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POSITIVE SOLUTIONS FOR NONLINEAR NEUMANN EIGENVALUE PROBLEMS

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ABSTRACT. We consider a parametric nonlinear Neumann problem driven by the p-Laplacian plus an L^∞-potential. We study the dependence of positive solutions on the parameter $\lambda > 0$, when the reaction term has a superdiffusive kind of behaviour. We prove a bifurcation type theorem, showing the existence of a critical parameter value $\lambda^* > 0$, such that for $\lambda > \lambda^*$, the problem has at least two positive solutions, for $\lambda = \lambda^*$ the problem has at least one positive solution and finally for $\lambda \in (0, \lambda^*)$, no positive solution exists.

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Neumann eigenvalue problem:

$$(P)_\lambda \qquad \begin{cases} -\Delta_p u(z) + \beta(z)\big|u(z)\big|^{p-2}u(z) \;=\; \lambda f\big(z, u(z)\big) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} \;=\; 0 \quad \text{on } \Omega. \end{cases}$$

Here Δ_p stands for the p-Laplace differential operator, defined by

$$\Delta_p u(z) \;=\; \operatorname{div}\big(\|\nabla u(z)\|^{p-2}\nabla u(z)\big) \quad \forall u \in W^{1,p}(\Omega)$$

(with $1 < p < +\infty$). Also, $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$, $\beta \in L^\infty(\Omega)$ is a potential function and $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory reaction (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous). For the potential function β, we require that the corresponding nonlinear Neumann eigenvalue problem with potential β of the p-Laplacian, has a positive principal eigenvalue $\widehat{\lambda}_1(\beta)$. Our aim is to determine the dependence of the positive solutions on the parameter $\lambda > 0$. This problem was investigated in the context of semilinear (i.e., $p = 2$) and nonlinear (i.e., $p \neq 2$) Dirichlet eigenvalue problems by Delgado-Suárez [4], Maya-Shivaji [17], Rabinowitz [22] (semilinear Dirichlet problems) and by Brock-Itturiaga-Ubilla [3], Dong [5], Guo [11], Hu-Papageorgiou

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[13], Perera [21], Takeuchi [23, 24] (nonlinear Dirichlet problems). Delgado-Suárez [4] and Takeuchi [23, 24] deal with logistic equations of superdiffusive type and so their reaction term has the special form:

$$\lambda\zeta^{q-1}(1-\zeta^r) \quad \forall \zeta \geqslant 0,$$

with $1 < p < q$, $r > 0$. In addition Takeuchi [23, 24] requires that $p \geqslant 2$. Hu-Papageorgiou [13] and Perera [21] extend to p-Laplace equations the work of Maya-Shivaji [17] and also relax significantly the hypotheses on the reaction $f(z,\zeta)$. Moreover, in Hu-Papageorgiou [13] the primitive of the reaction is nonsmooth and so the problem is multivalued (hemivariational inequality). The approach in Hu-Papageorgiou [13] is degree theoretic based on the degree theory for operators of monotone type. The work of Dong [5], extends to p-Laplacian equations the semilinear result of Rabinowitz [22]. We emphasize that none of the aforementioned works, proves a bifurcation theorem describing the precise dependence of the positive solutions on the parameter $\lambda > 0$. They show that there is a parameter value $\overline{\lambda} > 0$, such that for all $\lambda > \overline{\lambda}$, problem $(P)_\lambda$ has at least two solutions. They do not show the optimality of $\overline{\lambda} > 0$, i.e., that below it, no positive solution exists and in addition they do not study what happens when $\lambda = \overline{\lambda}$. Only Brock-Itturiaga-Ubilla [3] have such a bifurcation result but under stronger hypotheses on the reaction $f(z,\zeta)$. Namely $f(z,\zeta) > 0$ for almost all $z \in \Omega$ and all $\zeta > 0$ (see the proofs of Lemmata 3.1 and 3.2) and that $f(z,\cdot)$ is $(p-1)$-sublinear near $+\infty$ (see H_4 and $H_5(i)$). To the best of our knowledge no such results exist for the Neumann problems. As for some other multiplicity results for the Neumann problems we refer to the works of Gasiński-Papageorgiou [7, 9, 8, 10].

Our approach is variational bases on the critical point theory, coupled with suitable truncation techniques.

2. MATHEMATICAL PRELIMINARIES AND HYPOTHESES

Suppose that X is a Banach space and X^* is its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). For a given $\varphi \in C^1(X)$, we say that φ satisfies the Palais-Smale condition, if the following is true:

"Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\big\{\varphi(x_n)\big\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence."

Using this compactness type condition, we can have the following minimax theorem, known in the literature as the "mountain pass theorem".

Theorem 2.1. *If φ satisfies the Palais-Smale condition, $x_0, x_1 \in X$, $\varrho > 0$, $\|x_1 - x_0\| > \varrho$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \ < \ \inf\big\{\varphi(x) : \ \|x - x_0\| = \varrho\big\} \ = \ \eta_\varrho$$

and

$$c \ = \ \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma \ = \ \big\{\gamma \in C\big([0,1]; X\big) : \ \gamma(0) = x_0, \ \gamma(1) = x_1\big\},$$

then $c \geqslant \eta_\varrho$ and c is a critical value of φ.

In the analysis of problem $(P)_\lambda$, in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the ordered Banach space $C^1_n(\overline{\Omega})$, defined by

$$C^1_n(\overline{\Omega}) \ = \ \left\{u \in C^1(\overline{\Omega}) : \ \frac{\partial u}{\partial n}(z) = 0 \text{ for all } z \in \partial\Omega\right\}.$$

One can show that

$$W^{1,p}(\Omega) \ = \ \overline{C^1_n(\overline{\Omega})}^{\|\cdot\|},$$

where $\|\cdot\|$ denotes the usual norm of the Sobolev space $W^{1,p}(\Omega)$. The space $C^1_n(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ \ = \ \big\{u \in C^1_n(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\big\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ \ = \ \big\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\big\}.$$

Let $\beta \in L^\infty(\Omega)$ and consider the following nonlinear "weighted" eigenvalue problem:

$$\text{(2.1)} \qquad \begin{cases} -\Delta_p u(z) + \beta(z)\big|u(z)\big|^{p-2}u(z) \ = \ \lambda\big|u(z)\big|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} \ = \ 0 \quad \text{on } \Omega. \end{cases}$$

We point out that the potential function β may change sign. Problem (2.1) was studied in details by Mugnai-Papageorgiou [20]. Among other things, they proved that problem (2.1) has a smallest eigenvalue $\widehat{\lambda}_1(\beta)$ which is isolated, simple and admits the following variational characterization:

$$\text{(2.2)} \qquad \widehat{\lambda}_1(\beta) \ = \ \inf\left\{\frac{\sigma(u)}{\|u\|_p^p} : \ u \in W^{1,p}(\Omega), \ u \neq 0\right\},$$

where $\sigma \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ is defined by

$$\sigma(u) \ = \ \|\nabla u\|_p^p + \int_\Omega \beta(z)\big|u(z)\big|^p \, dz \quad \forall u \in W^{1,p}(\Omega).$$

The infimum in (2.2) is attained at the L^p-normalized eigenfunction $\widehat{u}_1$ (i.e., $\|\widehat{u}_1\|_p = 1$), which corresponds to $\widehat{\lambda}_1(\beta)$ (recall that $\widehat{\lambda}_1(\beta)$ is simple). It is clear that we can always assume that $\widehat{u}_1 \geqslant 0$ (note that in (2.2) we can replace u by $|u|$). Nonlinear regularity theory and the nonlinear maximal principle of Vázquez [25], imply that $\widehat{u}_1 \in \operatorname{int} C_+$. For details and generalizations, we refer to Mugnai-Papageorgiou [20].

The hypotheses on β are the following:

$\underline{H(\beta)}$: $\beta \in L^\infty(\Omega)$, $\widehat{\lambda}_1(\beta) > 0$.

Remark 2.2. If $\beta \in L^\infty(\Omega)$ and $\beta(z) \geqslant 0$ for almost all $z \in \Omega$, $\beta \neq 0$, then by virtue of Lemma 1 of Iannizzotto-Papageorgiou [15], we have that $\widehat{\lambda}_1(\beta) > 0$. But also sign changing potentials β can give $\widehat{\lambda}_1(\beta) > 0$.

The hypotheses on the reaction f are the following:

$\underline{H(f)}$: $f\colon \Omega \times \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i): there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in [p, p^*)$, such that

$$\big|f(z,\zeta)\big| \ \leqslant\ a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0,$$

where

$$p^* \ =\ \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geqslant N; \end{cases}$$

(ii): we have that

$$\limsup_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{p-1}} \ \leqslant\ 0 \quad \text{uniformly for almost all } z \in \Omega$$

and there exists $v_0 \in L^r(\Omega)$, such that $v_0(z) \geqslant 0$ for almost all $z \in \Omega$, $v_0 \neq 0$ and

$$\int_\Omega F\big(z, v_0(z)\big)\, dz \ >\ 0,$$

where

$$F(z,\zeta) \ =\ \int_0^\zeta f(z,s)\, ds;$$

(iii): $\displaystyle\lim_{\zeta \to 0^+} \frac{f(z,\zeta)}{\zeta^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iv): there exists $\tau > p$, such that for almost all $z \in \Omega$, the function $\zeta \longmapsto \frac{f(z,\zeta)}{\zeta^{\tau-1}}$ is strictly decreasing on $(0, +\infty)$;

(v): there exists $q > p$, such that for every $\varrho > 0$, we can find $\gamma_\varrho > 0$ for which we have that for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta) + \gamma_\varrho \zeta^{q-1}$ is nondecreasing on $[0, \varrho]$.

Remark 2.3. Since we are interested in positive solutions and all the above hypotheses concern only positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may (and will) assume that $f(z,\zeta) = 0$ for almost all $z \in \Omega$ and all $\zeta \leqslant 0$. As we illustrate in the examples that follow, these hypotheses incorporate as special cases important classes of nonlinearities, such as superdiffusive reactions (see Takeuchi

[23, 24]). Also, in contrast to Rabinowitz [22] and Dong [5], we do not require the existence of $\xi > 0$, such that $f(z,\zeta) < 0$ for almost all $z \in \Omega$, all $\zeta \geqslant \xi$.

Example 2.4. The following functions satisfy hypotheses $H(f)$. For the sake of simplicity, we drop the z-dependence.

$$f_1(\zeta) \;=\; \zeta^{q-1}(1-\zeta^{\eta}) \quad \forall \zeta \geqslant 0,$$

with $p < q$, $\eta > 0$ and $q + \eta < p^*$,

$$f_2(\zeta) \;=\; c\zeta^{p-1}\ln(1+\zeta) - \zeta^{q-1} \quad \forall \zeta \geqslant 0,$$

with $p < q < p^*$, $c > 0$,

$$f_3(\zeta) \;=\; \begin{cases} \zeta^{p-1}\ln(1+\zeta) & \text{if} \quad \zeta \in [0,1], \\ c\zeta^{\eta-1} & \text{if} \quad \zeta > 1, \end{cases}$$

with $1 < \eta < p$, $c = \ln 2 > 0$.

Note that f_1 is the reaction of superdiffusive logistic equations. Such equations arise in models of mathematical biology (see Gurtin-Mac Camy [12]). For the p-Laplacian, they were studied by Takeuchi [23, 24], with $p \geqslant 2$.

From Aizicovici-Papageorgiou-Staicu [2] (Proposition 3), we have

Proposition 2.5. *If $u_1, u_2 \in \operatorname{int} C_+$ with $u_1 \leqslant u_2$, $h_1, h_2 \in L^\infty(\Omega)$, $h_1 \leqslant h_2$, $\widehat{\xi} > 0$ and $p < q$ satisfy*

$$-\Delta_p u_k(z) + \beta(z)u_k(z)^{p-1} + \widehat{\xi} u_k(z)^{q-1} \;=\; h_k(z) \quad \textit{in } \Omega,\ k = 1,2$$

and for every nonempty, compact $K \subseteq \Omega$, we can find $\gamma_K > 0$, such that

$$\gamma_K \;\leqslant\; h_2(z) - h_1(z) \quad \textit{for almost all } z \in K,$$

then $u_2 - u_1 \in \operatorname{int} C_+$.

By a positive solution of $(P)_\lambda$, we mean a function $u \in W^{1,p}(\Omega) \setminus \{0\}$, such that $u(z) \geqslant 0$ for almost all $z \in \Omega$, which is a weak solution of $(P)_\lambda$. From nonlinear regularity (see Hu-Papageorgiou [14] and Lieberman [16]), we have that $u \in C_+ \setminus \{0\}$ and

$$-\Delta_p u(z) + \beta(z)u(z)^{p-1} \;=\; \lambda f\big(z, u(z)\big) \quad \text{for almost all } z \in \Omega.$$

Let $\varrho = \|u\|_\infty$ and let q and $\gamma_\varrho > 0$ be as postulated by hypothesis $H(f)(v)$. Then

$$-\Delta_p u(z) + \beta(z)u(z)^{p-1} + \lambda\gamma_\varrho u(z)^{q-1} \;=\; \lambda\big(f\big(z,u(z)\big) + \gamma_\varrho u(z)^{q-1}\big) \;\geqslant\; 0$$

for almost all $z \in \Omega$, so

$$\Delta_p u(z) \;\leqslant\; \big(\|\beta\|_\infty + \lambda\gamma_\varrho \varrho^{q-p}\big)u(z)^{p-1} \quad \text{for almost all } z \in \Omega,$$

so $u \in \operatorname{int} C_+$ (see Vázquez [25]).

Therefore, every positive solution of $(P)_\lambda$ belongs in $\operatorname{int} C_+$.

As we already indicated, by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. Also, if $\zeta \in \mathbb{R}$, then

$$\zeta^+ = \max\{\zeta, 0\} \quad \text{and} \quad \zeta^- = \max\{-\zeta, 0\}.$$

For every $u \in W^{1,p}(\Omega)$, we set

$$u^+(\cdot) = u(\cdot)^+ \quad \text{and} \quad u^-(\cdot) = u(\cdot)^-.$$

We know that $u^+, u^- \in W^{1,p}(\Omega)$ and $u = u^+ - u^-$, $|u| = u^+ + u^-$. By $|\cdot|_N$ we denote the Lebesgue measure in $\mathbb{R}^N$. For any $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ measurable, we define

$$N_h(u)(\cdot) = h\big(\cdot, u(\cdot)\big) \quad \forall u \in W^{1,p}(\Omega).$$

Finally $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ is the nonlinear map, defined by

$$\big\langle A(u), y\big\rangle = \int_\Omega \|\nabla u\|^{p-2}(\nabla u, \nabla y)_{\mathbb{R}^N}\, dz \quad \forall u, y \in W^{1,p}(\Omega).$$

From Iannizzotto-Papageorgiou [15, Proposition 2], we know that A is maximal monotone and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W^{1,p}(\Omega)$ and

$$\limsup_{n\to+\infty} \big\langle A(u_n), u_n - u\big\rangle \leqslant 0,$$

then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$.

3. BIFURCATION-TYPE RESULT

In this section, we study the dependence on the parameter $\lambda > 0$ of the positive solutions of $(P)_\lambda$. At the end, we have a bifurcation-type result describing this dependence.

Let

$$\mathcal{Y} = \big\{\lambda > 0 : \text{ problem } (P)_\lambda \text{ has a positive solution}\big\}.$$

We set $\lambda_* = \inf \mathcal{Y}$.

Proposition 3.1. *If hypotheses $H(\beta)$ and $H(f)$ hold, then $\lambda_* > 0$.*

Proof. Hypotheses $H(f)(i), (ii)$ and (iii) imply that we can find $c_1 > 0$, such that

$$f(z,\zeta) \leqslant c_1\zeta^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{3.1}$$

Let $\lambda \in \mathcal{Y}$. Then problem $(P)_\lambda$ has a solution $u \in \operatorname{int} C_+$. We have

$$\sigma(u) = \lambda \int_\Omega f(z,u)u\, dz \leqslant \lambda c_1 \|u\|_p^p \tag{3.2}$$

(see (3.1)). Suppose that $\lambda \in \big(0, \frac{\widehat{\lambda}_1(\beta)}{c_1}\big)$ (see hypotheses $H(\beta)$). Then from (3.2), we have

$$\sigma(u) < \widehat{\lambda}_1(\beta)\|u\|_p^p,$$

which contradicts (2.2). Hence $\lambda_* \geqslant \frac{\widehat{\lambda}_1(\beta)}{c_1} > 0$. ∎

Proposition 3.2. *If hypotheses $H(\beta)$ and $H(f)$ hold, then $\mathcal{Y} \neq \emptyset$. Moreover, if $\lambda \in \mathcal{Y}$, $\mu > \lambda$, then $\mu \in \mathcal{Y}$.*

Proof. By virtue of hypotheses $H(f)(i)$ and (ii), for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$, such that

$$F(z,\zeta) \ \leqslant\ \frac{\varepsilon}{p}\zeta^p + c_\varepsilon \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \geqslant 0. \tag{3.3}$$

Let $\varphi_\lambda \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem $(P)_\lambda$, defined by

$$\varphi_\lambda(u) \ =\ \frac{1}{p}\sigma(u) - \lambda \int_\Omega F\big(z,u(z)\big)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1\big(W^{1,p}(\Omega)\big)$ and for all $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\varphi_\lambda(u) \ &\geqslant\ \frac{1}{p}\sigma(u) - \frac{\lambda\varepsilon}{p}\|u^+\|_p^p - \lambda c_\varepsilon|\Omega|_N \\ &\geqslant\ \frac{\widehat{\lambda}_1(\beta) - \lambda\varepsilon}{p}\|u\|_p^p - \lambda c_\varepsilon|\Omega|_N\end{aligned}$$

(see (3.3) and (2.2)). Choosing $\varepsilon \in \big(0, \frac{\widehat{\lambda}_1(\beta)}{\lambda}\big)$, we obtain

$$\varphi_\lambda(u) \ \geqslant\ c_2\|u\|_p^p - \lambda c_\varepsilon|\Omega|_N \quad \forall u \in W^{1,p}(\Omega), \tag{3.4}$$

for some $c_2 > 0$.

Using (3.4), we can show that φ_λ is coercive. We argue by contradiction. So, suppose that φ_λ is not coercive. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ and $M > 0$, such that

$$\|u_n\| \ \longrightarrow\ +\infty \quad \text{and} \quad \varphi_\lambda(u_n) \ \leqslant\ M \quad \forall n \geqslant 1. \tag{3.5}$$

From (3.4) and (3.5) it follows that the sequence $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega)$ is bounded. Hence $\|\nabla u_n\|_p \longrightarrow +\infty$ (see (3.5)). We have

$$\frac{1}{p}\|\nabla u_n\|_p^p \ \leqslant\ \frac{1}{p}\big(\|\beta\|_\infty + \lambda\varepsilon\big)c_3 + c_4 \quad \forall n \geqslant 1$$

for some $c_3, c_3 > 0$ (see (3.3)), so

$$\text{the sequence } \{\nabla u_n\}_{n\geqslant 1} \subseteq\ L^p(\Omega;\mathbb{R}^N) \text{ is bounded,}$$

a contradiction. This proves that φ_λ is coercive. Also, exploiting the compactness of the embedding $W^{1,p}(\Omega) \subseteq L^p(\Omega)$, we can easily check that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$, such that

$$\varphi_\lambda(u_0) \ =\ \inf_{u\in W^{1,p}(\Omega)} \varphi_\lambda(u). \tag{3.6}$$

Consider the integral functional $I_F \colon L^r(\Omega) \longrightarrow \mathbb{R}$, defined by

$$I_F(v) \ =\ \int_\Omega F\big(z,v(z)\big)\,dz \quad \forall v \in L^r(\Omega).$$

By virtue of Krasnoselskii's theorem (see e.g., Gasiński-Papageorgiou [6, p. 407]), we have that I_F is continuous. Also, by hypothesis $H(f)(ii)$, $I_F(v_0) > 0$. Since $W^{1,p}(\Omega)$ is dense in $L^r(\Omega)$, we can find $\widehat{v} \in W^{1,p}(\Omega)$, such that $I_F(\widehat{v}) > 0$. Therefore, for large $\lambda > 0$, we will have

$$\varphi_\lambda(u_0) \leqslant \frac{1}{p}\sigma(\widehat{v}) - \lambda I_F(\widehat{v}) < 0 = \varphi_\lambda(0)$$

(see (3.6)), so $u_0 \neq 0$.

From ((3.6)), we have

$$\varphi'_\lambda(u_0) = 0,$$

so

$$A(u_0) + \beta|u_0|^{p-2}u_0 = \lambda N_f(u_0). \tag{3.7}$$

Acting on (3.7) with $-u_0^- \in W^{1,p}(\Omega)$, we obtain $u_0 \geqslant 0$, $u_0 \neq 0$. So, from (3.7), we have

$$\begin{cases} -\Delta_p u_0(z) + \beta(z) u_0(z)^{p-1} = \lambda f\big(z, u_0(z)\big) & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial n} = 0 \quad \text{on } \Omega \end{cases}$$

(see Motreanu-Papageorgiou [19]), so $\lambda \in \mathcal{Y}$ for large $\lambda > 0$ and so $\mathcal{Y} \neq \emptyset$.

Next suppose that $\lambda \in \mathcal{Y}$ and let $u_\lambda \in \operatorname{int} C_+$ be a positive solution of $(P)_\lambda$. For $\mu > \lambda$, let $\vartheta \in (0,1)$ be such that $\lambda = \vartheta^{\tau-p}\mu$ with $\tau > p$ as in hypothesis $H(f)(iv)$. Let $\underline{u} = \vartheta u_\lambda \in \operatorname{int} C_+$. We have

$$\begin{aligned} -\Delta_p \underline{u}(z) + \beta(z)\underline{u}(z)^{p-1} &= \lambda \vartheta^{p-1} f\big(z, u_\lambda(z)\big) \\ &\leqslant \vartheta^{\tau-1}\mu f\big(z, u_\lambda(z)\big) \\ &\leqslant \mu f\big(z, \underline{u}(z)\big) \quad \text{for almost all } z \in \Omega \end{aligned} \tag{3.8}$$

(see hypotheses $H(f)(iv)$). We introduce the following truncation of the reaction $f(z,\zeta)$:

$$g(z,\zeta) = \begin{cases} f\big(z, \underline{u}(z)\big) & \text{if } \zeta \leqslant \underline{u}(z), \\ f(z,\zeta) & \text{if } \underline{u}(z) < \zeta. \end{cases} \tag{3.9}$$

This is a Carathéodory function. We set

$$G(z,\zeta) = \int_0^\zeta g(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_\mu \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}_\mu(u) = \frac{1}{p}\sigma(u) - \mu \int_\Omega G\big(z, u(z)\big)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

We have

$$\widehat{\varphi}_\mu(u) \geqslant \frac{1}{p}\sigma(u) - \mu \int_{\{\underline{u} \leqslant u\}} F\big(z, u(z)\big)\,dz - c_5 \quad \forall u \in W^{1,p}(\Omega), \tag{3.10}$$

for some $c_5 > 0$ (see (3.9) and hypothesis $H(f)(i)$).

From (3.10), as before using (3.3) with $\varepsilon \in \big(0, \frac{\widehat{\lambda}_1(\beta)}{\mu}\big)$, we show that $\widehat{\varphi}_\mu$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$, such that

$$\widehat{\varphi}_\mu(\widehat{u}) \;=\; \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_\mu(u),$$

so

$$\widehat{\varphi}'_\mu(\widehat{u}) \;=\; 0$$

and thus

$$A(\widehat{u}) + \beta|\widehat{u}|^{p-2}\widehat{u} \;=\; \mu N_g(\widehat{u}). \tag{3.11}$$

On (3.11) we act with $(\underline{u} - \widehat{u})^+ \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned}\big\langle A(\widehat{u}),\ (\underline{u}-\widehat{u})^+\big\rangle + \int_\Omega \beta|\widehat{u}|^{p-2}\widehat{u}(\underline{u}-\widehat{u})^+\,dz \\ = \mu\int_\Omega f(z,\underline{u})(\underline{u}-\widehat{u})^+\,dz \\ \geqslant \big\langle A(\underline{u}),(\underline{u}-\widehat{u})^+\big\rangle + \int_\Omega \beta\underline{u}^{p-1}(\underline{u}-\widehat{u})^+\,dz\end{aligned}$$

(see (3.9) and (3.8)), so

$$\big\langle A(\underline{u}) - A(\widehat{u}), (\underline{u}-\widehat{u})^+\big\rangle + \int_\Omega \beta\big(\underline{u}^{p-1} - |\widehat{u}|^{p-2}\widehat{u}\big)(\underline{u}-\widehat{u})^+\,dz \;\leqslant\; 0$$

and thus

$$\big|\{\underline{u} > \widehat{u}\}\big|_N \;=\; 0,$$

i.e., $\underline{u} \leqslant \widehat{u}$.

Then (3.11) becomes

$$A(\widehat{u}) + \beta\widehat{u}^{p-1} \;=\; \mu N_f(\widehat{u})$$

(see (3.9)), so

$$\begin{cases} -\Delta_p\widehat{u}(z) + \beta(z)\big|\widehat{u}(z)\big|^{p-1} \;=\; \mu f\big(z,\widehat{u}(z)\big) & \text{in } \Omega, \\ \dfrac{\partial\widehat{u}}{\partial n} \;=\; 0 \quad \text{on } \Omega \end{cases}$$

(see Motreanu-Papageorgiou [19]) and thus

$$\widehat{u} \in \operatorname{int} C_+ \text{ is a positive solution of } (P)_\mu$$

and so $\mu \in \mathcal{Y}$. $\square$

Proposition 3.3. *If hypotheses $H(\beta)$ and $H(f)$ hold and $\lambda > \lambda_*$, then problem $(P)_\lambda$ has at least two nontrivial positive smooth solutions*

$$u_0, \widehat{u} \;\in\; \operatorname{int} C_+, \quad u_0 \leqslant \widehat{u}, \quad u_0 \neq \widehat{u}.$$

Proof. Let $\eta \in (\lambda_*, \lambda) \cap \mathcal{Y}$ and let $u_\eta \in \operatorname{int} C_+$ be a positive solution of problem $(P)_\eta$. Let $\vartheta \in (0,1)$ be such that $\eta = \vartheta^{\tau-p}\lambda$ ($\tau > p$ is as in hypothesis $H(f)(iv)$). Let $\underline{u} = \vartheta u_\eta \in \operatorname{int} C_+$. As in the proof of Proposition 3.2, we truncate $f(z,\cdot)$ at $\underline{u}(z)$, introduce the corresponding C^1-functional $\widehat{\varphi}_\lambda \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ and via the direct method, we obtain $u_0 \in \operatorname{int} C_+$, such that

$$\widehat{\varphi}_\lambda(u_0) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\varphi}_\lambda(u) \quad \underline{u} \leqslant u_0 \tag{3.12}$$

and u_0 solves problem $(P)_\lambda$.

Let $\varrho = \|u_0\|_\infty$ and let $\gamma_\varrho > 0$ be as postulated by hypothesis $H(f)(v)$. Then

$$\begin{aligned}
&-\Delta_p \underline{u}(z) + \beta(z)\underline{u}(z)^{p-1} + \eta\gamma_\varrho \underline{u}(z)^{q-1} \\
&= \vartheta^{p-1}\eta f\big(z, u_\eta(z)\big) + \eta\gamma_\varrho \underline{u}(z)^{q-1} \\
&= \eta\left(\frac{\vartheta^{\tau-1}}{\vartheta^{\tau-p}} f\big(z, u_\eta(z)\big) + \gamma_\varrho \underline{u}(z)^{q-1}\right) \\
&\leqslant \eta\left(\frac{\lambda}{\eta} f\big(z, \underline{u}(z)\big) + \gamma_\varrho \underline{u}(z)^{q-1}\right) \\
&= \lambda f\big(z, \underline{u}(z)\big) + \eta\gamma_\varrho \underline{u}(z)^{q-1} \\
&\leqslant \lambda f\big(z, \underline{u}(z)\big) + \lambda\gamma_\varrho \underline{u}(z)^{q-1} \\
&\leqslant \lambda f\big(z, u_0(z)\big) + \lambda\gamma_\varrho u_0(z)^{q-1} \\
&= -\Delta_p u_0(z) + \beta(z) u_0(z)^{p-1} + \lambda\gamma_\varrho u_0(z)^{q-1}
\end{aligned} \tag{3.13}$$

for almost all $z \in \Omega$ (we have used hypothesis $H(f)(iv) - (v)$ and the facts that $\vartheta^{\tau-p} = \frac{\eta}{\lambda}$, $\eta < \lambda$ and $\underline{u} \leqslant u_0$). We set

$$\begin{aligned}
h_1(z) &= \vartheta^{p-1}\eta f\big(z, u_\eta(z)\big) + \eta\gamma_\varrho \underline{u}(z)^{q-1} \\
h_2(z) &= \lambda f\big(z, u_0(z)\big) + \lambda\gamma_\varrho u_0(z)^{q-1}.
\end{aligned}$$

Then $h_1, h_2 \in L^\infty(\Omega)$ (recall that $u_0, \widehat{u}, u_\eta \in \operatorname{int} C_+$). Choose

$$m_0 \in \big(0, \min_{\overline{\Omega}} \underline{u}\big)$$

(recall that $\underline{u} \in \operatorname{int} C_+$). Then

$$\begin{aligned}
&h_1(z) + (\lambda - \eta)\gamma_\varrho m_0^{q-1} \\
&= \vartheta^{p-1}\eta f\big(z, u_\eta(z)\big) + \eta\gamma_\varrho \underline{u}(z)^{q-1} + (\lambda-\eta)\gamma_\varrho m_0^{q-1} \\
&\leqslant \vartheta^{p-1}\eta f\big(z, u_\eta(z)\big) + \eta\gamma_\varrho \underline{u}(z)^{q-1} + (\lambda-\eta)\gamma_\varrho \underline{u}(z)^{q-1} \\
&= \vartheta^{p-1}\eta f\big(z, u_\eta(z)\big) + \lambda\gamma_\varrho \underline{u}(z)^{q-1} \\
&\leqslant \lambda f\big(z, \underline{u}(z)\big) + \lambda\gamma_\varrho \underline{u}(z)^{q-1} \\
&\leqslant \lambda f\big(z, u_0(z)\big) + \lambda\gamma_\varrho u_0(z)^{q-1} \\
&= h_2(z) \quad \text{for almost all } z \in \Omega
\end{aligned}$$

(we have used hypotheses $H(f)(iv)$, (v) and the fact that $\eta < \lambda$), thus

$$(\lambda - \eta) m_0^{q-1} \leqslant (h_2 - h_1)(z) \quad \text{for almost all } z \in \Omega.$$

So, we can apply Proposition 2.5 and infer that

$$u_0 - \underline{u} \in \operatorname{int} C_+. \tag{3.14}$$

Let us set

$$[\underline{u}) = \{u \in W^{1,p}(\Omega) : \underline{u}(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}.$$

From the definition of $\widehat{\varphi}_\lambda$ (see (3.9)), we have

$$\widehat{\varphi}_\lambda\big|_{[\underline{u})} = \varphi_\lambda\big|_{[\underline{u})} - c_6 \tag{3.15}$$

for some $c_6 \in \mathbb{R}$. Then from (3.12), (3.14) and (3.15), it follows that u_0 is a local $C_n^1(\overline{\Omega})$-minimizer of φ_λ, hence from Motreanu-Papageorgiou [19], it follows that u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_λ.

Hypothesis $H(f)(iii)$ implies that for a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that

$$F(z, \zeta) \leqslant \frac{\varepsilon}{p} |\zeta|^p \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta. \tag{3.16}$$

Let $u \in C_n^1(\overline{\Omega})$ and assume that $\|u\|_{C_n^1(\overline{\Omega})} \leqslant \delta$. Then

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \frac{1}{p}\sigma(u) - \frac{\varepsilon\lambda}{p}\|u\|_p^p \\ &\geqslant \frac{1}{p}\big(\widehat{\lambda}_1(\beta) - \varepsilon\lambda\big)\|u\|_p^p \geqslant 0 \end{aligned}$$

(using (3.16), (2.2) and choosing $\varepsilon \in \big(0, \frac{\widehat{\lambda}_1(\beta)}{\lambda}\big)$), so

$$u = 0 \text{ is a local } C_n^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda,$$

so also

$$u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda$$

(see Mugnai-Papageorgiou [20]). Without any loss of generality, we may assume that

$$0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u_0)$$

(the analysis is similar if the opposite inequality holds). In addition, we may assume that $u_0 \in \operatorname{int} C_+$ is an isolated critical point of φ_λ (otherwise, we already have a whole sequence of distinct positive solutions of $(P)_\lambda$ converging in $W^{1,p}(\Omega)$ to u_0). As in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\varrho \in (0, \|u_0\|))$, such that

$$\varphi_\lambda(0) = 0 \leqslant \varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \varrho\} = \eta_\varrho. \tag{3.17}$$

Since φ_λ is coercive (see the proof of Proposition 3.2), it satisfies the Palais-Smale condition. This fact and (3.17), permit the use of the mountain pass theorem (see Theorem 2.1) and we can find $\widehat{u} \in W^{1,p}(\Omega)$, such that

$$\varphi_\lambda(0) \;=\; 0 \;\leqslant\; \varphi_\lambda(u_0) \;<\; \eta_\varrho \;\leqslant\; \varphi_\lambda(\widehat{u}) \tag{3.18}$$

(see (3.17)) and so

$$\varphi'_\lambda(\widehat{u}) \;=\; 0. \tag{3.19}$$

From (3.18), we have that $\widehat{u} \notin \{0, u_0\}$, while from (3.19), it follows that $\widehat{u} \in \operatorname{int} C_+$ solves problem $(P)_\lambda$. So, $\widehat{u}$ is the second nontrivial positive smooth solution of $(P)_\lambda$ distinct from u_0. $\square$

Next we examine what happens at the critical parameter value $\lambda^* > 0$ ("bifurcation point").

Proposition 3.4. *If hypotheses $H(\beta)$ and $H(f)$ hold, then $\lambda^* \in \mathcal{Y}$, i.e., $\mathcal{Y} = [\lambda^*, +\infty)$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{Y}$ be a sequence, such that $\lambda_n \searrow \lambda_*$. Let $u_n = u_{\lambda_n} \in \operatorname{int} C_+$ for $n \geqslant 1$ be the sequence of corresponding positive solutions of problems $(P)_{\lambda_n}$. We have

$$A(u_n) + \beta u_n^{p-1} \;=\; \lambda_n N_f(u_n) \quad \forall n \geqslant 1. \tag{3.20}$$

Hypotheses $H(f)(i)$ and (ii) imply that for a given $\varepsilon > 0$, we can find $\widehat{c}_\varepsilon > 0$, such that

$$f(z,\zeta) \;\leqslant\; \varepsilon(\zeta^+)^{p-1} + \widehat{c}_\varepsilon \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.21}$$

Acting on (3.20) with $u_n \in \operatorname{int} C_+$, we have

$$\begin{aligned}
\sigma(u_n) \;&=\; \lambda_n \int_\Omega f(z,u_n)u_n\,dz \\
&\leqslant\; \lambda_n \varepsilon \|u_n\|_p^p + \widehat{c}_\varepsilon \|u_n\|_p \\
&\leqslant\; \frac{\lambda_n \varepsilon}{\widehat{\lambda}_1(\beta)}\sigma(u_n) + \widehat{c}_\varepsilon \|u_n\|_p
\end{aligned}$$

(see (3.21), (2.2)), so choosing $\varepsilon < \frac{\widehat{\lambda}_1(\beta)}{\lambda_*}$ and since $\lambda_* < \lambda_n$ for all $n \geqslant 1$, we can find $n_0 \geqslant 1$, such that

$$c_7 \sigma(u_n) \;\leqslant\; \widehat{c}_\varepsilon \|u_n\|_p \quad \forall n \geqslant n_0, \tag{3.22}$$

for some $c_7 = c_7(\varepsilon) > 0$.

From (3.22) and (2.2), it follows that the sequence $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega)$ is bounded and then from (3.22) and hypothesis $H(\beta)$, we have that the sequence $\{\nabla u_n\}_{n\geqslant 1} \subseteq$

$L^p(\Omega;\mathbb{R}^N)$ is bounded. So, the sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.23}$$

$$u_n \longrightarrow u_* \quad \text{in } L^p(\Omega), \tag{3.24}$$

with $u_* \geqslant 0$. Acting on (3.20) with $u_n - u_*$, passing to the limit as $n \to +\infty$ and using (3.23), we obtain

$$\lim_{n\to+\infty} \big\langle A(u_n), u_n - u \big\rangle \;=\; 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W^{1,p}(\Omega) \tag{3.25}$$

(since A is of type $(S)_+$; see Iannizzotto-Papageorgiou [15]). Passing to the limit as $n \to +\infty$ in (3.20) and using (3.25), we have

$$A(u_*) + \beta u_*^{p-1} \;=\; \lambda_* N_f(u_*),$$

so $u_* \in C_+$ solves problem $(P)_\lambda$. We need to show that $u_* \neq 0$. Arguing indirectly, suppose that $u_* = 0$. We set

$$y_n \;=\; \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W^{1,p}(\Omega), \tag{3.26}$$

$$y_n \longrightarrow y \quad \text{in } L^p(\Omega). \tag{3.27}$$

From (3.20), we have

$$A(y_n) + \beta y_n^{p-1} \;=\; \frac{\lambda_n N_f(u_n)}{\|u_n\|^{p-1}} \quad \forall n \geqslant 1. \tag{3.28}$$

From hypotheses $H(f)(iii)$, we can find $\delta > 0$, such that

$$\big|f(z,\zeta)\big| \;\leqslant \zeta^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0,\delta].$$

From hypotheses $H(f)(i)$, we have

$$\big|f(z,\zeta)\big| \;\leqslant \widehat{c}\zeta^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant \delta,$$

with some $\widehat{c} > 0$. Therefore, finally

$$\big|f(z,\zeta)\big| \;\leqslant\; c_8\big(|\zeta|^{p-1} + |\zeta|^{r-1}\big) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \tag{3.29}$$

with some $c_8 > 0$. From (3.26) and (3.29), we see that the sequence

$$\left\{\frac{\lambda_n N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geqslant 1} \;\subseteq\; L^{r'}(\Omega)$$

is bounded. So, we may assume that

$$\frac{\lambda_n N_f(u_n)}{\|u_n\|^{p-1}} \longrightarrow h \quad \text{weakly in } L^{r'}(\Omega). \tag{3.30}$$

Using hypothesis $H(f)(iii)$ and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 31), we have

$$h = 0. \tag{3.31}$$

Acting on (3.28) with $y_n - y$, passing to the limit as $n \to +\infty$ and using (3.30), we have

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y \rangle = 0,$$

so

$$y_n \longrightarrow y \quad \text{in } W^{1,p}(\Omega), \tag{3.32}$$

hence $\|y\| = 1$, $y \geqslant 0$. Passing to the limit as $n \to +\infty$ in (3.28) and using (3.32), (3.30) and (3.31), we obtain

$$A(y) + \beta y^{p-1} = 0,$$

so

$$\begin{cases} -\Delta_p y(z) + \beta(z) y(z)^{p-1} = 0 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \Omega \end{cases} \tag{3.33}$$

(see Motreanu-Papageorgiou [19]). Since $y \neq 0$ (see (3.32)) and $\widehat{\lambda}_1(\beta) > 0$ (see hypothesis $H(\beta)$), from (3.33) we reach a contradiction (see (2.2)). Therefore $u_* \neq 0$ and so $\lambda_* \in \mathcal{Y}$. □

So, we can state the following bifurcation-type theorem for problem $(P)_\lambda$.

Theorem 3.5. *If hypotheses $H(\beta)$ and $H(f)$ hold, then there exists $\lambda_* > 0$, such that:*
(a) *for all $\lambda > \lambda_*$, problem $(P)_\lambda$ has at least two nontrivial positive solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \quad u_0 \leqslant \widehat{u}, \quad u_0 \neq \widehat{u};$$

(b) *for $\lambda = \lambda_*$, problem $(P)_\lambda$ has at least one nontrivial positive solution $u_* \in \operatorname{int} C_+$;*
(c) *for $\lambda \in (0, \lambda_*)$, problem $(P)_\lambda$ has no positive solutions.*

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