1 \leq n \leq n_2\}}, \frac{\varepsilon\Delta_2|\Phi|}{7\max\{nl_{2n} : n_1 \leq n \leq n_2\}\max\{\alpha_2'(t) : t \in J\}}\right\}.$$

Let $\{(u_i,v_i)\}_{i\in I_1}$ be at most countable sequence of pairwise disjoint intervals $(u_i,v_i) \subset [0,t_*]$ such that $\sum_{i\in I_1}(v_i-u_i) < \delta$ and set $\mathcal{M} = \sum_{i\in I_1}(u_i,v_i)$. Let $i \in I_1$ and $n \geq n_1$. Then (cf. (22) and (H_4))

$$\begin{aligned} &|\Phi|\int_{u_i}^{v_i}\Big(g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_{2n}(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))\Big)\,dt \\ &\leq \left|\int_{u_i}^{v_i}\Big(g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_{2n}(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))\Big)x_n'(t)\,dt\right| \\ &\leq \int_{x_n(v_i)}^{x_n(u_i)}(g_{1n}(s)+g_2(s))\,ds + H_1(u_i,v_i) + H_2(u_i,v_i) \end{aligned}$$

where

$$H_1(u_i,v_i) \leq \begin{cases} K\displaystyle\int_{u_i}^{v_i} h_1(\varphi(\alpha_1(t)))\,dt & \text{if } v_i \leq \tau_1 \\ K\displaystyle\int_{u_i}^{\tau_1} h_1(\varphi(\alpha_1(t)))\,dt + \frac{1}{\Delta_1}\int_{x_n(\alpha_1(v_i))}^{a} h_1(s)\,ds & \text{if } u_i < \tau_1 < v_i \\ \displaystyle\frac{1}{\Delta_1}\int_{x_n(\alpha_1(v_i))}^{x_n(\alpha_1(u_i))} h_1(s)\,ds & \text{if } u_i \geq \tau_1 \end{cases}$$

and

$$H_2(u_i,v_i) \le \begin{cases} K\displaystyle\int_{u_i}^{v_i} h_2(\varphi(\alpha_2(t)))\,dt & \text{if } v_i \le \tau_2 \\ K\displaystyle\int_{u_i}^{\tau_2} h_2(\varphi(\alpha_2(t)))\,dt + \frac{1}{\Delta_2}\int_{x_n(\alpha_2(v_i))}^{a} h_{2n}(s)\,ds & \text{if } u_i < \tau_2 < v_i \\ \displaystyle\frac{1}{\Delta_2}\int_{x_n(\alpha_2(v_i))}^{x_n(\alpha_2(u_i))} h_{2n}(s)\,ds & \text{if } u_i \ge \tau_2. \end{cases}$$

Since $|x_n(v_i) - x_n(u_i)| \le K(v_i - u_i)$, and

$$|x_n(\alpha_j(v_i)) - x_n(\alpha_j(u_i))| \le K \max\{\alpha_j'(t) : t \in J\}(v_i - u_i),$$

we have

$$\sum_{i\in I_1} |x_n(v_i) - x_n(u_i)| \le K \sum_{i\in I_1}(v_i - u_i) = Km(\mathcal{M}),$$

and

$$\begin{aligned} \sum_{i\in I_1} |x_n(\alpha_j(v_i)) - x_n(\alpha_j(u_i))| &\le K \max\{\alpha_j'(t) : t \in J\} \sum_{i\in I_1}(v_i - u_i) \\ &= K \max\{\alpha_j'(t) : t \in J\} m(\mathcal{M}). \end{aligned}$$

Consequently (cf. (6) and (7)),

$$\begin{aligned} &|\Phi| \int_{\mathcal{M}} \Big(g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))\Big)\,dt \\ &\le \sum_{i\in I_1} \left(\int_{x_n(v_i)}^{x_n(u_i)} (g_{1n}(s) + g_2(s))\,ds + H_1(u_i,v_i) + H_2(u_i,v_i) \right) \\ &\le \int_{\mathcal{S}_0} (g_1(s) + g_2(s))\,ds + nl_{1n} \min\left\{\frac{2}{n}, m(\mathcal{S}_0)\right\} + K \int_{\mathcal{M}_1\cap[0,\tau_1]} h_1(\varphi(\alpha_1(t)))\,dt \\ &\quad + \frac{1}{\Delta_1}\int_{\mathcal{S}_1} h_1(s)ds + K \int_{\mathcal{M}_1\cap[0,\tau_2]} h_2(\varphi(\alpha_2(t)))dt \\ &\quad + \frac{1}{\Delta_2}\left(\int_{\mathcal{S}_2} h_2(s)ds + nl_{2n} \min\left\{\frac{2}{n}, m(\mathcal{S}_2)\right\} \right), \end{aligned}$$

where $m(\mathcal{S}_0) \le Km(\mathcal{M}) < K\delta \le \delta_0$ and

$$m(\mathcal{S}_j) \le K \max\{\alpha_j'(t) : t \in J\} m(\mathcal{M}) < \delta_0 \max\{\alpha_j'(t) : t \in J\}, \quad j = 1,2.$$

Then (see (25) and (26))

$$\int_{\mathcal{S}_0} (g_1(s) + g_2(s))\,ds < \frac{\varepsilon|\Phi|}{7}, \quad nl_{1n} \min\left\{\frac{2}{n}, m(\mathcal{S}_0)\right\} < \frac{\varepsilon|\Phi|}{7},$$

$$K \int_{\mathcal{M}_1\cap[0,\tau_1]} h_1(\varphi(\alpha_1(t)))\,dt < \frac{\varepsilon|\Phi|}{7}, \quad \frac{1}{\Delta_1}\int_{\mathcal{S}_1} h_1(s)\,ds < \frac{\varepsilon|\Phi|}{7},$$

$$K\int_{\mathcal{M}_1\cap[0,\tau_2]} h_2(\varphi(\alpha_2(t)))\,dt < \frac{\varepsilon|\Phi|}{7},\quad \frac{1}{\Delta_2}\int_{\mathcal{S}_2} h_2(s)\,ds < \frac{\varepsilon|\Phi|}{7},$$

and

$$\frac{1}{\Delta_2} n l_{2n} \min\left\{\frac{2}{n}, m(\mathcal{S}_2)\right\} < \frac{\varepsilon|\Phi|}{7}.$$

Therefore

$$\int_{\mathcal{M}} \Big(g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))\Big)\,dt < \varepsilon, \quad n \ge n_1.$$

By Lemma 3 with $w_n(t) = f_{1n}(t, x_n(t), x_n(\alpha_1(t))) + f_{2n}(t, x_n(t), x_n(\alpha_2(t)))$, $p_n(t) = g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))$ for $n \ge n_1$ and $[\alpha,\beta] = [0, t_*]$, (23) and (24) are true. In addition, since $t_* \in (0,\gamma)$, it follows that

$$\lim_{n\to\infty}\int_0^t \Big(f_{1n}(s, x_n(s), x_n(\alpha_1(s))) + f_{2n}(s, x_n(s), x_n(\alpha_2(s)))\Big)\,ds$$
$$= \int_0^t \Big(f_1(s, x(s), x(\alpha_1(s))) + f_2(s, x(s), x(\alpha_2(s)))\Big)\,ds$$

for $t \in [0,\gamma)$. The sequence $\{x_n'(0)\}$ is bounded, and so we can assume that this sequence is convergent, say $\lim_{n\to\infty} x_n'(0) = C$. Then taking the limit as $n \to \infty$ in the equalities

$$x_n(t) = x_n(0) + x_n'(0)t + \mu\int_0^t\int_0^s q(v)\Big(f_{1n}(v, x_n(v), x_n(\alpha_1(v))) + f_{2n}(v, x_n(v), x_n(\alpha_2(v)))\Big)dv\,ds, \quad t \in J,$$

we get

$$x(t) = a + Ct + \mu\int_0^t\int_0^s q(v)\Big(f_1(v, x(v), x(\alpha_1(v))) + f_2(v, x(v), x(\alpha_2(v)))\Big)\,dv\,ds \tag{27}$$

for $t \in [0,\gamma)$. Hence $x \in C^1([0,\gamma))$.

We are going to prove that $\gamma = T$. Assume $\gamma < T$. From $x_n(\gamma) - x_n((T+\gamma)/2) = x_n'(c_n)(\gamma - T)/2$, where $c_n \in (\gamma, (T+\gamma)/2)$, and $\lim_{n\to\infty} x_n(\gamma) = \lim_{n\to\infty} x_n((T+\gamma)/2) = 0$ we see that $\lim_{n\to\infty} x_n'(c_n) = 0$. Then integrating the inequalities (for $n \ge n_1$)

$$2x_n''(t)x_n'(t) \ge 2\mu_T Q\Big(g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_{2n}(x_n(\alpha_2(t)))\Big)x_n'(t) \tag{28}$$

from 0 to c_n and applying the same procedure as in the proof of Lemma 1 (now with x_n and c_n instead of x and T, respectively), we have

$$(x_n'(c_n))^2 - (x_n'(0))^2 \ge \mu_T\left(2Ax_n'(0) - B - 2l_{1n} - \frac{2l_{2n}}{\Delta_2}\right),$$

where A is given by (18) and B by (19). Letting $n \to \infty$ yields

$$-C^2 \ge \mu_T(2AC - B) \tag{29}$$

since $\lim_{n\to\infty}(l_{1n}+(l_{2n}/\Delta_2))=0$. We recall that $C=\lim_{n\to\infty}x_n'(0)<0$. From (29) we deduce that

$$-C\le \mu_T A+\sqrt{\mu_T^2A^2+\mu_T B}.$$

On the other hand x_n' is nondecreasing and

$$a-(1/n)-x_n(c_n)=x_n(0)-x_n(c_n)=-x_n'(\eta_n)c_n,$$

where $\eta_n\in(0,c_n)$. Hence

$$x_n'(0)\le x_n'(\eta_n)=\frac{n(x_n(c_n)-a)+1}{nc_n}$$

and then from $\lim_{n\to\infty}x_n(c_n)=0$ and $1/c_n>2/(T+\gamma)$ it follows that

$$C<-\frac{2a}{T+\gamma}<-\frac{a}{T}.$$

Consequently, $a/T<-C\le\mu_T A+\sqrt{\mu_T^2A^2+\mu_T B}$ and then $\mu_T>a^2/[(2aA+BT)T]$, contrary to the definition (17) of μ_T. Hence $\gamma=T$.

Since $\{x_n'(T)\}_{n\ge n_1}$ is bounded, we can assume that the sequence $\{x_n'(T)\}_{n\ge n_1}$ is convergent, say $\lim_{n\to\infty}x_n'(T)=D$ (≤ 0). Repeated integration of (28) now from 0 to T gives

$$(x_n'(T))^2-(x_n'(0))^2\ge\mu_T\left(2Ax_n'(0)-B-2l_{1n}-\frac{2l_{2n}}{\Delta_2}\right),$$

and letting $n\to\infty$ we obtain $(0\ge)$ $D^2-C^2\ge\mu_T(2AC-B)$. Then

$$D^2-C^2\ge\frac{(2AC-B)a^2}{(2aA+BT)T}$$

and

$$C^2+\frac{2a^2A}{(2aA+BT)T}C-\frac{a^2B}{(2aA+BT)T}-D^2\le 0. \tag{30}$$

Now from the inequalities $x_n(t)\le a-(1/n)+[(3-2na)/(2nT)]t$ for $t\in J$ and $n\ge n_1$ which follows from $x_n(a)=a-(1/n)$, $x_n(T)=1/(2n)$ and $x_n''\ge 0$ on $(0,T)$, we have

$$x(t)\le a-\frac{at}{T}\quad\text{for } t\in J. \tag{31}$$

Suppose $D=0$. Since $x_n'(0)\le(3-2na)/(2nT)$ for $n\ge n_1$, we see that $C=\lim_{n\to\infty}x_n'(0)\le -a/T$. On the other hand (30) implies $C\ge -a/T$. Therefore $C=-a/T$. Now (27) and (31) give $x(t)=a-(at/T)$ for $t\in J$ and

$$\mu\int_0^t\int_0^s q(v)\Big(f_1(v,x(v),x(\alpha_1(v)))+f_2(v,x(v),x(\alpha_2(v)))\Big)\,dv\,ds=0$$

for $t\in J$ and $\mu\in[0,\mu_T]$. We see that in this case $x(t)=a-(at/T)$ is a solution of BVP (1), (2) in the set $\mathcal{E}$. Let $D<0$. Then there exists $n_3\ge n_1$ such that $x_n'(T)\le D/2$ for $n\ge n_3$. To prove that the sequence

$$\{f_{1n}(t,x_n(t),x_n(\alpha_1(t)))+f_{2n}(t,x_n(t),x_n(\alpha_2(t)))\}_{n\ge n_3}\subset L_1(J)$$

has uniformly continuous integrals on J we can use the same procedure as above, now with $D/2$ instead of Φ. Then $f_1(t,x(t),x(\alpha_1(t))) + f_2(t,x(t),x(\alpha_2(t))) \in L_1(J)$, by Lemma 3, and (27) shows that $x \in C^1(J)$. We know that x is decreasing on J, and so there exists a unique $t_0 \in (0,T)$ such that $x(t_0) = b$ and, if $x(\alpha_2(T)) < c$, then there exists a unique $t_1 \in (0,T)$ such that $x(\alpha_2(t_1)) = c$. From (27) we see that x satisfies (1) for $t \in (0,T)$ with the exception at most two points t_0 and t_1. Hence x is a solution of BVP (1), (2) in the set $\mathcal{E}$. □

From Theorem 1 it follows immediately the following result for the solvability of BVP (3), (4) in the set $\mathcal{E}$.

Corollary 1. *Suppose that assumptions* (H_1) *and*

(H_6) $f_* : J \times D_1 \to [0,\infty)$ *is continuous with* $D_1 = (0,b) \cup (b,a)$, $f_*(t,x) \le g_*(x)$, $(t,x) \in J \times D_1$, *where* $g_* \ge 0$ *is continuous on* D_1, $\int_0^a g_*(s)\,ds < \infty$ *and* $\lim_{v\to 0} v g_*(b+v) = 0$

are satisfied. If $0 \le \mu \le a^2/[2QT\int_0^a g_*(s)\,ds]$ *then BVP* (3), (4) *has a solution in the set* $\mathcal{E}$.

Remark 1. Assumption (H_5) can be replaced by the following slightly weaker assumption: There exist sequences $\{c_n\}$, $\{c_n^*\}$, $\{d_n\}$, $\{d_n^*\}$ of positive numbers, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} c_n^* = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d_n^* = 0$ such that

$$\lim_{n\to\infty} (c_n g_1(b-c_n) + d_n g_1(b+d_n)) = 0, \quad \lim_{n\to\infty} (c_n^* h_2(c-c_n^*) + d_n^* h_2(c+d_n^*)) = 0.$$

Remark 2. It follows from our considerations that the assertion of Theorem 1 is true also in the cases that f_1 and f_2 have a finite number of singularities on $(0,a)$ in the phase variables x and y, respectively, supposing that singularities are of the type as $x = b$ and $y = c$ in assumptions (H_2), (H_3) and (H_5).

Example 1. Consider the BVP

$$\begin{aligned} x''(t) = \mu(1+|\sin(t^{-1}(T-t)^{-1})|)\Bigg(&\frac{K}{|a-2x(t)|^\alpha} + \frac{L}{(x(t))^\beta(a-x(t))^\gamma} \\ &+ \frac{M}{(a-x(t-\frac{T}{4}))^\delta} + \frac{N}{|a-4x(\frac{4t}{3}-\frac{T}{2})|^\varepsilon}\Bigg), \end{aligned} \tag{32}$$

$$x(t) = a\left(1+\frac{t}{T}\right) \quad \text{for } t \in [-T/2, 0], \quad x(T) = 0, \tag{33}$$

where K, L, M, N are nonnegative constants, $K+L+M+N > 0$, α, δ, $\varepsilon \in (-\infty,1)$ and β, $\gamma \in (0,1)$. Assumptions (j), (jj) and (H_1)–(H_5) are satisfied with $\alpha_1(t) = t-(T/4)$, $\alpha_2(t) = (4t/3)-(T/2)$, $\Delta_1 = 1$, $\Delta_2 = 4/3$, $\tau_1 = T/4$, $\tau_2 = 3T/8$, $\varphi(t) = a(1+(t/T))$, $b = a/2$, $c = a/4$, $Q = 2$, $g_1(x) = K/|a-2x|^\alpha + L/[x^\beta(a-x)^\gamma]$, $h_1(y) =$

$M/(a-y)^\delta$, $g_2(x)=0$ and $h_2(y)=N/|a-4y|^\varepsilon$. Consequently, by Theorem 1, BVP (32),(33) has a solution in the set $\mathcal{E}$ if $0 \le \mu \le \mu_T$, where μ_T is defined by (17) with

$$A = 2T\left(\frac{M}{4^{1-\delta}(1-\delta)a^\delta} + \frac{3N(3^{1-\varepsilon}-1)}{16(1-\varepsilon)a^\varepsilon}\right),$$

and

$$B = 4\left(\frac{Ka^{1-\alpha}}{1-\alpha} + La^{1-\beta-\gamma}\mathrm{B}(1-\beta,1-\gamma) + \frac{Ma^{1-\delta}}{1-\delta} + \frac{3Na^{1-\varepsilon}(1+3^{1-\varepsilon})}{16(1-\varepsilon))}\right),$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the beta function.

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