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OPERATOR VALUED MEASURES FOR OPTIMAL FEEDBACK CONTROL OF INFINITE DIMENSIONAL SYSTEMS

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ABSTRACT. In this paper we present a brief review of some recent results on weak compactness in the space of operator valued measures. These results are then applied to optimal structural feedback control for deterministic and partially observed stochastic systems on infinite dimensional spaces. Existence of optimal structural feedback controls for standard as well as nonstandard control problems are presented. The objects being controlled are the measure valued functions (and functionals thereof) induced by the stochastic systems.

Keywords: Space of operator valued measures, Weak compactness, Semigroups of bounded linear operators, Optimal structural control

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1. INTRODUCTION

In physical sciences and engineering, involving control theory and optimization, one has the freedom to choose from a given class of controls the best one that minimizes or maximizes certain functionals representing the measure of performance. The controls may be measurable functions taking values from a Banach space (regular controls) or may be vector or operator valued measures [5,6,7,9,10,11,14,19]. For control theory in finite and infinite dimensional Banach spaces based on regular controls the reader is referred to the recent book of Fattorini [10] where the presentation is encyclopedic. Vector measures can be used as controls and Operator valued measures as structural feedback controls. In the study of existence theory, compactness is a very useful tool, particularly in the area of optimization, optimal control, system identification, Kalman Filtering, Structural control etc. [5,6,7,8,9,10,11].

Necessary and sufficient conditions for weak compactness in the space of vector measures has been a subject of great interest over half a century. One of the seminal results in this topic is the well known Bartle-Dunford-Schwartz theorem [1, Theorem 5, p. 105], for countably additive bounded vector measures with values in Banach spaces satisfying, along with their duals, the Radon-Nikodym property. This result was extended to finitely additive vector measures by Brooks [2] and Brooks and Dinculeanu [1, Corollary 6, 106].

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Recently the author extended some of these results to a certain class of operator valued measures [12, 13]. In this paper we present these results briefly before considering control problems involving operator valued measures as controls. The rest of the paper is organized as follows. In section 2, we present the basic properties of operator valued measures. In section 3, we present certain results on compactness. Section 4 is the main part of the paper where we consider the control problems (feedback).

2. OPERATOR VALUED MEASURES

Let D be a compact Hausdorff space and Σ an algebra of subsets of D, $\{X, Y\}$ a pair of B-spaces and $\mathcal{L}(Y, X)$ is the space of bounded linear operators from Y to X. The function

$$B : \Sigma \longrightarrow \mathcal{L}(Y, X)$$

is generally a finitely additive (f.a.) set function with values in $\mathcal{L}(Y, X)$. This class, denoted by $M_{ba}(\Sigma, \mathcal{L}(Y, X))$, is called the space of operator valued measures. Clearly, this is a B-space with respect to the topology induced by the supremum of the operator norm on Σ.

In case of operator valued measures there are several notions of countable additivity related to the topology used. Here we are interested in countable additivity in the strong operator topology. This is defined as follows:

Definition 2.1 (countable additivity). An element $B \in M_{ba}(\Sigma, \mathcal{L}(Y, X))$ is said to be countably additive in the strong operator topology $(ca - \tau_{so})$ if for any family of pair wise disjoint sets $\{\sigma_i\} \in \Sigma, \sigma_i \subset D, \cup \sigma_i \in \Sigma$, and for every $y \in Y$,

$$\lim_{n \to \infty} |B(\bigcup \sigma_i)y - \sum_{i=1}^{n} B(\sigma_i)y|_X = 0.$$

Definition 2.2 (strong variation). The variation of B on J in the strong operator topology is given by:

$$|B|_s(J) = \sup\Big\{ |\sum_{i=1}^{n} B(\sigma_i)y_i|_X, y_i \in B_1(Y), \\ \{\sigma_i, 1 \le i \le n,\} \in \Pi_\Sigma(J), n \in N \Big\},$$

where $\Pi_\Sigma(J)$ denotes the class of all Σ-measurable disjoint partitions of J.

Note that if $Y = R$, B reduces to an X valued vector measure and the above expression gives the standard semivariation of vector measures. Similarly, if $X = R$, B is a Y^* valued vector measure and the variation reduces to standard variation.

There are other notions of variations such as uniform and weak which we do not use here.

3. WEAK COMPACTNESS

Now we present some recent results on the characterization of conditionally weakly compact sets in the space of operator valued measures. The first result presented here involves Hilbert spaces and nuclear operator valued measures.

Theorem 3.1. Let $\{X, Y\}$ be a pair of separable Hilbert spaces with complete orthonormal basis $\{x_i, y_i\}$. A set $\Gamma \subset M_{ba}(\Sigma, \mathcal{L}_1(Y, X))$ is conditionally weakly compact if, and only if, the following conditions hold:

(c1): Γ is bounded,

(c2): for each $\sigma \in \Sigma$, $\sum_{i=1}^{\infty} |(M(\sigma)y_i, x_i)_Y|$ is convergent uniformly with respect to $M \in \Gamma$,

(c3): for each $i \in N$, the set of scalar valued measures $\{\mu_M(\cdot) \equiv (M(\cdot)y_i, x_i), M \in \Gamma\}$ is a conditionally weakly compact subset of $M_{ba}(\Sigma)$.

Proof. [Ahmed 12, Theorem 3.2, PMD, (2010)].

This result was recently extended to more general spaces of operator valued measures [13]. Here we consider $\{X, Y\}$ to be a pair of Banach spaces and replace the space of nuclear operators by $\mathcal{L}(Y, X)$, the space of bounded linear operators. Let

$$M_{casbsv}(\Sigma, \mathcal{L}(Y, X)) \subset M_{ba}(\Sigma, \mathcal{L}(Y, X))$$

denote the space of operator valued measures countably additive in the strong operator topology having bounded semi variations (variation in the strong operator topology). To proceed further, we introduce the subject of integration of vector valued functions with respect to operator valued measures. The most general theory of integration was introduced by Dobrakov [3,4]. This generalizes the theory of Lebesgue integral, Bochner integral, Bartle bilinear integral and Dinculeanu integral etc. For a detailed survey on this topic See [18].

Dobrakov Integral: For any $f \in B_\infty(D, Y)$ and $T \in M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$ the integral,

$$I_T(f) \equiv \int_D T(ds)f(s) \ \in X,$$

is well defined in the sense of Dobrakov [3, 4]. As usual the integral is first defined for simple functions $\mathcal{S}(D, Y)$ and then extended to $B_\infty(D, Y)$ by density argument. The most important point is that the limit is taken in the sense of unconditional convergence of the sum arising from the simple functions. This limit is the Dobrakov integral. It is based on the notion of unconditional convergence of infinite series in B-spaces and Orlicz-Pettis Theorem [1]. This is unlike the Lebesgue and Bochner integrals which are based on absolute convergence. This is where the main difference is.

It is interesting to note that for any $T \in M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ and $\sigma \in \Sigma$ the set function given by

$$\hat{T}(\sigma) \equiv \sup\Big\{ | \int_\sigma T(ds)f(s)|_X, f \in \mathcal{S}(D,Y), \| f \|_\infty \leq 1 \Big\}$$

is a countably subadditive submeasure. In fact this also coincides with the variation in the strong operator topology of T on J and $\hat{T}(D) = |T|_s$.

A general result on Weak Compactness: Now we are prepared to present a general result characterizing weakly compact sets in $M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$. Let $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ and $f \in B_\infty(D,Y)$. Define the set

$$\Gamma(f) \equiv \big\{ \mu \in M_{ba}(\Sigma, X) : \mu(\sigma) = \int_\sigma T(ds)f(s), \sigma \in \Sigma, T \in \Gamma \big\}.$$

It is easy to verify that $\Gamma(f) \subset M_{cabv}(\Sigma, X)$, the space of countably additive X valued vector measures having bounded variation.

Theorem 3.2. Suppose D is a compact Hausdorff $\mathcal{F}$-space, and $\{X, Y\}$ is a pair of B-spaces with X being reflexive. Then a set $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ is conditionally weakly compact if, and only if, the following conditions hold:

(i): Γ is bounded in the sense that $\sup\{\hat{T}(D) \equiv |T|_s, T \in \Gamma\} < \infty$.

(ii): For each $f \in B_\infty(D,Y)$, the set $\{|\mu|(\cdot), \mu \in \Gamma(f)\}$ is uniformly c.a.

Proof. For proof see [Ahmed 13, Theorem 1].

4. APPLICATIONS TO OPTIMAL FEEDBACK CONTROL

The primary objective of this paper is to apply the weak compactness results to optimal control problems in infinite dimension. We present two applications to structural control problems in infinite dimension. The first example deals with deterministic systems on general Banach spaces. The second example deals with stochastic systems on Hilbert spaces.

4.1. **Deterministic System.**

Consider the structural control system on a real Banach space X

$$dx = Axdt + B(dt)y + f(x)dt, x(0) = \xi \tag{4.1}$$

$$y = Lx + \eta \text{ (output)} \tag{4.2}$$

over the time interval $t \in [0,T]$. The state space X is a reflexive B-space and the output space Y is any real Banach space. The operator $L \in \mathcal{L}(X,Y)$ represents the sensor and $\eta \in B_\infty(I,Y)$ is a deterministic perturbation. The objective functional is given by

$$J(B) \equiv \int_0^T \ell(t, x(t))dt + |B|_s, \tag{4.3}$$

where $|B|_s$ denotes the semivariation (variation in the strong operator topology) of B over the set I. The admissible set of structural controls is given by a set $\Gamma \subset M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$. The objective is to find a control that minimizes this functional.

Let $\mathcal{G}_0(M,\omega)$ denote the class of infinitesimal generators $\{A\}$ of C_0-semigroups of linear operators on X with stability parameters (M,ω) for $M \geq 1$ and $\omega \in R$.

Theorem 4.1. Suppose $A \in \mathcal{G}_0(M,\omega)$ generating the semigroup $S(t), t \geq 0$, compact for $t > 0$, Γ a weakly compact subset of $M_{casbsv}(\Sigma_I, \mathcal{L}(Y,X))$, f locally Lipschitz with at most linear growth, $L \in \mathcal{L}(X,Y)$, $\eta \in B_\infty(I,Y)$. There exists $\nu \in M^+_{cabv}(\Sigma_I)$ such that $|B|_s(\sigma) \leq \nu(\sigma)$ for $\sigma \in \Sigma_I$ uniformly w.r.t $B \in \Gamma$. The cost integrand ℓ is measurable in t and lower semicontinuous in x on X and there exists $\alpha \in L^+_1(I)$ and $\beta \geq 0$ satisfying

$$|\ell(t,x)| \leq \alpha(t) + \beta |x|^p_X, \text{ for any } p \in (0,\infty).$$

Then, there exists a $B_o \in \Gamma$ at which J attains its minimum.

Proof. For detailed proof see [13, Theorem 1] see also [5, 19]. □

Remark 4.1. (i): Assumption on compactness of the s.g $S(t)$ can be relaxed by some additional (stronger) assumptions on Γ. (ii): It was shown in [5, Lemma 2.3; see also Remark 3.4, p. 105], that for each element of $M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ there exists a nonnegative countably additive measure dominating its semivariation.

4.2. **Stochastic System.** Here we consider some standard and nonstandard optimal control problems for a partially observed stochastic system given by

$$dx = Axdt + B(dt)y + f(x)dt + g(x)dW, x(0) = \xi \tag{4.4}$$

$$dy = Lxdt + dv, y(0) = 0, \text{ (output)} \tag{4.5}$$

Naturally, now both the state space X and the output space Y are assumed to be separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ be a complete filtered probability space with $\mathcal{F}_t, t \geq 0$, being an increasing family of complete sub sigma algebras of the sigma algebra $\mathcal{F}$. Let E be another separable Hilbert space, and the process $W \equiv \{W(t), t \geq 0\}$ a Q-Wiener process on E adapted to $\mathcal{F}_t$ and that $\mathbf{E}\{(W(t),e)^2\} = t(Qe,e)$ for $e \in E$. The process v in the output equation is an Y-valued $\mathcal{F}_t$ adapted Wiener process with covariance operator Q_0, that is, for every $y \in Y$, $\mathbf{E}\{(v(t),y)^2\} = t(Q_0 y, y)$. For convenience of notation we let Σ denote Σ_I, the sigma algebra of subsets of I.

Let $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$ denote the admissible set of structural controls.

The basic assumptions used are:

(H1): $A \in \mathcal{G}_0(M,\omega)$ generating a C_0-s.g $S(t), t \geq 0$.
(H2): $\exists\, \nu \in M^+_{cabv}(\Sigma)$ such that for each $\sigma \in \Sigma$, $|B|_s(\sigma) \leq \nu(\sigma)$ for all $B \in \Gamma$.

(H3): Both $f : X \longrightarrow X$ and $g : X \longrightarrow \mathcal{L}(E, X)$ are globally Lipschitz with Lipschitz constant $K = K_Q > 0$, satisfying

$$\left\{|f(x) - f(z)|_X^2 + \parallel g(x) - g(z) \parallel_Q^2\right\} \leq K|x - y|_X^2,$$

where $\parallel g \parallel_Q^2 \equiv Tr(gQg^*)$.

(H4): For the output dynamics, $L \in \mathcal{L}(X, Y)$ and v is an Y valued Wiener process, independent of the Wiener process W, with incremental covariance $Q_0 \in \mathcal{L}_1^+(Y)$, the class of positive nuclear operators in Y.

Let $B_\infty^a(I, L_2(\Omega, X))$ denote the vector space of $\mathcal{F}_t$-adapted X valued random processes $\{x(t), t \geq 0\}$ having (norm) bounded second moments in the sense that

$$\parallel x \parallel^2_{B_\infty^a(I,L_2(\Omega,X))} \equiv \sup\{\mathbf{E}|x(t)|_X^2, t \in I\} < \infty.$$

Since the filtration is complete, it is easy to verify that $B_\infty^a(I, L_2(\Omega, X))$ is a Banach with respect to the norm topology $\parallel \cdot \parallel_{B_\infty^a(I,L_2(\Omega,X))}$.

Our next objective is to prove sequential continuity of the solution map $B \longrightarrow x^B(t)$ from the space $M_{casbsv}(\Sigma_I, \mathcal{L}(Y, X))$ to the Hilbert space $L_2(\Omega, X)$. This will allow us to prove weak continuity of the induced probability measures on $\mathcal{B}(X)$, Borel subsets of X, with respect to the operator valued measures B.

Theorem 4.2. Consider the system (4.4)–(4.5) and suppose the assumptions (H1)–(H4) hold and that the semigroup $S(t), t > 0$, is compact. Then,

(a): for each $\mathcal{F}_0$ measurable initial state $\xi \in L_2(\Omega, X)$ and control measure $B \in \Gamma \subset M_{casbsv}(\Sigma_I, \mathcal{L}(Y, X))$, the system (4.4) has a unique mild solution $x^B \in B_\infty^a(I, L_2(\Omega, X))$. And the corresponding output process $y \equiv y^B \in B_\infty^a(I, L_2(\Omega, Y))$.

(b): for each $t \in I$, the solution map $B \longrightarrow x^B(t)$ is sequentially continuous with respect to the weak topology on $M_{casbsv}(\Sigma_I, \mathcal{L}(Y, X))$ and the norm topology on $L_2(\Omega, X)$.

Proof. Essentially the proof of the existence and uniqueness part is similar to that given in [5, Theorem 3.5] see also [20]. Since the output operator $L \in \mathcal{L}(X, Y)$ and, by assumption (H4), v is a continuous square integrable martingale, the output y^B, given by

$$y^B(t) = \int_0^t Lx^B(s)ds + v(t), t \in I,$$

is continuous P-a.s and an element of $B_\infty^a(I, L_2(\Omega, Y))$. We present briefly an outline of the proof of continuity. Let $B_n \xrightarrow{w} B_o$ in $M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$ and let $\{x_n, x_o\} \in$

$B_\infty^a(I, L_2(\Omega, X))$ denote the corresponding mild solutions satisfying the following integral equations

$$x_n(t) = S(t)\xi + \int_0^t S(t-s)B_n(ds)y_n(s) + \int_0^t S(t-s)f(x_n)ds \\ + \int_0^t S(t-s)g(x_n)dW(s), t \in I, \tag{4.6}$$

and

$$x_o(t) = S(t)\xi + \int_0^t S(t-s)B_o(ds)y_o(s) + \int_0^t S(t-s)f(x_o)ds \\ + \int_0^t S(t-s)g(x_o)dW(s), t \in I, \tag{4.7}$$

respectively. Subtracting equation (4.6) from (4.7) and rearranging terms we obtain the following expression:

$$x_o(t) - x_n(t) = e_n(t) + \int_0^t S(t-s)B_n(ds)(y_o(s) - y_n(s)) \\ + \int_0^t S(t-s)f(x_o) - f(x_n)ds + \int_0^t S(t-s)[g(x_o) - g(x_n)]dW(s), \tag{4.8}$$

for $t \in I$, where

$$e_n(t) \equiv \int_0^t S(t-s)[B_o(ds) - B_n(ds)]y_o(s), t \in I. \tag{4.9}$$

After long but straightforward computation using the assumptions (H1)–(H3), it follows from equations (4.8) and (4.9) that

$$\mathbf{E}|x_o(t) - x_n(t)|_X^2 \leq 4\mathbf{E}|e_n(t)|_X^2 + C\int_0^t \mathbf{E}|x_o(s) - x_n(s)|_X^2 ds, \tag{4.10}$$

for $t \in I$, where

$$C = 4(\tilde{M})^2\big\{(\parallel L \parallel_{\mathcal{L}(X,Y)} \nu(I))^2 T + (1+T)K^2\big\},$$

with $\tilde{M} \equiv \sup\{\parallel S(t) \parallel_{\mathcal{L}(X)}, t \in I\}$. Then by Gronwall inequality, it follows from (4.10) that

$$\mathbf{E}|x_o(t) - x_n(t)|_X^2 \leq 4\mathbf{E}|e_n(t)|_X^2 + 4C\exp(CT)\int_0^t \mathbf{E}|e_n(s)|_X^2 ds, \tag{4.11}$$

for $t \in I$. We show that for each $t \in I$, $e_n(t) \xrightarrow{s} 0$ in X P-a.s. We prove this by showing that

$$(e_n(t), z) = \int_0^t \big(S^*(t-s)z, (B_o(ds) - B_n(ds))y_o(s)\big) \to 0$$

P-as, uniformly with respect to $z \in B_1(X)$. Denote the retract of the Ball $B_r(Y)$ by $\Phi_r(y)$, and define $y_{o,r} = \Phi_r(y_o)$. Clearly $y_{o,r} \in B_\infty(I, Y)$ P-a.s and $y_{o,r} \xrightarrow{s} y_o$ in

$B^a_\infty(I, L_2(\Omega, Y))$. Using the retract we can write

$$\begin{aligned}(e_n(t), z) &= \int_0^t \big(S^*(t-s)z, (B_o(ds) - B_n(ds))y_o(s)\big) \\ &= \int_0^t \big(S^*(t-s)z, (B_o(ds) - B_n(ds))y_{o,r}(s)\big) \\ &\quad + \int_0^t \big(S^*(t-s)z, (B_o(ds) - B_n(ds))(y_o(s) - y_{o,r}(s)\big) \equiv I_1 + I_2\end{aligned}$$

for $t \in I$. By our assumption $S(t)$, and hence $S^*(t)$, is compact and since $y_{o,r} \in B_\infty(I, Y)$ P-almost surely, it follows from weak convergence of B_n to B_o that I_1 converges to zero P-a.s uniformly on the unit ball $B_1(X)$. Considering I_2, it follows from (H2) that

$$|I_2|^2 \le (2\tilde{M}|z|_X)^2 \nu(I) \int_0^t |y_o(s) - y_{o,r}(s)|_Y^2 \nu(ds), t \in I,\ P - a.s.$$

Since $y_{o,r}(t) \xrightarrow{s} y_o(t)$ in Y for each $t \in I$ P-a.s, and they are elements of $B_\infty(I, L_2(\Omega, Y))$, and ν is a nonnegative countably additive measure having bounded variation, I_2 converges to zero as $r \to \infty$ uniformly with respect to $z \in B_1(X)$. Thus $(e_n(t), z) \to 0$ P-almost surely uniformly on the unit ball $B_1(X)$ and hence, for each $t \in I$, $e_n(t) \xrightarrow{s} 0$ in X with probability one $(P - a.s)$. Since the semivariations of $\{B_n, B_o\}$ are dominated by the measure ν, it follows from the expression for e_n given by (4.9) that

$$|e_n(t)|_X^2 \le 4(\tilde{M})^2 \nu(I) \int_0^T |y_o(s)|_Y^2 \nu(ds), t \in I,$$

P-a.s. Recalling that $y_o \in B^a_\infty(I, L_2(\Omega, Y))$, it follows from the inequality (4.11) and the Lebesgue bounded convergence theorem that for each $t \in I$,

$$\mathbf{E}|x_o(t) - x_n(t)|_X^2 \longrightarrow 0.$$

This proves the continuity as stated. □

For any $B \in \Gamma$ and $t \in I$, let $\mu_t^B \equiv \mathcal{L}(x^B(t))$ denote the probability law or measure induced by the random element $x^B(t)$. That is, for any $G \in \mathcal{B}(X)$, $\mu_t^B(G) = P\{x^B(t) \in G\}$. Let $C_0(X)$ denote the Banach space of bounded continuous real valued function on X endowed with the standard sup norm topology $\sup\{|\varphi(x)|, x \in X\}$; and let $\mathcal{M}_0(X)$ denote the space of probability measures defined on $\mathcal{B}(X)$. Define for each $t \in I$, the reachable set

$$\mathcal{R}(t) \equiv \{\nu \in \mathcal{M}_0(X) : \nu = \mu_t^B \text{ for some } B \in \Gamma\}. \tag{4.12}$$

This is the set of measures induced by the solution process of the system (4.4)–(4.5) at time t. As a byproduct of Theorem 4.2, we have the following result.

Theorem 4.3. Consider the system (4.4)–(4.5) and suppose the assumptions of Theorem 4.2 hold. Then, for each $t \in I$, the reachable set $\mathcal{R}(t) \subset \mathcal{M}_0(X)$ is compact in the weak topology.

Proof. It follows from Theorem 4.2 that whenever $B_n \xrightarrow{w} B_o$, we have $x_n(t) \xrightarrow{s} x_o(t)$ in $L_2(\Omega, X)$ for each $t \in I$. Hence, for each $t \in I$, we can extract a subsequence $\{x_{n_k}(t)\}$ of the sequence $\{x_n(t)\}$ such that $x_{n_k}(t) \xrightarrow{s} x_o(t)$ in X P-a.s. Thus, for any $\varphi \in C_0(X)$, we have

$$\varphi(x_{n_k}(t)) \to \varphi(x_o(t)) \ P - a.s.$$

Since $\varphi \in C_0(X)$, it follows from this and Lebesgue bounded convergence theorem that

$$\mathbf{E}\varphi(x_{n_k}(t)) \to \mathbf{E}\varphi(x_o(t)).$$

This is equivalent to weak convergence in the sense that

$$\int_X \varphi(x)\mu_t^{n_k}(dx) \to \int_X \varphi(x)\mu_t^o(dx)$$

for every $\varphi \in C_0(X)$, where $\mu_t^{n_k} = \mathcal{L}(x_{n_k}(t))$ and $\mu_t^o = \mathcal{L}(x_o(t))$. And it is often written as $\mu_t^{n_k} \xrightarrow{w} \mu^o$ in $\mathcal{M}_0(X)$. This completes the proof. □

Now we are prepared to consider some interesting control problems where the controls are the operator valued measures considered as structural controls. Henceforth by control we mean an operator valued measure from an admissible class $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$.

Problem 1 (Target Seeking): Let C be any closed subset of X and suppose we want a control policy that maximizes the concentration of probability mass on C at the terminal time T. In other words, this is a most desirable target that we want to reach (or hit) at time $T \in (0, \infty)$ with maximum probability. Thus the objective is to find a control $B \in \Gamma$ such that

$$J_1(B) \equiv \mu_T^B(C)$$

is maximized. Note that this is equivalent to the problem $\sup\{\tilde{J}_1(\mu) \equiv \mu(C), \mu \in \mathcal{R}(T)\}$. We prove the following result.

Theorem 4.4. Suppose the assumptions of Theorem 4.3 hold. Then there exists a control $B_o \in \Gamma$ at which J_1 attains its maximum.

Proof. By Theorem 4.3, $\mathcal{R}(T)$ is weakly compact. So it suffices to prove that $\tilde{J}_1$ as defined above is upper semicontinuous on it. Let $\{\mu_n\}$ be any sequence (a net) from $\mathcal{R}(T)$. Since this set weakly compact there exists a subsequence (subnet), relabeled as the original sequence, and an element $\mu_o \in \mathcal{R}(T)$ such $\mu_n \xrightarrow{w} \mu_o$ in $\mathcal{M}_0(X)$. Then it follows from [15, Theorem 6.1, p. 40] that

$$\overline{\lim}\, \mu_n(C) \le \mu_o(C)$$

which is the same as $\overline{\lim}\, \tilde{J}_1(\mu_n) \le \tilde{J}_1(\mu_o)$. Thus $\tilde{J}_1$ is weakly upper semicontinuous on $\mathcal{R}(T)$ and therefore J_1 is weakly upper semicontinuous on Γ. Since $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$ is weakly compact J_1 attains its maximum on it. This completes the proof. □

Problem 2 (Evasion): Let D be any open subset of X and suppose we want a control policy that minimizes the probability of hitting D at time T. In other words, this is a danger zone and we want to avoid it at time T. Thus the objective is to find a control $B \in \Gamma$ such that

$$J_2(B) \equiv \mu_T^B(D)$$

is minimized. Again, this is equivalent to the problem

$$\inf\{\tilde{J}_2(\mu) \equiv \mu(D), \mu \in \mathcal{R}(T)\}.$$

Theorem 4.5. Suppose the assumptions of Theorem 4.3 hold. Then there exists a control $B_o \in \Gamma$ at which J_2 attains its minimum.

Proof. The proof follows immediately from the previous result, by noting that

$$\mu(D) = 1 - \mu(X \setminus D).$$

Hence the functional $\tilde{J}_2(\mu) \equiv \mu(D)$ is weakly lower semicontinuous and so attains its minimum on $\mathcal{R}(T)$. Thus by theorem 4.3, $B \longrightarrow J_2(B)$ is weakly lower semicontinuous on Γ and consequently there exists a control $B_o \in \Gamma$ at which J_2 attains its minimum. □

Formulation of certain control problems requires the probability measure valued functions. In particular, we are interested in the evolution of measures $\mu^B \equiv \{\mu_t^B, t \geq 0\}$, induced by the solutions of the stochastic systems (4.4)–(4.5) as discussed here. In this vain we introduce the set $\mathcal{R}$ as follows. Let $L_\infty^w(I, \mathcal{M}_0(X))$ denote the class of weak star measurable $\mathcal{M}_0(X)$ valued functions and define the reachable set of solutions as

$$\mathcal{R} \equiv \left\{\mu \in L_\infty^w(I, \mathcal{M}_0(X)) : \mu = \mu^B, \text{for some } B \in \Gamma\right\}. \tag{4.13}$$

Note that $\mathcal{R} = \Pi_{t \in I}\mathcal{R}(t)$. It follows from general topology, in particular Tychonoff's product theorem, that any product of compact topological spaces is compact in the product topology. According to this result $\mathcal{R}$ is compact in the product topology induced by the weak topology of each factor space. Here we present an elementary but instructive proof.

Theorem 4.6. Suppose the assumptions of Theorem 4.3 hold. Then the set $\mathcal{R}$ is a weak star sequentially compact subset of $L_\infty^w(I, \mathcal{M}_0(X))$.

Proof. Let $\{\mu^n\} \in \mathcal{R}$, and $\varphi \in L_1(I, C_0(X))$. Then the duality pairing

$$\mu^n(\varphi) \equiv \int_{I \times X} \varphi(t, x)\mu_t^n(dx)dt$$

is well defined. By Fubinis theorem, we can write this as an iterated integral

$$\mu^n(\varphi) = \int_I \left(\int_X \varphi(t, x)\mu_t^n(dx)\right)dt.$$

Define $f_n(t) \equiv \int_X \varphi(t,x)\mu_t^n(dx), t \in I$. Since for almost all $t \in I$, $\varphi(t,\cdot) \in C_0(X)$, and $\mathcal{R}(t)$ is weakly compact, there exists a $\mu_t^o \in \mathcal{R}(t)$ such that along a subsequence if necessary,

$$f_n(t) \equiv \int_X \varphi(t,x)\mu_t^n(dx) \to \int_X \varphi(t,x)\mu_t^o(dx) \equiv f_o(t)$$

for almost all $t \in I$. This way we can construct μ^o point wise for each $t \in I$. Clearly f_o, being the almost everywhere limit of a sequence measurable functions $\{f_n\}$, is measurable and so μ^o is weakly measurable. Further, $|f_n(t)| \leq |\varphi(t,\cdot))|_{C_0(X)} \equiv g_\varphi(t)$. Clearly, $g_\varphi \in L_1^+(I)$ and therefore by Lebesgue dominated convergence theorem we may conclude that

$$\lim_{n\to\infty} \int_I f_n(t)\ dt = \int_I f_o(t)\ dt$$

and hence

$$\lim_{n\to\infty} \int_{I\times X} \varphi(t,x)\mu_t^n(dx)dt = \int_{I\times X} \varphi(t,x)\mu_t^o(dx)dt.$$

Note that we have $\mu^o \in \mathcal{R}$. Thus $\mathcal{R}$ is weak star compact as stated. □

Remark 4.2. In the proof above, we can replace the sequential convergence by net convergence.

Problem 3: Let us revisit the Evasion problem. Let $D \subset X$ be an open set (considered as the forbidden or danger zone). The objective is to stay away from D as far as possible for the entire period of time $I = [0,T]$. We formulate this with a little more generality. Let λ be a nonnegative countably additive measure on $\Sigma \equiv \Sigma_I$ having bounded total variation and consider the functional

$$J_3(B) \equiv \int_I \mu_t^B(D)\lambda(dt).$$

Corollary 4.7. Suppose the assumptions of Theorem 4.6 hold. Then there exists a control $B \in \Gamma$ at which J_3 attains its minimum and so the evasion Problem-3 has a solution.

Proof. We prove that J_3 is weakly lower semicontinuous on Γ. Let $B_n, B_o \in \Gamma$ such that $B_n \xrightarrow{w} B_o$ in $M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$. Let $\mu^n, \mu^o \in \mathcal{R}$ denote the corresponding measure valued functions. Then, for each $t \in I$, $\mu_t^n, \mu_t^o \in \mathcal{R}(t)$ with $\mu_t^n \xrightarrow{w} \mu_t^o$ possibly along a subsequence. Since D is an open set, it follows from Theorem 4.5 that

$$\underline{\lim}\, \mu_t^n(D) \geq \mu_t^o(D).$$

Clearly, $0 \leq \mu_t^n(D), \mu_t^o(D) \leq 1$. Since the measure λ has bounded total variation on I, using Fatou Lemma, it is easy to verify that

$$J_3(B_o) \equiv \int \mu_t^o(D)\lambda(dt) \leq \underline{\lim} \int_I \mu_t^n(D)\lambda(dt) \equiv \underline{\lim}\, J_3(B_n).$$

Thus J_3 is weakly lower semicontinuous and hence it attains its minimum on Γ. This completes the proof. □

Remark 4.3. Let $\{D_i, 1 \leq i \leq m\}$ be a family of open sets in X and $\{t_i, 1 \leq i \leq m\} \subset I$ and $F : R^m \longrightarrow R$, and consider the cylinder function

$$J(B) \equiv F(\mu^B_{t_1}(D_1), \mu^B_{t_2}(D_2), \cdots, \mu^B_{t_m}(D_m))$$

to be minimized. If F is monotone nondecreasing in its arguments and lower semicontinuous and bounded away from $-\infty$, the functional J is weakly lower semicontinuous and bounded away from $-\infty$ and so attains its minimum on Γ.

Remark 4.4. Let $\varphi_i \in C_0(X), 1 \leq i \leq m$ and $\{t_i, 1 \leq i \leq m\} \subset I$ and F as in Remark 4.3, and consider the cylinder function

$$J(B) \equiv F(\mu^B_{t_1}(\varphi_1), \mu^B_{t_2}(\varphi_2), \cdots, \mu^B_{t_m}(\varphi_m)).$$

If F is lower semicontinuous on R^m and bounded away from $-\infty$, J attains its minimum on Γ. Similarly, if F is upper semicontinuous and bounded away from $+\infty$, then J attains its maximum on Γ.

Remark 4.5. Let $C(X)(\supset C_0(X))$ denote the space of continuous not necessarily bounded real valued functions. This is an algebra. For $p \geq 0$, define $\lambda_p(x) \equiv (1/2)(1 + |x|^p_X), x \in X$. Clearly $\lambda_p \in C(X)$. Define the vector space

$$C_p(X) \equiv \big\{\varphi \in C(X) : \parallel \varphi \parallel_p \equiv \sup\{\frac{|\varphi(x)|}{\lambda_p(x)}, x \in X\} < \infty\big\}.$$

It is easy to see that $C_p(X)$ is a Banach space with respect to the norm topology $\parallel \cdot \parallel_p$. Let $\mathcal{M}^s_p(X)$ denote the space of signed measures on $\mathcal{B}(X)$ such that

$$\parallel \mu \parallel_p \equiv \int_X \lambda_p(x)|\mu|(dx) < \infty.$$

It is clear that with respect to this norm topology $\mathcal{M}^s_p(X)$ is also a Banach space. By use of Riesz representation theorem one can justify that the dual of $C_p(X)$ is $\mathcal{M}^s_p(X)$. Thus, in view of Theorem 4.2(a), the reachable set $\mathcal{R}$ is a subset of $L^w_\infty(I, \Pi^2_0(X)) \subset L^w_\infty(I, \mathcal{M}_0(X)) \subset L^w_\infty(I, \mathcal{M}^s_0(X))$ where $\Pi^2_0(X) \subset \mathcal{M}_0(X)$ is the space of probability measures possessing finite second moments. We can use this observation to consider several other control problems.

Problem 4: A problem similar to the classical control problem can be stated as follows. Suppose a desired path $\mu^d \in L^w_\infty(I, \mathcal{M}_0(X))$ and a measure $\nu \in \mathcal{M}_0(X)$ are given. The objective is to find a control measure $B \in \Gamma$ that forces the system to follow the given path μ^d and reach at time T the target measure ν as close as possible. This problem can be formulated using the Prohorov metric $\varrho_P : \mathcal{M}_0(X) \times \mathcal{M}_0(X) \longrightarrow [0, \infty)$. The cost functional is then defined by

$$J(B) \equiv \int_0^T \alpha\ \varrho_P(\mu^B_t, \mu^d_t)\ \lambda(dt) + \beta\ \varrho_P(\mu^B_T, \nu) \tag{4.14}$$

where $\alpha, \beta \geq 0$ and $\lambda \in M^+_{cabv}(\Sigma)$.

Theorem 4.8. Suppose the assumptions of theorem 4.6 hold and $\alpha, \beta \geq 0$ and $\lambda \in M^+_{cabv}(\Sigma)$. Then there exists an optimal control measure in Γ at which the cost functional (4.14) attains its minimum.

Proof. Define the functional Ψ on $L^w_\infty(I, \mathcal{M}_0(X))$ by

$$\Psi(\mu) \equiv \int_0^T \alpha\ \varrho_P(\mu_t, \mu^d_t)\ \lambda(dt) + \beta\ \varrho_P(\mu_T, \nu). \tag{4.15}$$

By virtue of Theorem 4.3 it is clear from the proof of Theorem 4.6, that weak lower semicontinuity of $B \to J(B)$ on $\Gamma \subset M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$ is equivalent to weak star lower semicontinuity of Ψ on $\mathcal{R} \subset L^w_\infty(I, \mathcal{M}_0(X))$. Thus it suffices to prove the later. Since the Prohorov metric topology is equivalent to the topology of weak (actually weak star) convergence, it is immediate that Ψ is lower semicontinuous with respect to the weak star topology on $L^w_\infty(I, \mathcal{M}_0(X))$. By Theorem 4.6, $\mathcal{R} \subset L^w_\infty(I, \mathcal{M}_0(X))$ is weak star sequentially compact and therefore Ψ attains its minimum on $\mathcal{R}$, and by the equivalence, J attains its minimum on Γ proving existence of optimal control. □

In control theory time optimal control is very interesting. The objective is to achieve certain goal in minimum time or maximize the time to disaster. We consider this in the next problem.

Problem 5 (Time Optimal Control). Considering the system (4.4)–(4.5), suppose the initial measure μ_0 induced by the random element $x(0) \equiv \xi$ has the support $K_0 \subset X$ a closed set. Let $K_1 \subset X$ be another closed set containing K_0 in its interior. It is clear that the support of $\mu^B_t, t \geq 0$, starting from K_0 at time $t = 0$, will evolve with time (shrinking/expanding) and may at some time extend beyond K_1. In other words, some mass of the measure may leak out of K_1. We call this exit time. Our objective is to find a control that maximizes this exit time. Since $t \to \mu^B_t$ is only weakly measurable and not necessarily continuous, we formulate this problem as follows:

$$J(B) \equiv \inf\{t \geq 0 : \int_0^t \mu^B_s(X \setminus K_1)\ ds > 0\}. \tag{4.16}$$

We wish to find a control that maximizes this functional thereby maximizing the exit time. Here we use the convention $\inf(\emptyset) = +\infty$. We consider J to be an extended real valued function on Γ.

Theorem 4.9. Suppose the assumptions of Theorem 4.3 hold, $\{K_0, K_1\}$ are closed subsets of X with K_1 having nonempty interior and $K_0 \subset \text{int}(K_1)$, and that the set

$$T_B \equiv \{t \geq 0 : \int_0^t \mu^B_s(X \setminus K_1)\ ds > 0\} \neq \emptyset$$

for all $B \in \Gamma$. Then there exists a control $B \in \Gamma$ at which J given by the expression (4.16) attains its maximum.

Proof. We show that the functional $B \to J(B)$ given by (4.16) is weakly upper semicontinuous. Let $B_n \xrightarrow{w} B_o$ in $M_{casbsv}(\Sigma, \mathcal{L}(Y,X))$. Since K_1 is a closed set, by Theorem 4.5, we have

$$\underline{\lim}\, \mu_s^{B_n}(X \setminus K_1) \geq \mu_s^{B_o}(X \setminus K_1).$$

Then by careful examination, it is easy to see that

$$\begin{aligned} &\{t \geq 0 : \underline{\lim} \int_0^t \mu_s^{B_n}(X \setminus K_1)\, ds > 0\} \\ &\supseteq \{t \geq 0 : \int_0^t \underline{\lim}\, \mu_s^{B_n}(X \setminus K_1)\, ds > 0\} \\ &\supseteq \{t \geq 0 : \int_0^t \mu_s^{B_o}(X \setminus K_1)\, ds > 0\}. \end{aligned}$$

Hence we have

$$\begin{aligned} \inf\{t \geq 0 : \underline{\lim} \int_0^t \mu_s^{B_n}(X \setminus K_1)\, ds > 0\} \\ \leq \inf\{t \geq 0 : \int_0^t \underline{\lim}\, \mu_s^{B_n}(X \setminus K_1)\, ds > 0\} \\ \leq \inf\{t \geq 0 : \int_0^t \mu_s^{B_0}(X \setminus K_1)\, ds > 0\}. \end{aligned} \tag{4.17}$$

By definition of limit inferior, for every $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that for all $t \geq 0$, and $n \geq n_\varepsilon$,

$$\int_0^t \mu_s^{B_n}(X \setminus K_1) ds \geq -\varepsilon + \underline{\lim} \int_0^t \mu_s^{B_n}(X \setminus K_1) ds, \ \forall\, n \geq n_\varepsilon, \tag{4.18}$$

and consequently, for all $n \geq n_\varepsilon$,

$$\begin{aligned} J(B_n) &\equiv \inf\{t \geq 0 : \int_0^t \mu_s^{B_n}(X \setminus K_1)\, ds > 0\} \\ &\leq \inf\{t \geq 0 : -\varepsilon + \underline{\lim} \int_0^t \mu_s^{B_n}(X \setminus K_1)\, ds > 0\}. \end{aligned}$$

Since this is valid for all $n \geq n_\varepsilon$, and $\varepsilon(> 0)$ can be chosen arbitrarily small, from this it is easy to justify that

$$\overline{\lim}\, J(B_n) \leq \inf\{t \geq 0 : \underline{\lim} \int_0^t \mu_s^{B_n}(X \setminus K_1)\, ds > 0\}. \tag{4.19}$$

Now using (4.17) in (4.19) we obtain $\overline{\lim}\, J(B_n) \leq J(B_o)$ proving upper semicontinuity of the map $B \longrightarrow J(B)$. Since Γ is weakly compact, J attains its maximum on Γ. This completes the proof. □

Problem 6 (Control of Hausdorff Dimension). In applications it is often desirable to determine the complexity of the support of the measure, in particular, when the probability measures are defined on infinite dimensional spaces. Usually the complexity can be measured in terms of Hausdorff (or fractal) dimension. Clearly, the larger the dimension, the greater is the complexity. For example, in the study of stability, often Hausdorff dimension is used to quantify the complexity of attractors and invariant sets etc.

Here we wish to formulate an objective functional that includes a measure of complexity of the support and then find a control that minimizes such a functional. Let $\mathcal{K}(X)$ denote the hyper space of compact subsets (including the empty set) of the separable Hilbert space X and suppose it is furnished with the metric topology determined by the well known Hausdorff metric ρ_H. It is well known that $(\mathcal{K}(X), \rho_H)$ is a Polish space. Let $d_H : \mathcal{K}(X) \longrightarrow [0, \infty]$ denote the Hausdorff dimension function which is a nonnegative extended real valued set function. Consider the functional

$$h(B, C) \equiv d_H(C) + (\beta/T) \int_0^T \mu_t^B(X \setminus C)\lambda(dt) \tag{4.20}$$

defined on $\Gamma \times \mathcal{K}(X)$ where β is a large positive number, $\lambda \in M^+_{cabv}(\Sigma)$, and Γ is the class of admissible controls contained in $M_{casbsv}(\Sigma, \mathcal{L}(Y, X))$. Then we define the cost functional as

$$J(B) \equiv \inf\{h(B, C), C \in \mathcal{K}(X)\}. \tag{4.21}$$

We must choose $B \in \Gamma$ that minimizes this functional.

It is known that the Hausdorff dimension function is not lower semicontinuous with respect to the metric ρ_H. In fact it is a Baire class 2 function [Mattila and Mauldin, 16] which is highly discontinuous. Thus our technique will not work for this functional. So as in [Ahmed, 17, p. 204], we replace the Hausdorff dimension function $d_H(\cdot)$ by any other suitable set function $\eta : \mathcal{K}(X) \longrightarrow [0, \infty]$ satisfying certain properties similar to the Hausdorff dimension function. And in place of the expression (4.20) we use the following functional

$$\ell(B, C) \equiv \eta(C) + (\beta/T) \int_0^T \mu_t^B(X \setminus C)\lambda(dt) \tag{4.22}$$

and define the cost functional by

$$J(B) \equiv \inf\{\ell(B, C), C \in \mathcal{K}(X)\}. \tag{4.23}$$

Let $\mathcal{N}$ denote the class of sets comprised of singletons, finite sets and the empty set. We introduce the following assumptions on the set function η:

(P1): $\eta(F) = 0$, for all $F \in \mathcal{N}$, $\eta(C_1) \le \eta(C_2)$ for all $C_1, C_2 \in \mathcal{K}(X), C_1 \subset C_2$.
(P2): η is coercive with respect to Hausdorff dimension in the sense that $\lim_{d_H(C) \to \infty} \eta(C) = \infty$.

Theorem 4.10. Suppose the assumptions of Theorem 4.6 hold and the set function η satisfies the properties (P1) and (P2). Then, there exists a control measure in Γ at which J attains its minimum.

Proof. For the proof we follow the same procedure as in [17, Theorem 7.8.8]. First we prove that for every $B \in \Gamma$, the function $C \longrightarrow \ell(B,C)$ is lower semicontinuous on $(\mathcal{K}(X), \rho_H)$. Since $\ell(B,C) \leq \eta(C) + (\beta/T)\lambda(I)$ for all $C \in \mathcal{K}(X)$ and for all $B \in \Gamma$, it is clear that $C \longrightarrow \ell(B,C)$ is coercive. By assumption η is lower semicontinuous. So it suffices to verify that the second term of the expression (4.22) is lower semicontinuous. Let $\{C_n, C_o\} \in \mathcal{K}(X)$ and suppose $C_n \xrightarrow{\rho_H} C_o$. Then for every $\varepsilon > 0$, there exists an integer n_ε such that $C_n \subset C_o^\varepsilon$ for all $n \geq n_\varepsilon$ where $C_o^\varepsilon \equiv \{x \in X : d(x, C_o) \leq \varepsilon\}$. Clearly, for all $n \geq n_\varepsilon$ we have

$$(\beta/T) \int_0^T \mu_t^B(X \setminus C_o^\varepsilon)\lambda(dt) \leq (\beta/T) \int_0^T \mu_t^B(X \setminus C_n)\lambda(dt)$$

and hence

$$(\beta/T) \int_0^T \mu_t^B(X \setminus C_o^\varepsilon)\lambda(dt) \leq \underline{\lim}\Big\{(\beta/T) \int_0^T \mu_t^B(X \setminus C_n)\lambda(dt)\Big\}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from this that the second component of (4.22) is lower semicontinuous on $(\mathcal{K}(X), \rho_H)$. Thus for each B, the functional $C \longrightarrow \ell(B,C)$ is lower semicontinuous in the Hausdorff metric topology. Clearly, $\ell(B,C) \geq 0$. It is clear from these facts that for each fixed $B \in \Gamma$, $\ell(B, \cdot)$ attains its minimum. This shows that $J(B)$ given by (4.23) is well defined. Now let $\{B_n\} \subset \Gamma$ be a minimizing sequence for J. From the above analysis there exists a corresponding sequence $\{C_n\} \subset \mathcal{K}(X)$ such that $J(B_n) = \ell(B_n, C_n)$. Since $\{B_n, C_n\} \subset \Gamma \times \mathcal{K}(X)$ is the minimizing sequence there exists a set $D \in \mathcal{K}(X)$ such that $C_n \subset D$ for all $n \in N$. It is clear that $(\mathcal{K}(D), \rho_H)$, furnished with the Hausdorff metric topology, is a compact Polish space. Since $\{B_n, C_n\} \subset \Gamma \times \mathcal{K}(D)$, there exists a subsequence, relabeled as the original sequence, and $\{B_o, C_o\} \in \Gamma \times \mathcal{K}(D)$ such that $B_n \xrightarrow{w} B_o$, and $C_n \xrightarrow{\rho_H} C_o$. Hence, for every $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that $C_n \subset C_o^\varepsilon$ for all $n \geq n_\varepsilon$ where $C_o^\varepsilon \equiv \{x \in X : d(x, C_o) \leq \varepsilon\}$. Let $\{\mu^n, \mu^o\}$ denote the measures corresponding to $\{B_n, B_o\}$ respectively. By weak convergence of μ^n to μ^o (along a subsequence if necessary), it is easy to see that

$$\mu_t^o(X \setminus C_o^\varepsilon) \leq \underline{\lim}\, \mu_t^n(X \setminus C_o^\varepsilon) \leq \underline{\lim}\, \mu_t^n(X \setminus C_n). \tag{4.24}$$

Integrating with respect to the positive measure λ and using Fatou Lemma, it follows from the above inequality that

$$\int_0^T \mu_t^o(X \setminus C_o^\varepsilon)\ \lambda(dt) \leq \underline{\lim} \int_0^T \mu_t^n(X \setminus C_n)\ \lambda(dt). \tag{4.25}$$

By our assumption, η is lower semicontinuous on $(\mathcal{K}(X), \rho_H)$. Hence we have

$$\begin{aligned} \eta(C_o) + (\beta/T) \int_0^T \mu_t^o(X \setminus C_o^\varepsilon)\ \lambda(dt) \\ \leq \underline{\lim}\Big\{\eta(C_n) + (\beta/T) \int_0^T \mu_t^n(X \setminus C_n)\ \lambda(dt)\Big\} \end{aligned} \tag{4.26}$$

for every $\varepsilon > 0$. Since $\varepsilon(> 0)$ is otherwise arbitray, it follows from (4.26) that

$$J(B_o) \leq \underline{\lim}\, J(B_n).$$

Thus J is weakly lower semicontinuous and since Γ is weakly compact, it attains its minimum on Γ. This completes the proof. □

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