Dynamic Systems and Applications 21 (2012) 269-292

OPTIMAL CONTROL FOR A CLASS OF DYNAMIC VISCOELASTIC CONTACT PROBLEMS WITH ADHESION

KRZYSZTOF BARTOSZ AND PIOTR KALITA

Faculty of Mathematics and Computer Science, Institute of Computer Science,
ul. Łojasiewicza 6, 30-348 Kraków, Poland

ABSTRACT. We study optimal control of problem governed by a coupled system of hemivariational inequality (HVI) and ordinary differential equation (ODE). The problem models viscoelastic, adhesive contact between a body and a foundation. We employ the Kelvin-Voigt viscoelastic law and consider the general nonmonotone and multivalued subdifferential boundary conditions. The bonding field at the contact surface changes according to ODE based law. We provide conditions which guarantee existence of optimal solutions for general Bolza type optimal control problem.

AMS (MOS) Subject Classification. 35L70, 35R45, 49J20, 49J40, 74D10

1. INTRODUCTION

The aim of this article is to present existence results for optimal control problem governed by a system of second order (in time) Partial Differential Inclusion (or hemivariational inequality) of type $u'' + Au' + Bu + \gamma^* \partial J(\gamma u) \ni f$ and ODE. In the inclusion B is a linear elasticity operator and A is a nonlinear and coercive viscosity operator of monotone type. Moreover J is the locally Lipschitz functional defined on the space of boundary functions, γ is the trace operator and ∂ denotes the subdifferential in the sense of Clarke. The system models a contact process between a body and a foundation. The boundary of the body is divided into three disjoint regions: classical Dirichlet and Neumann region and the region of contact. On the boundary of contact we consider two boundary conditions: one between normal component of displacement and normal component of the stress and the other one between tangential components of those quantities. Both relations are assumed to be governed by nonmonotone and multivalued subdifferential boundary conditions through which the inclusion is coupled with the ODE that describes the change of adhesion forces on the boundary of contact.

Mechanical problems with subdifferential boundary conditions in nonconvex and nonmonotone case lead to the notion of hemivariational inequality (HVI) which was

This paper is dedicated to prof. S. Migórski on the occasion of his 50th birthday. This work has been supported by the National Science Centre of Poland under grant no. N N201 604640.

Received November 15, 2011 1056-2176 $15.00

introduced by Panagiotopoulos in early 1980s as a generalization of a variational inequality. For the examples and detailed explanation the reader is referred to [26] and [22]. Existence of solutions for the second order in time inclusion modeling the nonmonotone contact between the viscoelastic body and a foundation was proved in [20]. Regularity results for such problems were obtained in [17] and the asymptotic behavior of solutions was studied in [16].

The processes of adhesion are very important in industry for several reasons. Firstly, in the car industry, aviation and space exploration there are many settings where nonmetallic parts are glued together. This is the case of composite materials which are made of layers of different materials. Composite materials may undergo delamination where different layers debond and move relatively to each other. In such situations there is a need to add adhesion to the description of the contact process. Secondly, in several industrial applications a glue is added to prevent relative motion of the contacting surfaces. The glue introduces tension that opposes the separation of the surfaces in the normal direction and opposes the relative motion in the tangential directions. Analysis of models for such problems is mainly done in a quasistatic case and can be found in [10, 14, 28, 31, 32, 33]. Finally, a new application of the contact theory concerns the medical field of arthoplasty where bonding between the bone implant and the tissue is of considerable importance since debonding may lead to decrease in the person ability to use the artificial limb or joint (cf. [29, 30]). Artificial implants of knee and hip prostheses (both cemented and cement-less) demonstrate that the adhesion is important at the bone-implant interface.

The study of adhesive contact is very recent in the mathematical literature. The novelty lies in the introduction of the ODE governed adhesive field on the contact surface. The idea of the introduction of a surface internal variable (called the bonding field or the adhesion field) is based on thermodynamic derivation and is originated in [11, 12, 34]. The bonding field β is a dimensionless variable which describes the pointwise fractional density of active bonds on the contact surface. Following [11, 12], the evolution of the bonding field is governed by an ordinary differential equation depending on the displacement and considered on contact surface. The conditions that guarantee the existence and uniqueness of solutions for the coupled (HVI)-(ODE) system that governs the adhesive contact of elastic body were given only recently in [4] and [5].

Optimal control for the HVIs has been studied in Chapter 8 of [26], [15] [27] (elliptic problems), [8] (shape design for elliptic hemivariational inequalities), [21] (parabolic problems), [24], [18], [13] (hyperbolic problems), [9] (coupled hyperbolic - parabolic systems).

In this paper we examine optimal control problems for a system of second order (in time) evolution inclusion and ordinary differential equation. For such a system we deal with Bolza problems. We remark that since the system under consideration has generally many solutions, the state of the problem for a given control can be not determined uniquely. The content of the paper is as follows. Section 2 contains the preliminary material. In Section 3 we present the physical setting and variational formulation of the problem. Section 4 summarizes existence results of [4] and [5]. Optimal control problem and the main result on the existence of optimal solutions and controls are provided in Section 5.

2. NOTATION AND PRELIMINARIES

In this section we recall some notation suitable for mathematical formulations of mechanical contact problems, cf. [3, 14, 23, 25, 26, 32].

We denote by $\mathcal{S}_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$). We define the inner products and the corresponding norms by

$$u \cdot v = u_i v_i, \qquad |v| = (v \cdot v)^{1/2} \qquad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma : \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\|_{\mathcal{S}_d} = (\tau : \tau)^{1/2} \qquad \text{for all } \sigma, \tau \in \mathcal{S}_d.$$

We adopt the summation convention over repeated indices.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let n denotes the outward unit normal vector to Γ. The assumption that Γ is Lipschitz ensures that n is defined a.e. on Γ. We use the following spaces

$$H = L^2(\Omega; \mathbb{R}^d), \qquad \mathcal{H} = \big\{ \tau = \{\tau_{ij}\} \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \big\} = L^2(\Omega; \mathcal{S}_d),$$
$$H_1 = \{ u \in H \mid \varepsilon(u) \in \mathcal{H} \} = H^1(\Omega; \mathbb{R}^d), \qquad \mathcal{H}_1 = \{ \tau \in \mathcal{H} \mid \operatorname{div} \tau \in H \},$$

where $\varepsilon\colon H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathcal{S}_d)$ and $\operatorname{div}\colon \mathcal{H}_1 \to L^2(\Omega;\mathbb{R}^d)$ denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{div}\sigma = \{\sigma_{ij,j}\} \tag{2.1}$$

and the index following a comma indicates a partial derivative. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \qquad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma : \tau \, dx,$$
$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\operatorname{div}\sigma, \operatorname{div}\tau)_H.$$

For every $v \in H_1$ we denote by v its trace γv on Γ, where $\gamma\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ is the trace map. Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_N and v_T the usual normal and the tangential components of v on the boundary Γ, i.e. $v_N = v \cdot n$ and $v_T = v - v_N n$. Similarily, for a regular (say C^1) tensor field $\sigma\colon \Omega \to \mathcal{S}_d$, we define its normal and tangential components by $\sigma_N = (\sigma n) \cdot n$ and $\sigma_T = \sigma n - \sigma_N n$.

We recall some definitions (see [7]). Let X be a reflexive Banach space and X^* its dual. The Clarke generalized directional derivative of a locally Lipschitz function

$h : X \to \mathbb{R}$ at the point $x \in X$ in the direction $v \in X$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The Clarke generalized gradient of h at x denoted by $\partial h(x)$ is a subset of X^* given by $\partial h(x) = \{\zeta \in X^* : h^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}$ for all $v \in X\}$. The locally Lipschitz function $h : X \to \mathbb{R}$ is called regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one sided directional derivative $h'(x; v)$ exists and satisfies $h'(x; v) = h^0(x; v)$ for all $v \in X$. Finally $T : X \to X^*$ is called *pseudomonotone* if for any sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $x_n \to x$ weakly in X and $\limsup_{n \to \infty} \langle T x_n, x_n - x \rangle \leq 0$ we have $T x_n \to T x$ weakly in X^* and $\langle T x_n, x_n \rangle \to \langle T x, x \rangle$.

Finally by C we will denote the generic constant dependent only on the problem data.

3. PROBLEM FORMULATION

3.1. **Physical setting.** The physical setting and the process are as follows. The set Ω is occupied by a viscoelastic body in $\mathbb{R}^d$ which is referred to as the reference configuration. We assume that the boundary Γ of Ω is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C with $m(\Gamma_D) > 0$, where m denotes the boundary measure.

As a result of applied volume forces and surface tractions the mechanical system evolves over the time interval $[0, T]$ where $T > 0$. We denote by $u(x, t) = (u_1(x, t), \ldots, u_d(x, t))$ the displacement at time $t \in [0, T]$ of a particle $x = (x_1, \ldots, x_d) \in \Omega$ and by $\sigma(x, t) = (\sigma_{ij}(x, t))$ the stress tensor at time t and position x. In the model the material is assumed to be viscoelastic and its response satisfies the constitutive law:

$$\sigma(u, u') = \mathcal{C}(\varepsilon(u')) + \mathcal{G}(\varepsilon(u)) \quad \text{on} \quad \Omega \times (0, T), \tag{3.1}$$

where $\mathcal{C}$ and $\mathcal{G}$ are prescribed nonlinear viscosity and linear elasticity operators, respectively and ε is defined by (2.1). We remark that in linear viscoelasticity the above law takes the form of the Kelvin-Voigt relation

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}(u') + g_{ijkl} \varepsilon_{kl}(u) \quad \text{on} \quad \Omega \times (0, T),$$

where $\mathcal{C} = \{c_{ijkl}\}$ and $\mathcal{G} = \{g_{ijkl}\}$, i, j, k, $l = 1, \ldots, d$ are the viscosity and elasticity tensors, respectively, which may be functions of position.

The dynamic equation of motion represents momentum conservation (cf. [25, 14]) and it governs the evolution of the state of the body

$$u'' - \operatorname{div} \sigma(u, u') = f_1 \quad \text{on} \quad \Omega \times (0, T),$$

where $f_1 = f_1(x,t)$ is the density of applied volume forces. We suppose that the mass density is constant, conveniently set equal to one.

Next we describe the boundary conditions. The body is assumed to be held fixed on the part Γ_D of the surface, so the displacement $u = 0$ on $\Gamma_D \times (0,T)$. On the part Γ_N a prescribed surface force $f_2 = f_2(x,t)$ is applied, thus we have the condition $\sigma n = f_2$ on $\Gamma_N \times (0,T)$. The body may come in adhesive contact over the part Γ_C of its surface. So we define a function $\beta : \Gamma_C \times [0,T] \to [0,1]$, which represents a bonding field between the boundary Γ_C and the surface of the foundation. When $\beta = 0$ all bonds are inactive and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and a fracture β of the bonds is active. Evolution of β is governed by the following ordinary differential equation

$$\beta'(t) = F(t, u(t), \beta(t)) \quad \text{on} \quad \Gamma_C \times (0,T).$$

The conditions on the contact boundary are naturally divided to ones in the normal direction and those in the tangential direction (Section 5.4 of [14]). They are given by the subdifferential boundary conditions of the form

$$\begin{aligned} -\sigma_N(t) &\in \partial j_N(t, \beta(t), u_N(t)) \quad \text{on} \quad \Gamma_C \times (0,T) \\ -\sigma_T(t) &\in \partial j_T(t, \beta(t), u_T(t)) \quad \text{on} \quad \Gamma_C \times (0,T), \end{aligned}$$

where the functions $j_T \colon \Gamma_C \times (0,T) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $j_N \colon \Gamma_C \times (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz in their last variables, and ∂j_T and ∂j_N denote the Clarke subdifferentials of $j_T(x,t,\beta,\cdot)$ and $j_N(x,t,\beta,\cdot)$, respectively. The component σ_T represents the friction force on the contact surface. Here and below we skip occasionally the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. We remark that both j_N and j_T depend on the adhesive field β. Note also that the explicit dependence of the functions j_N and j_T with respect to the time variable allows to model situations when the frictional contact conditions depend on the temperature, which plays the role of a parameter, and which evolution in time is prescribed.

Finally, in the evolution problem we need to prescribe the initial conditions for the displacement and the velocity, i.e.

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{in} \quad \Omega,$$

where u^0 and u^1 denote the initial displacement and the initial velocity, respectively. The function $\beta^0 : \Gamma_C \to [0,1]$ denotes the initial bonding field.

Collecting all the equations and conditions, we obtain the following formulation of the mechanical problem: find the displacement field $u \colon \Omega \times (0,T) \to \mathbb{R}^d$, the stress field $\sigma \colon \Omega \times (0,T) \to \mathcal{S}_d$ and the bonding field $\beta \colon \Gamma_C \times (0,T) \to [0,1]$ such that

$$u'' - \operatorname{div} \sigma(u, u') = f_1 \quad \text{in} \quad \Omega \times (0,T), \tag{3.2}$$

$$\sigma(u, u') = \mathcal{C}(t, \varepsilon(u')) + \mathcal{G}(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T), \tag{3.3}$$

$$u = 0 \qquad \text{on } \Gamma_D \times (0,T), \tag{3.4}$$

$$\sigma n = f_2 \qquad \text{on } \Gamma_N \times (0,T), \tag{3.5}$$

$$-\sigma_N(t) \in \partial j_N(t, \beta(t), u_N(t)) \qquad \text{on } \Gamma_C \times (0,T), \tag{3.6}$$

$$-\sigma_T(t) \in \partial j_T(t, \beta(t), u_T(t)) \qquad \text{on } \Gamma_C \times (0,T), \tag{3.7}$$

$$\beta'(t) = F(t, u(t), \beta(t)) \qquad \text{on } \Gamma_C \times (0,T), \tag{3.8}$$

$$\beta(0) = \beta^0 \qquad \text{on } \Gamma_C, \tag{3.9}$$

$$u(0) = u^0, \quad u'(0) = u^1 \qquad \text{in } \Omega. \tag{3.10}$$

The above problem represents the *classical formulation* of the adhesive viscoelastic frictional contact problem. The conditions (3.6) and (3.7) introduce the main difficulty to the problem since they are "nondifferentiable" and belong to the conditions we meet in the part of mechanics called *nonsmooth mechanics*. This is a reason for which the problem (3.2)–(3.10) has no classical solution, i.e. solution which possesses all the necessary classical derivatives and satisfies the relations in the usual sense at each point and at each time instance. We are forced to formulate the above problem in a weak sense.

3.2. **Variational formulation of the problem.** In order to obtain the variational formulation of the problem (3.2)–(3.10), we introduce

$$V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_D\}.$$

This space is the closed subspace of H_1 and so it is a Hilbert space with the inner product and the corresponding norm given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}} \text{ for } u, v \in V.$$

From the Korn inequality $\|v\|_{H_1} \leq C\|\varepsilon(v)\|_{\mathcal{H}}$ for $v \in V$ with $C > 0$ (cf. Section 6.3 of [23]), it follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|$ are the equivalent norms on V. Identifying H with its dual, we have an evolution triple of spaces (V, H, V^*) with dense, continuous and compact embeddings. For this evolution triple we define the spaces $\mathcal{V} = L^2(0,T;V)$, $\widehat{\mathcal{H}} = L^2(0,T;H)$, $\mathcal{V}^* = L^2(0,T;V^*)$ and $\mathcal{W} = \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\}$. Norms in $\mathcal{V}, \widehat{\mathcal{H}}, \mathcal{V}^*$ will be denoted by $\|\cdot\|_{\mathcal{V}}, \|\cdot\|_{\widehat{\mathcal{H}}}, \|\cdot\|_{\mathcal{V}^*}$, while the duality brackets between V and V^* and $\mathcal{V}$ and $\mathcal{V}^*$ will be denoted by $\langle \cdot, \cdot \rangle$. We also put $Q = \{f \in L^2(\Gamma_C) : 0 \leq f(x) \leq 1 \text{ a.e. on } \Gamma_C\}$ and $\mathcal{Q} = \{f : [0,T] \to Q\}$.

We need the following hypotheses on the data of the problem (3.2)–(3.10).

$\underline{H(\mathcal{C})}$: The viscosity operator $\mathcal{C} \colon \Omega \times (0,T) \times \mathcal{S}_d \to \mathcal{S}_d$ satisfies the assumptions:

(i) $\mathcal{C}(\cdot, \cdot, \varepsilon)$ is measurable on $\Omega \times (0,T)$ for all $\varepsilon \in S_d$,

(ii) $\mathcal{C}(x, t, \cdot)$ is continuous on $\mathcal{S}_d$ for a.e. $(x,t) \in \Omega \times (0,T)$,

(iii) $\|\mathcal{C}(x,t,\varepsilon)\|_{\mathcal{S}_d} \leq c_1 (b(x,t) + \|\varepsilon\|_{\mathcal{S}_d})$ for $\varepsilon \in \mathcal{S}_d$, a.e. $(x,t) \in \Omega \times (0,T)$ with $b \in L^2(\Omega \times (0,T))$, $c_1 > 0$,

(iv) $(\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq 0$ for all $\varepsilon_1,\ \varepsilon_2 \in \mathcal{S}_d$ and a.e. $(x,t) \in \Omega \times (0,T)$,

(v) $\mathcal{C}(x,t,\varepsilon) : \varepsilon \geq c_2 \|\varepsilon\|^2_{\mathcal{S}_d}$ for all $\varepsilon \in \mathcal{S}_d$ and a.e. $(x,t) \in \Omega \times (0,T)$ with $c_2 > 0$,

(vi) $\mathcal{C}$ is of the form $\mathcal{C}(x,t,\varepsilon) = \mathbb{C}(x,t)\varepsilon$ where $\mathbb{C} = \{C_{ijkl}\}$ with $i,j,k,l \in \{1,\ldots,d\}$ is a viscosity tensor,

(vii) $\|\mathbb{C}(x,t)\|_{\mathcal{L}(\mathcal{S}_d,\mathcal{S}_d)} \leq c_1$ a.e. $(x,t) \in U$ with $c_1 > 0$.

By a simple observation (vi) implies (ii), (vi) and (vii) imply (iii), and (v) and (vi) imply (iv). In the sequel we consider two sets of assumptions: (i)–(v) that lead to pseudomonotone case and the stronger ones (i),(v)–(vii) that lead to linear case.

$\underline{H(\mathcal{G})}$: The elasticity operator $\mathcal{G}\colon \Omega \times \mathcal{S}_d \to \mathcal{S}_d$ is of the form $\mathcal{G}(x,\varepsilon) = \mathbb{E}(x)\varepsilon$ (the Hooke law) with a symmetric and nonnegative elasticity tensor $\mathbb{E}$, i.e. $\mathbb{E} = \{G_{ijkl}\}$, $i,\ j,\ k,\ l = 1,\ldots,d$ with $G_{ijkl} \in L^\infty(\Omega)$, $G_{ijkl} = G_{jikl} = G_{lkij}$ and $G_{ijkl}(x)\chi_{ij}\chi_{kl} \geq 0$ for all symmetric tensors $\chi = \{\chi_{ij}\}$ and for a.e. $x \in \Omega$.

$\underline{H(j_N)}$: $j_N\colon \Gamma_C \times (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $j_N(\cdot,\cdot,r,s)$ is measurable for all $r,\ s \in \mathbb{R}$ and $j_N(\cdot,\cdot,0,0) \in L^1(\Gamma_C \times (0,T))$;

(ii) $j_N(x,t,r,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $r \in \mathbb{R}$;

(iii) $|\partial j_N(x,t,r,s)| \leq c_N\,(1 + |s|)$ for all a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r,\ s \in \mathbb{R}$ with $c_N > 0$;

(iv) either $j_N(x,t,r,\cdot)$ or $-j_N(x,t,r,\cdot)$ is regular for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $r \in \mathbb{R}$;

(v) $j^0_N(x,t,\cdot,\cdot;z)$ is upper semicontinuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $z \in \mathbb{R}$, where j^0_N denotes the Clarke directional derivative of $j_N(x,t,r,\cdot)$ in the direction z.

$\underline{H(j_T)}$: $j_T\colon \Gamma_C \times (0,T) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a function such that

(i) $j_T(\cdot,\cdot,r,\xi)$ is measurable for all $(r,\xi) \in \mathbb{R}\times\mathbb{R}^d$ and $j_T(\cdot,\cdot,0,0) \in L^1(\Gamma_C\times(0,T))$;

(ii) $j_T(x,t,r,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $r \in \mathbb{R}$;

(iii) $|\partial j_T(x,t,r,\xi)| \leq c_T\,(1 + |\xi|)$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $r \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ with $c_T > 0$;

(iv) either $j_T(x,t,r,\cdot)$ or $-j_T(x,t,r,\cdot)$ is regular for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $r \in \mathbb{R}$;

(v) $j^0_T(x,t,\cdot,\cdot;z)$ is upper semicontinuous for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $z \in \mathbb{R}^d$, where j^0_T denotes the Clarke directional derivative of $j_T(x,t,r,\cdot)$ in the direction z.

$\underline{H(F)}$: The adhesive evolution rate function $F\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

(i) $F(\cdot,\cdot,\xi,r)$ is measurable on $\Gamma_C \times (0,T)$ for all $(\xi,r) \in \mathbb{R}^d \times \mathbb{R}$ and $F(x,t,\cdot,\cdot)$ is continuous on $\mathbb{R}^d \times \mathbb{R}$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$;

(ii) $|F(x,t,\xi_1,r_1) - F(x,t,\xi_2,r_2)| \le L_F(|\xi_1-\xi_2| + |r_1-r_2|)$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $\xi_i \in \mathbb{R}^d$, $r_i \in [0,1]$, $i = 1, 2$ with $L_F > 0$;

(iii) $F(x,t,\xi,0) = 0$, $F(x,t,\xi,r) \ge 0$ for $r \le 0$ and $F(x,t,\xi,r) \le 0$ for $r \ge 1$, for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $\xi \in \mathbb{R}^d$.

In what follows whenever we assume both $H(j_N)(iv)$ and $H(j_T)(iv)$, we mean that for a.e. $(x,t) \in \Gamma_C \times (0,T)$ and all $r \in \mathbb{R}$ either "both $j_N(x,t,r,\cdot)$ and $j_T(x,t,r,\cdot)$ are regular" or "both $-j_N(x,t,r,\cdot)$ and $-j_T(x,t,r,\cdot)$ are regular".

The above hypotheses are coherent and it seems they are realistic with respect to the physical data. They are not surprising from the mathematical point of view and appear to be adequate in the given framework. The concrete examples of functions j_T, j_N and F which satisfy the hypotheses $H(j_T)$, $H(j_N)$ and $H(F)$ are given in Section 3.5 of [4] and in Section 5 of [5].

The mass forces, boundary tractions and initial displacement, velocity and the bonding field are the following

$\underline{H(f)}$: $f_1 \in L^2(0,T;H)$, $f_2 \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$, $u^0 \in V$, $u^1 \in H$, $\beta^0 \in Q$.

Assuming $H(f)$, we define $f \in \mathcal{V}^*$ by

$$\langle f(t), v\rangle = (f_1(t), v)_H + (f_2(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \text{for} \quad v \in V \text{ and a.e. } t \in (0,T).$$

We remark that if $H(f)$ is satisfied, then (H_0) holds, where

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u^0 \in V$, $u^1 \in H$, $\beta^0 \in Q$.

To obtain the variational formulation of the problem (3.2)–(3.10), let $v \in V$. Using the equation of motion (3.2) and the Green formula (assuming the regularity of the functions involved), we have

$$\langle u''(t), v\rangle + (\sigma(t), \varepsilon(v))_{\mathcal{H}} = (f_1(t), v)_H + \int_\Gamma \sigma(t)n \cdot v \, d\Gamma \quad \text{for} \quad v \in V \text{ and } t \in (0,T).$$

Taking into account the boundary condition on Γ_N, we obtain

$$\langle u''(t), v\rangle + (\sigma(t), \varepsilon(v))_{\mathcal{H}} - \int_{\Gamma_C} \sigma(t)n \cdot v \, d\Gamma = \langle f(t), v\rangle.$$

On the other hand, using the notation of Section 2, we have

$$\int_{\Gamma_C} \sigma(t)n \cdot v \, d\Gamma = \int_{\Gamma_C} (\sigma_N v_N + \sigma_T \cdot v_T) \, d\Gamma.$$

By the definition of the generalized directional derivative it follows from the multivalued relations (3.6) and (3.7) on $\Gamma_C \times (0,T)$, that

$$-\sigma_N(x,t)\, z \le j_N^0(x,t,\beta(x,t), u_N(x,t); z) \quad \text{for all} \quad z \in \mathbb{R},$$

$$-\sigma_T(x,t) \cdot \eta \le j_T^0(x,t,\beta(x,t), u_T(x,t); \eta) \quad \text{for all} \quad \eta \in \mathbb{R}^d$$

for a.e. $(x,t) \in \Gamma_C \times (0,T)$.

Hence we obtain the following variational formulation of the problem (3.2)–(3.10). It is called *adhesive viscoelastic frictional contact problem.*

Problem (AVFC): find a displacement field $u\colon (0,T) \to V$ and a bonding field $\beta\colon (0,T) \to L^2(\Gamma_C)$ such that $u \in \mathcal{V}$, $u' \in \mathcal{W}$ and $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_C)) \cap \mathcal{Q}$ and

$$\begin{cases} \langle u''(t), v\rangle + (\sigma(t), \varepsilon(v))_{\mathcal{H}} + \\ + \displaystyle\int_{\Gamma_C} \Big(j_N^0(x,t,\beta(x,t),u_N(t);v_N) + j_T^0(x,t,\beta(x,t),u_T(t);v_T) \Big)\, d\Gamma(x) \geq \\ \geq \langle f(t), v\rangle \ \text{ for all } \ v \in V \text{ and a.e. } t \in (0,T) \\ \sigma(t) = \mathcal{C}\varepsilon(t,u'(t)) + \mathcal{G}\varepsilon(u(t)) \ \text{ for a.e. } t \in (0,T) \\ \beta'(x,t) = F(x,t,u(x,t),\beta(x,t)) \ \text{ a.e. on } \ \Gamma_C \times (0,T) \\ \beta(0) = \beta^0 \ \text{ on } \ \Gamma_C \times (0,T) \\ u(0) = u^0, \ \ u'(0) = u^1 \ \text{ in } \ \Omega. \end{cases}$$

Problem (AVFC) is a system in which the hemivariational inequality for the displacement field is coupled with the ordinary differential equation for the bonding field. The existence theorem for Problem (AVFC) is given in [5] and recalled in Section 4.

3.3. **Evolution inclusion.** In this subsection we formulate an evolution inclusion associated with Problem (AVFC). To this end we define auxiliary operators and formulate lemmas on their properties. We refer to [4], Lemma 27 (see also Lemma 5 in [5]) and Lemma 31 (see also Lemma 9 in [5]) for the proof of Lemma 3.1 and Lemma 3.5 respectively. The proofs of lemmata 3.3 and 3.4 can be found in [19].

First we consider the Cauchy problem for the ordinary differential equation on the contact surface. Under the assumption that the displacement is given on the contact part of the boundary, we establish the existence of solutions to (3.8)–(3.9) and provide the result on the continuous dependence of the adhesion field on the displacement.

Lemma 3.1. *Assume that $H(F)$ holds and $z \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$. Then for every $\beta^0 \in Q$ there exists $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_C)) \cap \mathcal{Q}$ a unique solution of the Cauchy problem*

$$\beta'(x,t) = F(x,t,z(x,t),\beta(x,t)) \quad \text{a.e. on } \Gamma_C \times (0,T) \tag{3.11}$$

$$\beta(x,0) = \beta^0(x) \quad \text{on } \Gamma_C. \tag{3.12}$$

Moreover, given $z_i \in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ and denoting the unique solutions corresponding to z_i, $i = 1, 2$ by $\beta_i \in W^{1,\infty}(0,T;L^2(\Gamma_C)) \cap \mathcal{Q}$, we have

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_C)} \leq d \int_0^t \|z_1(s) - z_2(s)\|_{L^2(\Gamma_C;\mathbb{R}^d)}\, ds$$

for a.e. $t \in (0,T)$ with $d > 0$.

Let the operator $\mathcal{R}\colon L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\times Q\to W^{1,\infty}(0,T;L^2(\Gamma_C))\cap\mathcal{Q}$ be defined by

$$\mathcal{R}(z,\beta^0)=\beta, \tag{3.13}$$

where $\beta\in W^{1,\infty}(0,T;L^2(\Gamma_C))$ is the unique solution of the Cauchy problem (3.11)–(3.12) corresponding to given $z\in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$, $\beta^0\in Q$.

Corollary 3.2. *From Lemma* 3.1*, it follows that under the hypothesis $H(F)$ for every $\beta^0\in Q$ the operator $\mathcal{R}(\cdot,\beta^0)$ is well defined and*

$$\|\mathcal{R}(z_1,\beta^0)-\mathcal{R}(z_2,\beta^0)\|_{L^\infty(0,T;L^2(\Gamma_C))}\le C\|z_1-z_2\|_{L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))} \tag{3.14}$$

for all $z_i\in L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))$ with some constant $C>0$.

Moreover we define the operators $A\colon(0,T)\times V\to V^*$ and $B\colon V\to V^*$ by

$$\langle A(t,u),v\rangle=(\mathcal{C}(x,t,\varepsilon(u)),\varepsilon(v))_{\mathcal{H}}\quad\text{for } u,v\in V \text{ and } t\in(0,T) \tag{3.15}$$

and

$$\langle Bu,v\rangle=(\mathcal{G}(x,\varepsilon(u)),\varepsilon(v))_{\mathcal{H}}\quad\text{for } u,v\in V. \tag{3.16}$$

Lemma 3.3. *Under the hypothesis $H(\mathcal{C})$(i)–(v) the operator $A\colon(0,T)\times V\to V^*$ defined by* (3.15) *satisfies*

$\underline{H(A)}$: $A\colon(0,T)\times V\to V^*$ *is such that:*

(i) *$A(\cdot,v)$ is measurable on $(0,T)$ for all $v\in V$;*
(ii) *$A(t,\cdot)$ is pseudomonotone for every $t\in(0,T)$;*
(iii) *$||A(t,v)||_{V^*}\le a(t)+b||v||$ a.e. t, for all $v\in V$ with $a\in L^2(0,T)$, $a\ge 0$, $b>0$;*
(iv) *$\langle A(t,v),v\rangle\ge c_2||v||^2$ a.e. $t\in(0,T)$, for all $v\in V$.*

Lemma 3.4. *Under the assumption $H(\mathcal{G})$ the operator $B\colon V\to V^*$ defined by* (3.16) *satisfies*

$\underline{H(B)}$: $B\colon V\to V^*$ *is a bounded, linear, monotone and symmetric operator (i.e. $B\in\mathcal{L}(V,V^*)$, $\langle Bv,v\rangle\ge 0$ for all $v\in V$, $\langle Bv,w\rangle=\langle Bw,v\rangle$ for all v, $w\in V$).*

In order to express the system (AVFC) in the form of evolution inclusion, we have to extend the pointwise relations (3.6) and (3.7) to relations between multifunctions defined on infinite dimensional spaces. To formulate an evolution inclusion corresponding to (AVFC), we consider the functional $J\colon(0,T)\times L^2(\Gamma_C)\times L^2(\Gamma_C;\mathbb{R}^d)\to\mathbb{R}$ defined by

$$J(t,w,u)=\int_{\Gamma_C} j_N(x,t,w(x),u_N(x))+j_T(x,t,w(x),u_T(x))\,d\Gamma(x) \tag{3.17}$$

for a.e. $t\in(0,T)$, $w\in L^2(\Gamma_C)$ and $u\in L^2(\Gamma_C;\mathbb{R}^d)$.

Lemma 3.5. *Under the hypothesis $H(j_N)$ and $H(j_T)$, the functional J defined by (3.17) satisfies*

$\underline{H(J)}$: $J\colon (0,T)\times L^2(\Gamma_C)\times L^2(\Gamma_C;\mathbb{R}^d)\to\mathbb{R}$ *is such that*

(i) $J(\cdot,w,u)$ *is measurable on* $(0,T)$ *for all* $w\in L^2(\Gamma_C)$, $u\in L^2(\Gamma_C;\mathbb{R}^d)$ *and* $J(\cdot,0,0)\in L^1(0,T)$*;*

(ii) $J(t,w,\cdot)$ *is well defined and locally Lipschitz (in fact, Lipschitz on bounded subsets of* $L^2(\Gamma_C;\mathbb{R}^d)$*) for a.e.* $t\in(0,T)$ *and all* $w\in L^2(\Gamma_C)$*;*

(iii) $\|\zeta\|_{L^2(\Gamma_C;\mathbb{R}^d)}\le c_0\left(1+\|u\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ *for a.e.* $t\in(0,T)$*, all* $\zeta\in\partial J(t,w,u)$*,* $w\in L^2(\Gamma_C)$ *and* $u\in L^2(\Gamma_C;\mathbb{R}^d)$ *with* $c_0>0$*;*

(iv) $J^0(t,\cdot,\cdot;v)\colon L^2(\Gamma_C)\times L^2(\Gamma_C;\mathbb{R}^d)\to\mathbb{R}$ *is upper semicontinuous, for a.e.* $t\in(0,T)$ *and all* $v\in L^2(\Gamma_C;\mathbb{R}^d)$*,*

where $J^0(t,w,u;v)$ *denotes the Clarke directional derivative of the function* $J(t,w,\cdot)$ *at a point* u *in the direction* v*, where* u, $v\in L^2(\Gamma_C;\mathbb{R}^d)$*. Moreover, we have*

$$J^0(t,w,u;v)=\int_{\Gamma_C}\left(j_N^0(x,t,w,u_N;v_N)+j_T^0(x,t,w,u_T;v_T)\right)\,d\Gamma(x),$$

for a.e. $t\in(0,T)$*, all* $w\in L^2(\Gamma_C)$ *and* u, $v\in L^2(\Gamma_C;\mathbb{R}^d)$*.*

We denote by Z the Sobolev space $H^\delta(\Omega;\mathbb{R}^d)$ with a fixed $\delta\in[\frac12,1)$. Let $\|\bar\gamma\|=\|\bar\gamma\|_{\mathcal{L}(Z,L^2(\Gamma;\mathbb{R}^d))}$ be the norm of the trace operator $\bar\gamma\colon Z\to L^2(\Gamma;\mathbb{R}^d)$ and let $c_e>0$ be the embedding constant of V into Z, i.e. $\|v\|_Z\le c_e\|v\|$ for all $v\in V$. Denoting by $i\colon V\to Z$ the embedding injection, we have $\gamma v=\bar\gamma(iv)$ for all $v\in V$. For simplicity we omit the notation of the embedding i and we write $\gamma v=\bar\gamma v$ for $v\in V$. So we have

$$V\subset Z\subset H\subset Z^*\subset V^*$$

with all embeddings being continuous. This also implies that

$$\mathcal{W}\subset\mathcal{V}\subset\mathcal{Z}\subset\widehat{\mathcal{H}}\subset\mathcal{Z}^*\subset\mathcal{V}^*$$

with continuous embeddings, where $\mathcal{Z}=L^2(0,T;Z)$ and $\mathcal{Z}^*=L^2(0,T;Z^*)$ denotes its dual. We denote by $\widetilde\gamma$ the Nemyckii operator corresponding to $\bar\gamma$ defined by $(\widetilde\gamma v)(t)=\bar\gamma(v(t))$ for $v\in\mathcal{Z}$. We know that $\widetilde\gamma\in\mathcal{L}(\mathcal{Z},L^2(0,T;L^2(\Gamma;\mathbb{R}^d)))$ is well defined, linear and bounded.

We consider the following evolution inclusion of second order

$$\begin{cases}\text{find u}\in\mathcal{V}\text{ with u}'\in\mathcal{W}\text{ such that for a.e. t}\in(0,\mathrm{T})\\ u''(t)+A(t,u'(t))+Bu(t)+\bar\gamma^*\left(\partial J(t,\mathcal{R}(\widetilde\gamma u,\beta^0)(t),\widetilde\gamma u(t))\right)\ni f(t),\\ u(0)=u^0,\ u'(0)=u^1,\end{cases}\tag{3.18}$$

where $A\colon(0,T)\times V\to V^*$, $B\colon V\to V^*$, $J\colon(0,T)\times L^2(\Gamma_C)\times L^2(\Gamma_C;\mathbb{R}^d)\to\mathbb{R}$ and $\mathcal{R}\colon L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\times Q\to W^{1,\infty}(0,T;L^2(\Gamma_C))\cap\mathcal{Q}$ are defined by (3.15), (3.16), (3.17) and (3.13), respectively.

Definition 3.6. A function $u \in \mathcal{V}$ solves (3.18) if and only if $u' \in \mathcal{W}$ and there exists $\xi \in \mathcal{Z}^*$ such that

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \xi(t) = f(t) \quad \text{a.e. } t \in (0,T) \\ \xi(t) \in \bar{\gamma}^* \left(\partial J(t, (\mathcal{R}(\widetilde{\gamma} u, \beta^0))(t), \widetilde{\gamma} u(t))\right) \quad \text{a.e. } t \in (0,T) \\ u(0) = u^0, \ u'(0) = u^1. \end{cases}$$

The reason to introduce the problem (3.18) is given below.

Lemma 3.7. *Assume $H(A)$, $H(B)$, (H_0), $H(j_N)$(i)-(iv) and $H(j_T)$(i)-(iv) and let $\mathcal{R}$ be the operator defined by (3.13). Then the following statements are equivalent*

(i) *u is a solution to the inclusion (3.18);*
(ii) *u and $\beta = \mathcal{R}(\widetilde{\gamma} u, \beta^0)$ solve the Problem (AVFC).*

Let us introduce the Nemyckii operators $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$, $\mathcal{B} : \mathcal{V} \to \mathcal{V}^*$, $\mathcal{N} : \mathcal{Z} \times L^2(\Gamma_C) \to 2^{\mathcal{V}^*}$ defined respectively by $(\mathcal{A}v)(t) = A(t, v(t))$, $(\mathcal{B}v)(t) = B(v(t))$ for $v \in \mathcal{V}$ and

$$\mathcal{N}(v, \beta^0) = \{w \in \mathcal{Z}^* : w(t) \in \bar{\gamma}^*(\partial J(t, \mathcal{R}(\widetilde{\gamma} v, \beta^0)(t), \widetilde{\gamma} v(t))) \ a.e.\ t \in [0,T]\}$$

for $v \in \mathcal{Z}, \beta^0 \in L^2(\Gamma_C)$. We can rewrite inclusion (3.18) as

$$u'' + \mathcal{A}u' + \mathcal{B}u + w = f, \ \textit{where } w \in \mathcal{N}(u, \beta^0).$$

In the sequel we will need the following

Lemma 3.8. *The following statements are valid:*

(i) *Under assumptions $H(\mathcal{C})$(i)–(v) the operator $\mathcal{A}$ satisfies the following condition: for every sequence $\{v_n\} \subset \mathcal{W}$ with $v_n \to v$ weakly in $\mathcal{W}$ and $\limsup_{n\to\infty} \langle \mathcal{A}v_n, v_n - v\rangle \leq 0$ it follows that $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$ and $\langle \mathcal{A}v_n, v_n\rangle \to \langle \mathcal{A}v, v\rangle$,*
(ii) *Under assumptions $H(\mathcal{C})$(i),(v)–(vii) the operator $\mathcal{A}$ is linear and continuous,*
(iii) *Under assumptions $H(\mathcal{G})$ the operator $\mathcal{B}$ is linear and continuous.*

Proof. (i) follows directly from Lemma 7 (e) in [20] (see also Lemma 15 (e) in [5]) and the thesis (iii) follows from Lemma 4. For the proof of (ii) let us take $u, v \in \mathcal{V}$ and estimate

$$\langle \mathcal{A}u, v\rangle \leq \int_0^T \langle A(t, u(t)), v(t)\rangle \, dt = \int_0^T (\mathbb{C}(x,t)\varepsilon(u(t)), \varepsilon(v(t)))_{\mathcal{H}} \, dt =$$
$$= \int_0^T \|\mathbb{C}(x,t)\varepsilon(u(t))\|_{\mathcal{H}} \|v(t)\| \, dt \leq c_1 \int_0^T \|u(t)\|\|v(t)\| \, dt \leq c_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}.$$

Therefore $\|\mathcal{A}u\|_{\mathcal{V}^*} \leq c_1 \|u\|_{\mathcal{V}}$ and we have the thesis. $\square$

4. **Existence of weak solution**

In order to formulate the main existence theorem we need to define the following constants: $c = \max\{c_N, c_T\}$, $c_0 = \sqrt{2}\, c \max\{1, \sqrt{\mathrm{m}(\Gamma_C)}\}$ and $m = c_0 \max\{1, \|\bar{\gamma}\|\}$ and consider the following constraint:

$$\frac{c_2}{2} - m\, c_e^2\, T\, \|\bar{\gamma}\| > 0. \tag{4.1}$$

We recall two existence theorems which are proved in [5] (see respectively Theorems 14 and 4)

Theorem 4.1. *Under the hypotheses* $H(A)$, $H(B)$, $H(J)$, (H_0) *and* (4.1), *if*

$$\mathcal{R}(\cdot, \beta^0)\colon L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d)) \to W^{1,\infty}(0,T;L^2(\Gamma_C)) \cap \mathcal{Q}$$

satisfies (3.14) *for all* $\beta^0 \in Q$, *then the evolution inclusion* (3.18) *has a solution.*

Theorem 4.2. *Assume the hypotheses* $H(\mathcal{C})$*(i)–(v),* $H(\mathcal{G})$, $H(j_N)$, $H(j_T)$, $H(f)$, $H(F)$ *and* (4.1). *Then Problem (AVFC) admits a solution* $\{u, \beta\}$ *such that*

$$u \in W^{1,2}(0,T;V) \cap C(0,T,V), \quad u' \in C(0,T;H), \quad u'' \in L^2(0,T;V^*),$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_C)) \cap \mathcal{Q}.$$

Moreover, the stress tensor corresponding to the solution satisfies $\sigma \in L^2(0,T;\mathcal{H})$ *with* $\operatorname{div}\sigma \in L^2(0,T;V^*)$.

5. **OPTIMAL CONTROL PROBLEMS**

In this section we formulate the control problem corresponding to Problem (AVFC) with the input data assumed to be the control variables. We provide the conditions which guarantee the existence of solution for the optimal control problem.

5.1. **Energy estimate.** First we formulate and prove a proposition which provides an a priori estimation for the solutions of the problem (3.18).

Proposition 5.1. *Under the hypotheses* $H(A)$, $H(B)$, $H(J)$, (H_0) *and* $H(F)$ *if* (4.1) *holds then there exists a positive constant* C *such that*

$$\begin{aligned} &\|\beta\|_{W^{1,\infty}(0,T;L^2(\Gamma_C))} + \|u\|_{C(0,T;V)} + \|u'\|_{\mathcal{W}} \leq \\ &\leq C(1 + \|u^0\| + \|u^1\|_H + \|f\|_{\mathcal{V}^*} + \|\beta^0\|_{L^2(\Gamma_C)}), \end{aligned} \tag{5.1}$$

where u *and* $\beta = \mathcal{R}(\widetilde{\gamma} u, \beta^0)$ *is a solution of* (3.18) *with a given data* f *and initial values* u^0, u^1 *and* β^0.

In order to prove the above proposition we use the following

Lemma 5.2. *Assume that the hypotheses $H(A)$, $H(B)$, $H(J)$, (H_0) and* (4.1) *hold, and $\mathcal{R}\colon L^2(0,T;L^2(\Gamma_C;\mathbb{R}^d))\times Q\to W^{1,\infty}(0,T;L^2(\Gamma_C))\cap\mathcal{Q}$ is an operator defined by (3.13). If u is a solution of* (3.18), *then*

$$\|u\|_{C(0,T;V)}+\|u'\|_{\mathcal{W}}\le C\left(1+\|u^0\|+\|u^1\|_H+\|f\|_{\mathcal{V}^*}\right) \tag{5.2}$$

with a positive constant C independent on β^0.

For the proof of Lemma 5.2 we refer to [5], Proposition 13.

Proof of Proposition 5.1 Let $H(F)$ hold and $z=\widetilde{\gamma}u$ where u is the solution of (3.18). Moreover, let $\beta\in W^{1,\infty}(0,T;L^2(\Gamma_C))\cap\mathcal{Q}$ be a unique solution of the Cauchy problem (3.11)–(3.12). We will show that

$$\|\beta\|_{L^\infty(0,T;L^2(\Gamma_C))}\le(1+L_FTe^{L_FT})\|\beta^0\|_{L^2(\Gamma_C)}. \tag{5.3}$$

Integrating (3.11) over $(0,t)$ we get for $t\in(0,T)$ and a.e. $x\in\Gamma_C$

$$|\beta(x,t)|\le|\beta^0(x)|+\int_0^t|F(x,s,z(x,s),\beta(x,s))|\,ds.$$

We use $H(F)(ii)$ and $H(F)(iii)$ to get

$$|\beta(x,t)|\le|\beta^0(x)|+\int_0^t|F(x,s,z(x,s),\beta(x,s))-F(x,s,z(x,s),0)|\,ds\le$$
$$\le|\beta^0(x)|+L_F\int_0^t|\beta(x,s)|\,ds.$$

Applying the Gronwall inequality we obtain for $t\in(0,T)$ and a.e. $x\in\Gamma_C$

$$|\beta(x,t)|\le|\beta^0(x)|(1+L_FTe^{L_FT}).$$

Integrating the square of above inequality over Γ_C and taking the square root we get (5.3).

Moreover we use the condition $H(F)(ii)$ to obtain the following estimate

$$\|\beta'(t)\|^2_{L^2(\Gamma_C)}=\|F(\cdot,t,u(\cdot,t),\beta(\cdot,t))\|^2_{L^2(\Gamma_C)}=$$
$$=\int_{\Gamma_C}|F(x,t,u(x,t),\beta(x,t))-F(x,t,u(x,t),0)|^2d\Gamma\le L_F\|\beta(t)\|^2_{L^2(\Gamma_C)},$$

for a.e. $t\in[0,T]$, which means, that

$$\|\beta'\|_{L^\infty(0,T;L^2(\Gamma_C))}\le L_F\|\beta\|_{L^\infty(0,T;L^2(\Gamma_C))}. \tag{5.4}$$

From (5.3) and (5.4) we have

$$\|\beta'\|_{L^\infty(0,T;L^2(\Gamma_C))}\le L_F(1+L_FTe^{L_FT})\|\beta^0\|_{L^2(\Gamma_C)}. \tag{5.5}$$

Combining (5.3) with (5.5) we get

$$\|\beta\|_{W^{1,\infty}(0,T;L^2(\Gamma_C))}\le(1+L_F)(1+L_FTe^{L_FT})\|\beta^0\|_{L^2(\Gamma_C)}. \tag{5.6}$$

From (5.2) and (5.6) we get the estimate (5.1). □

5.2. **Optimal control with respect to input data.** We introduce the spaces $\Phi = \mathcal{V}^* \times V \times H \times L^2(\Gamma_C)$, $\Psi = \mathcal{Z}^* \times V \times H \times L^2(\Gamma_C)$ and $Y = \mathcal{V} \times \mathcal{W} \times W^{1,\infty}(0,T;L^2(\Gamma_C))$. We endow Y with the product topology $\tau_Y = (w-\mathcal{V}) \times (w-\mathcal{W}) \times (w*-W^{1,\infty}(0,T;L^2(\Gamma_C))$, Φ with topology $\tau_\Phi = (w-\mathcal{V}^*) \times (w-V) \times (w-H) \times (s-L^2(\Gamma_C))$ and Ψ with topology $\tau_\Psi = (w-\mathcal{Z}^*) \times (s-V) \times (s-H) \times (s-L^2(\Gamma_C))$. In Φ and Ψ we define the sets $\bar{\Phi} = \mathcal{V}^* \times V \times H \times Q$ and $\bar{\Psi} = \mathcal{Z}^* \times V \times H \times Q$. We observe that $\bar{\Phi}$ is closed in τ_Φ and $\bar{\Psi}$ is closed in τ_Ψ. We also define the multivalued mappings $S : \bar{\Phi} \to Y$ as $S(\phi) = \{y \in Y : y$ solves (AVFC) with the data $\bar{\Phi} \ni \phi = (f, u^0, u^1, \beta^0)\}$ and R being the restriction of S on $\bar{\Psi}$.

The control problem is formulated as follows. Given $\emptyset \neq \Phi_{ad} \subset \bar{\Phi}$ (respectively $\Phi_{ad} \subset \bar{\Psi}$), representing the set of admissible controls, and an objective functional $\mathcal{F} : \bar{\Phi} \times Y \to \mathbb{R}$, find a control $\phi^* \in \Phi_{ad}$ and a state $y^* \in S(\phi^*) \subset Y$ such that

$$\mathcal{F}(\phi^*, y^*) = \inf\{\mathcal{F}(\phi, y) : \phi \in \Phi_{ad},\, y \in S(\phi)\}. \tag{5.7}$$

The proof of the existence of an optimal solution for (5.7) is based on a result on the continuous (in τ_Φ or respectively τ_Ψ) dependence of solution of (3.18) on the input data.

We show the following lemma.

Lemma 5.3. *Under assumptions $H(\mathcal{G}), H(j_N), H(j_T), H(F), H_0$ and $H(\mathcal{C})$(i),(v)–(vii) (respectively $H(\mathcal{C})$(i)–(v)) if for some sequence $\{\phi_k\}_k \subset \bar{\Phi}$ such that $\phi_k \to \phi$ in τ_Φ we have $y_k \in S(\phi_k)$ (respectively $\{\phi_k\}_k \subset \bar{\Psi}$ such that $\phi_k \to \phi$ in τ_Ψ we have $y_k \in R(\phi_k)$), then there exists the subsequence $y_k \to y$ in τ_Y, such that $y \in S(\phi)$ (respectively $y \in R(\phi)$).*

Proof. First we take the sequence $\phi_k = (f_k, u_k^0, u_k^1, \beta_k^0) \to (f, u^0, u^1, \beta^0) = \phi$ in τ_Φ (or respectively τ_Ψ). We observe that since $\bar{\Phi}$ is $\tau_\Phi-$closed (respectively $\bar{\Psi}$ is $\tau_\Psi-$closed) then $\phi \in \bar{\Phi}$ (respectively $\phi \in \bar{\Psi}$). We define (u_k, u_k') as (one of possibly many) solutions of the HVI

$$\begin{aligned} &\text{find } u_k \in \mathcal{V} \text{ with } u_k' \in \mathcal{W} \text{ such that for a.e. } t \in (0,T) \\ &u_k''(t) + A(t, u_k'(t)) + Bu_k(t) + \bar{\gamma}^*(\partial J(t, \mathcal{R}(\widetilde{\gamma} u_k, \beta_k^0)(t), \widetilde{\gamma} u_k(t))) \ni f_k(t), \\ &u_k(0) = u_k^0, \quad u_k'(0) = u_k^1. \end{aligned} \tag{5.8}$$

In addition let $\beta_k = \mathcal{R}(\widetilde{\gamma} u_k, \beta_k^0)$. By Proposition 5.1, $\{\beta_k\}$ is bounded in $W^{1,\infty}(0,T;L^2(\Gamma_C))$, $\{u_k\}$ is bounded in $C(0,T;V)$ and $\{u_k'\}$ is bounded in $\mathcal{W}$. Therefore we can extract subsequences $\{\beta_k\}$ and $\{u_k\}$ such that

$$\beta_k \to \beta \qquad w*-W^{1,\infty}(0,T;L^2(\Gamma_C)), \tag{5.9}$$

$$u_k \to u \qquad w-\mathcal{V}, \tag{5.10}$$

$$u_k' \to u' \qquad w-\mathcal{V}, \tag{5.11}$$

$$u_k'' \to u'' \qquad w - \mathcal{V}^*, \tag{5.12}$$

where $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_C))$, $u \in \mathcal{V}$ and $u' \in \mathcal{W}$. It suffices to show that (u, u', β) solves (AVFC) with (f, u^0, u^1, β^0). We perform this in 4 steps.

Step 1. Passing to the limit in initial conditions. By the continuity of embedding $\mathcal{W} \subset C(0,T;H)$ we have $u_k \to u$ weakly in $C(0,T;H)$ so for all $t \in [0,T]$ we have $u_k(t) \to u(t)$ weakly in H. Therefore in particular $u_k^0 = u_k(0) \to u(0)$ weakly in H, so $u(0) = u^0$. Similarly $u_k' \to u'$ weakly in $C(0,T;H)$ so $u_k'(0) \to u'(0)$ weakly in H and therefore $u'(0) = u^1$. Finally, since $\beta_k \to \beta$ weakly in $C(0,T;L^2(\Gamma_C))$ we have $\beta_k(0) \to \beta(0)$ weakly in $L^2(\Gamma_C)$ and therefore $\beta(0) = \beta^0$.

Step 2. Passing to the limit in ODE. Let us define $\bar{\beta} = \mathcal{R}(\widetilde{\gamma} u, \beta^0)$. We show that $\bar{\beta} = \beta$ and $\beta_k(t) \to \beta(t)$ strongly in $L^2(\Gamma_C)$ for all $t \in [0,T]$. We define $r_k := \beta_k - \bar{\beta}$. We observe that $r_k(0) = \beta_k^0 - \beta^0$ and $r_k'(x,t) = F(x,t,\widetilde{\gamma} u_k(x,t), \beta_k(x,t)) - F(x,t,\widetilde{\gamma} u(x,t), \bar{\beta}(x,t))$ almost everywhere on $\Gamma_C \times (0,T)$. Integrating above equation on the interval $(0,t)$ for all $t \in (0,T)$ and almost all $x \in \Gamma_C$, we obtain

$$|r_k(x,t)| \le \int_0^t |F(x,s,\widetilde{\gamma} u_k(x,s), \beta_k(x,s)) - F(x,s,\widetilde{\gamma} u(x,s), \bar{\beta}(x,s))|\, ds + |r_k(x,0)|.$$

By $H(F)(ii)$ we get

$$|r_k(x,t)| \le |r_k(x,0)| + L_F \int_0^t |r_k(x,s)|\, ds + L_F \int_0^t |\widetilde{\gamma} u_k(x,s) - \widetilde{\gamma} u(x,s)|\, ds.$$

By the Gronwall inequality we obtain

$$\begin{aligned} |r_k(x,t)| \le |r_k(x,0)| + L_F \int_0^t |\widetilde{\gamma} u_k(x,s) - \widetilde{\gamma} u(x,s)|\, ds + \\ + \int_0^t e^{(t-s)L_F} L_F \left(|r_k(x,0)| + L_F \int_0^s |\widetilde{\gamma} u_k(x,r) - \widetilde{\gamma} u(x,r)|\, dr \right) ds. \end{aligned}$$

Therefore, for some constant C_1 (depending on T) we have

$$|r_k(x,t)| \le C_1(|r_k(x,0)| + \int_0^t |\widetilde{\gamma} u_k(x,s) - \widetilde{\gamma} u(x,s)|\, ds).$$

Integrating the square of above inequality over Γ_C and using the Jensen inequality we get

$$\|r_k(t)\|_{L^2(\Gamma_C)} \le C_2(\|r_k(0)\|_{L^2(\Gamma_C)} + \|\widetilde{\gamma} u_k - \widetilde{\gamma} u\|_{L^2(0,T;L^2(\Gamma_C))}) \text{ with } C_2 > 0.$$

By the continuity of the trace, we can write (for some constant $C_3 > 0$)

$$\|r_k(t)\|_{L^2(\Gamma_C)} \le C_3(\|r_k(0)\|_{L^2(\Gamma_C)} + \|u_k - u\|_{\mathcal{Z}}) \text{ for all } t \in (0,T).$$

Now since $\mathcal{W} \subset \mathcal{Z}$ is compact and $r_k(0) \to 0$ strongly in $L^2(\Gamma_C)$, then also $\beta_k(t) \to \bar{\beta}(t)$ strongly on $L^2(\Gamma_C)$ for all $t \in (0,T)$ (and $\beta_k \to \bar{\beta}$ strongly in $C(0,T;L^2(\Gamma_C))$). By (5.9) we have $\bar{\beta} = \beta$.

Step 3. Passing to the limit in multivalued term. We can rewrite the inclusion (5.8) as

$$u_k'' + \mathcal{A}u_k' + \mathcal{B}u_k + w_k = f_k, \text{ where } w_k \in \mathcal{N}(u_k, \beta_k^0). \tag{5.13}$$

By $H(J)(iii)$ and the continuity of the trace we get

$$\|w_k(t)\|_{Z^*} \le c_0 \|\bar{\gamma}^*\|(1 + \|\bar{\gamma}\|\|u_k(t)\|_Z) \text{ for a.e. } t \in (0,T).$$

Integrating from 0 to T the square of above inequality and using the fact that embedding $V \subset Z$ is continuous we get

$$\|w_k\|_{\mathcal{Z}^*} \le C(1 + \|u_k\|_{\mathcal{V}}) \text{ for } C > 0.$$

Since $\{u_k\}$ is bounded in $\mathcal{V}$ then $\{w_k\}$ is bounded in $\mathcal{Z}^*$ and we can extract weakly$-\mathcal{Z}^*$ convergent subsequence $w_k \to w$. Our aim is to verify that $w \in \mathcal{N}(u, \beta^0)$. We observe that $w_k = \bar{\gamma}^* \xi_k$ for some $\xi_k \in L^2(0,T; L^2(\Gamma_C))$ and $\xi_k(t) \in \partial J(t, \mathcal{R}(\widetilde{\gamma} u_k, \beta_k^0)(t), \widetilde{\gamma} u_k(t)))$ almost everywhere on $(0,T)$. By compactness of embedding $\mathcal{W} \subset \mathcal{Z}$ we have $u_k \to u$ strongly in $\mathcal{Z}$ so also $\widetilde{\gamma} u_k \to \widetilde{\gamma} u$ strongly in $L^2(0,T; L^2(\Gamma_C))$. It follows that $\widetilde{\gamma} u_k(x,t) \to \widetilde{\gamma} u(x,t)$ almost everywhere in $\Gamma_C \times (0,T)$. Furthermore $\mathcal{R}(\widetilde{\gamma} u_k, \beta_k^0) = \beta_k$ converges to $\beta = \mathcal{R}(\widetilde{\gamma} u, \beta^0)$ strongly in $C(0,T; L^2(\Gamma_C))$. Finally we observe that since $w_k \to w$ weakly in $\mathcal{Z}^*$ then also (by density of $\widetilde{\gamma}(\mathcal{Z})$ in $L^2(0,T; L^2(\Gamma_C))$) we have $\xi_k \to \xi$ weakly in $L^2(0,T; L^2(\Gamma_C))$ and $\xi_k \to \xi$ weakly in $L^1(0,T; L^2(\Gamma_C))$. By the convergence theorem of Aubin and Cellina (cf. [1], Theorem 7.2.2) we get the desired thesis.

Step 4. Passing to the limit in the inclusion. It suffices to verify that $u'' + \mathcal{A}u' + \mathcal{B}u + w = f$. For this purpose let us pass with k to infinity in (5.13). By (5.12), $u_k'' \to u''$ weakly in $\mathcal{V}^*$, by (5.10) and continuity of $\mathcal{B}$ we have $\mathcal{B}u_k \to \mathcal{B}u$ weakly in $\mathcal{V}^*$. Moreover, by the assumption, $f_k \to f$ weakly in $\mathcal{V}^*$. Finally, by the continuity of embedding $\mathcal{Z}^* \subset \mathcal{V}^*$, we have $w_k \to w$ weakly in $\mathcal{V}^*$. It therefore suffices to verify that $\mathcal{A}u_k' \to \mathcal{A}u'$ weakly in $\mathcal{V}^*$. We do it separately for two cases:

Linear case. $(H(\mathcal{C})(i), (v) - -(viii))$ Since $u_k' \to u'$ weakly in $\mathcal{V}$ by (5.11) we get the thesis by the Lemma 3.8 (ii) and the fact that linear and continuous operators are also weakly continuous.

Nonlinear case. $(H(\mathcal{C})(i) - -(v))$ Hither and below the convergence of input data ϕ_k is assumed to hold in τ_Ψ. We know that $\mathcal{A}u_k' = f_k - u_k'' - \mathcal{B}u_k - w_k$ is bounded in $\mathcal{V}^*$. Therefore, for some subsequence, $\mathcal{A}u_k' \to \zeta$ weakly in $\mathcal{V}^*$. Since we have (5.11) and (5.12) by Lemma 3.8 (i) in order to obtain $\zeta = \mathcal{A}u'$ it suffices to show that $\limsup_{n\to\infty} \langle \mathcal{A}u_k', u_k' \rangle \le \langle \zeta, u' \rangle$. Let us estimate

$$\begin{aligned} \limsup_{n\to\infty} \langle \mathcal{A}u_k', u_k' \rangle &= \limsup_{n\to\infty} \langle f_k - u_k'' - \mathcal{B}u_k - w_k, u_k' \rangle \le \\ &\le \limsup_{n\to\infty} \langle f_k, u_k' \rangle - \liminf_{n\to\infty} \langle u_k'', u_k' \rangle - \liminf_{n\to\infty} \langle \mathcal{B}u_k, u_k' \rangle - \liminf_{n\to\infty} \langle w_k, u_k' \rangle. \end{aligned} \tag{5.14}$$

We deal with each term separately. By compact embedding $\mathcal{W} \subset \mathcal{Z}$ we have $u_k' \to u'$ strongly in $\mathcal{Z}$. Since $f_k \to f$ weakly in $\mathcal{Z}^*$, then $\lim_{n\to\infty}\langle f_k, u_k'\rangle = \langle f, u'\rangle$. Similarly, since $w_k \to w$ weakly in $\mathcal{Z}^*$, then $\lim_{n\to\infty}\langle w_k, {u_k}'\rangle = \langle w, u'\rangle$. Furthermore, we observe that $\langle u_k'', u_k'\rangle = \frac{1}{2}\|u_k'(T)\|_H^2 - \frac{1}{2}\|u_k'(0)\|_H^2$. By the weak convergence of $u_k'(T)$ to $u'(T)$ and the weak lower semicontinuity of the norm, we get

$$\liminf_{n\to\infty}\langle u_k'', u_k'\rangle \geq \liminf_{n\to\infty}\frac{1}{2}\|u_k'(T)\|_H^2 - \limsup_{n\to\infty}\frac{1}{2}\|u_k'(0)\|_H^2 \geq$$
$$\geq \frac{1}{2}\|u'(T)\|_H^2 - \limsup_{n\to\infty}\frac{1}{2}\|u_k^1\|_H^2 = \frac{1}{2}\|u'(T)\|_H^2 - \frac{1}{2}\|u'(0)\|_H^2 = \langle u'', u'\rangle$$

Finally we have $\int_0^T\langle Bu(t), u'(t)\rangle\, dt = \frac{1}{2}\langle Bu(T), u(T)\rangle - \frac{1}{2}\langle Bu(0), u(0)\rangle$ for $u \in \mathcal{W}$ from Lemma 3.8 (iii), and therefore

$$\liminf_{n\to\infty}\langle \mathcal{B}u_k, u_k'\rangle = \liminf_{n\to\infty}\int_0^T\langle Bu_k(t), u_k'(t)\rangle\, dt \geq \tag{5.15}$$
$$\geq \liminf_{n\to\infty}\frac{1}{2}\langle Bu_k(T), u_k(T)\rangle - \limsup_{n\to\infty}\frac{1}{2}\langle Bu_k(0), u_k(0)\rangle.$$

We use the continuous embedding $H^1(0,T;V) \subset C(0,T;V)$ to note that $u_k(T) \to u(T)$ weakly in V. From Lemma 3.4 if follows that $\langle Bu(T), u(T)\rangle \leq \liminf_{n\to\infty}\langle Bu_k(T), u_k(T)\rangle$. Coming back to (5.15) and taking advantage of the fact that $u_k(0) = u_k^0 \to u^0 = u(0)$ strongly in V and B is continuous, we arrive at

$$\liminf_{n\to\infty}\langle \mathcal{B}u_k, u_k'\rangle \geq \frac{1}{2}\langle Bu(T), u(T)\rangle - \frac{1}{2}\langle Bu(0), u(0)\rangle = \langle \mathcal{B}u, u'\rangle.$$

Applying above formulas in (5.14), we get the thesis

$$\limsup_{n\to\infty}\langle \mathcal{A}{u_k}', {u_k}'\rangle \leq \langle f - u'' - \mathcal{B}u - w, u'\rangle = \lim_{k\to\infty}\langle f_k - {u_k}'' - \mathcal{B}u_k - w_k, u'\rangle = \langle \zeta, u'\rangle.$$

The proof is complete. □

We need some additional hypotheses concerning the set of admissible controls and the objective functional.

<u>$H(\Phi_{ad})$</u>:

(i) $\Phi_{ad} \subset \bar{\Phi}$ is compact in the τ_Φ topology,
(ii) $\Phi_{ad} \subset \bar{\Psi}$ is compact in the τ_Ψ topology.

<u>$H(\mathcal{F})$</u>:

(i) $\mathcal{F}$ is lower semicontinuous with respect to the $\tau_\Phi \times \tau_Y$ topology,
(ii) $\mathcal{F}$ is lower semicontinuous with respect to the $\tau_\Psi \times \tau_Y$ topology.

We are now in a position to deliver an existence result for the optimal control problem (5.7).

Theorem 5.4. *Assume that one of the following hypotheses holds:*

(*) *$H(\mathcal{C})(i),(v)--(vii)$, $H(\mathcal{G})$, $H(j_N)$, $H(j_T)$, $H(f)$, $H(F)$, $H(\Phi_{ad})(i)$, $H(\mathcal{F})(i)$ and (4.1), or*

(**) *$H(\mathcal{C})(i–v)$, $H(\mathcal{G})$, $H(j_N)$, $H(j_T)$, $H(f)$, $H(F)$, $H(\Phi_{ad})(ii)$, $H(\mathcal{F})(ii)$ and (4.1).*

Then the problem (5.7) *admits an optimal solution.*

Proof. Let us suppose that $(*)$ holds. Let $\{(\phi_n, y_n)\}$ be a minimizing sequence for the problem (5.7), that is, $\phi_n \in \Phi_{ad}, y_n \in \mathcal{S}(\phi_n)$ and

$$\lim_{n\to\infty} \mathcal{F}(\phi_n, y_n) = \inf\{\mathcal{F}(\phi, y) : \phi \in \Phi_{ad},\ y \in \mathcal{S}(\phi)\} = m \in [-\infty, +\infty).$$

From $H(\Phi_{ad})(i)$ we may choose a subsequence $\{\phi_n\}$ such that $\phi_n \to \phi^*$ with respect to the topology τ_Φ and $\phi^* \in \Phi_{ad}$. From Lemma 3.8 (ii) and Lemma 5.3 we obtain $y_n \to y^*$ in Y with respect to the topology τ_Y and $y^* \in S(\phi^*)$. Thus, due to $H(\mathcal{F})(i)$, we have $m \le \mathcal{F}(\phi^*, y^*) \le \liminf_{n\to\infty} \mathcal{F}(\phi_n, y_n) = m$, which completes the proof with assumption $(*)$. The proof of Theorem 5.4 with assumption (**) is analogous. □

In order to make Theorem 5.4 useful, we need to analyze the assumptions $H(\Phi_{ad})(i)$, $H(\Phi_{ad})(ii)$ and $H(\mathcal{F})(i)$, $H(\mathcal{F})(ii)$. First, let us focus on assumptions on the admissible controls. If the set Φ_{ad} is a cartesian product $\Phi_{ad} = \Phi^1_{ad} \times \Phi^2_{ad} \times \Phi^3_{ad} \times \Phi^4_{ad}$, then conditions $H(\Phi_{ad})(i)$ and $H(\Phi_{ad})(ii)$ can be reformulated using the following equivalences:

$H(\Phi_{ad})(i)$ holds iff $\Phi^1_{ad}, \Phi^2_{ad}, \Phi^3_{ad}$ are weakly compact in $\mathcal{V}^*$, V and H respectively while $\Phi^4_{ad} \subset Q$ is compact in $L^2(\Gamma_C)$.

$H(\Phi_{ad})(ii)$ holds iff Φ^1_{ad} is weakly compact in $\mathcal{Z}^*$ while $\Phi^2_{ad}, \Phi^3_{ad}, \Phi^4_{ad} \subset Q$ are compact in V, H and $L^2(\Gamma_C)$ respectively.

Due to reflexivity of the spaces $\mathcal{V}^*$, $\mathcal{Z}^*$, V and H, the weak compactness in above properties is a consequence of boundedness and closedness of respective sets. As for strong compactness by a compact embedding $H^2(\Omega;\mathbb{R}^d) \subset H^1(\Omega;\mathbb{R}^d)$ if $\Phi^2_{ad} \subset H^2(\Omega;\mathbb{R}^d)$ is bounded in $H^2(\Omega;\mathbb{R}^d)$ then the strong compactness in V holds. In order to verify the strong compactness in L^2 spaces we need some simple criteria for strong compactness of sets in the spaces $L^2(\Theta;\mathbb{R}^N)$, where $(\Theta, N) \in \{(\Omega, d), (\Gamma_C, 1)\}$. In this context we provide a simple

Corollary 5.5. *If the set $\Phi \subset L^2(\Theta;\mathbb{R}^N)$ satisfies one of following conditions*

(i) *$\Phi \subset C(\Theta;\mathbb{R}^N)$, all functions from Φ are uniformly bounded and equicontinuous.*
(ii) *$\Phi \subset H^1(\Theta;\mathbb{R}^N)$, Φ is closed and bounded in $H^1(\Theta;\mathbb{R}^N)$.*
(iii) *$\Theta = \bigcup_{i=1}^K \overline{\Theta}_i$ where Θ_i are open and mutually disjoint sets and there exist compact sets $\Phi_i \subset L^2(\Theta_i;\mathbb{R}^N)$ such that for every $\phi \in \Phi$ we have $\phi|_{\Theta_i} \in \Phi_i$ for $i = 1 \ldots K$, with $K \in \mathbb{N}_+$*

(iv) *The set* Φ *is closed in* $L^2(\Theta;\mathbb{R}^N)$. *For every* $f\in\Phi$ *there exist* $K\in\mathbb{N}_+$ *and* $\{\Theta_i\}_{i=1}^K$, *where sets* Θ_i *are open and mutually disjoint and* $\Theta=\bigcup_{i=1}^K\overline{\Theta}_i$. *Furthermore there exists* $D_1>0$ *such that for every* $f\in\Phi$ *there exists* $h_0>0$ *such that for every* $0<h\leq h_0$ *we have* $m(\{x:\ dist(x,\bigcup_{i=1}^K\partial\Theta_i)\leq h\})\leq D_1 h$. *Moreover there exists* $D_2>0$ *and* $D_3>0$ *such that for all* $f\in\Phi$ *and almost every* $x\in\Theta$ *we have* $f(x)\leq D_2$ *and for every* $i\in\{1,\ldots,K\}$ $f|_{\Theta_i}$ *satisfies the Lipschitz condition with the constant* D_3.

then Φ *is compact in* $L^2(\Theta;\mathbb{R}^N)$.

Before we pass to the proof of Corollary 5.5, we cite the following lemma (cf. for instance [6], Theorem 2.2.8).

Lemma 5.6. *Let* $p\in[1,+\infty]$ *and* $X\subseteq L^p(\mathbb{R}^n)$. *Then* X *is relatively compact iff*

(a) X *is bounded,*
(b) $\forall\varepsilon>0\ \exists\delta>0\ \forall u\in X, h\in\mathbb{R}^n\ |h|<\delta\Rightarrow\|u(\cdot+h)-u(\cdot)\|^p_{L^p(\mathbb{R}^n)}<\varepsilon$,
(c) $\forall\varepsilon>0\ \exists r_\varepsilon>0\ \forall u\in X\ \|u\|_{L^p(\mathbb{R}^n\setminus B_{r_\varepsilon}(0))}<\varepsilon$.

Proof of Corollary 5.5. In the proof we limit to the case $\Theta=\Omega$ (extension to boundary case is purely technical). The compactness of the set Φ under the assumption (i) is a direct consequence of the Arzela-Ascoli theorem. By assumption (ii), it follows from the compact embedding $H^1(\Theta;\mathbb{R}^N)\subset L^2(\Theta;\mathbb{R}^N)$. Let us assume that (iii) holds and consider a sequence $\{\phi^m\}_{m=1}^\infty\subset\Phi$. Our goal is to construct a function $\phi\in\Phi$ such that a subsequence of $\{\phi^m\}$ tends to ϕ in $L^2(\Theta;\mathbb{R}^N)$. We define functions $\phi_i^m=\phi^m|_{\Theta_i}\in\Phi_i$ for $i=1\ldots K, m=1\ldots\infty$. By compactness of sets Φ_i for $i=1\ldots K$ and the fact that K is a finite number, we can extract a subsequence $\{\phi^{m_K}\}\subset\{\phi^m\}$ such that each sequence $\{\phi_i^{m_K}\}$, defined as $\phi_i^{m_K}=\phi^{m_K}|_{\Theta_i}$ converges to a function $\phi_i\in\Phi_i$ in the norm $L^2(\Theta_i;\mathbb{R}^N)$. We consider a function $\phi=\sum_{i=1}^K\chi_{\Theta_i}\psi_i$, where χ_{Θ_i} denotes the characteristic function of Θ_i and estimate the norm

$$\|\phi^{m_K}-\phi\|^2_{L^2(\Theta;\mathbb{R}^N)}=\int_\Theta|\phi^{m_K}(x)-\phi(x)|^2dx=\sum_{i=1}^K\int_{\Theta_i}|\phi_i^{m_K}(x)-\phi_i(x)|^2dx\to 0.$$

Finally, we assume that (iv) holds and use Lemma 5.6 with $p=2$ and $X=\Phi$. For simplicity we consider only the one-dimensional case. We have to verify properties (a), (b) and (c) in the reference to Φ. The conditions (a) and (c) obviously follow from (iv) and boundedness of Θ. In order to prove (b), we fix $\varepsilon>0$. Let

$$\delta=\frac{\sqrt{16D_1^2D_2^4+4m(\Theta)D_3^2\varepsilon}-4D_1D_2^2}{2m(\Theta)D_3^2} \tag{5.16}$$

and $f\in\Phi$. Let K and $\{\Theta_i\}_{i=1}^K$ be defined as in (iv). We denote $\mathcal{K}=\bigcup_{i=1}^K\partial\Theta_i$. For an arbitrary $h\in\mathbb{R}^d$ we define $\mathcal{K}_h=\{x\in\Omega:\ dist(x,\mathcal{K})\leq|h|\}$, $\Omega_1=\{x\in\Omega:\ \exists i\in$

$\{1 \ldots K\} : x, x+h \in \Theta_i\}$ and $\Omega_2 = \Omega \setminus \Omega_1$. It is easy to see that $\Omega_2 \subset \mathcal{K}_h$. Hence, from the Lipshitz condition in (iv) and (5.16) we have for $|h| < \delta$

$$\|f(\cdot + h) - f(\cdot)\|^2_{L^2(\Omega)} = \int_{\Omega_1} |f(x+h) - f(x)|^2 dx +$$
$$+ \int_{\Omega_2} |f(x+h) - f(x)|^2 dx \leq \int_{\Omega_1} D_3^2 |h|^2 dx + m(\Omega_2) 4 D_2^2 \leq$$
$$\leq m(\Omega) D_3^2 |h|^2 + m(\mathcal{K}_h) 4 D_2^2 \leq |h|^2 m(\Omega) D_3^2 + 4 D_1 D_2^2 |h| < \varepsilon$$

The condition (b) is verified and the proof is complete.

Now we move to the discussion of objective functionals. As example we may consider the following functional

$$F_1(\phi, y) = \int_0^T L(t, u(t), u'(t), \beta(t), \beta'(t))\, dt + G(\phi),$$

where G is prescribed functional on $\bar{\Phi}$ (or respectively $\bar{\Psi}$), that is lower semicontinuous with respect to τ_Φ (or respectively τ_Ψ) and L is prescribed functional on $[0,T] \times V \times V \times L^2(\Gamma_C) \times L^2(\Gamma_C)$ such that for a.e. $t \in [0,T]$ $L(t, \cdot, \cdot, \cdot, \cdot)$ is convex and lower semicontinuous and furthermore for some $M > 0, \alpha \in L^1(0,T;\mathbb{R})$ and a.e. $t \in [0,T]$ we have $L(t, v, w, \gamma, \delta) \geq \alpha(t) - M(\|v\| + \|w\| + \|\gamma\|_{L^2(\Gamma_C)} + \|\delta\|_{L^2(\Gamma_C)})$ for all $v, w \in V$ and $\gamma, \delta \in L^2(\Gamma_C)$ (see [2]).

Other examples of admissible functionals are the following

$$\begin{aligned}
F_2(\phi, y) &= \sum_{i=0}^{r} \left(\|u(t_i) - w_i^1\|^2 + \|u'(t_i) - w_i^2\|_H^2 + \|\beta(t_i) - w_i^3\|^2_{L^2(\Gamma_C)} \right), \\
F_3(\phi, y) &= \int_0^T \int_\Omega \left(|u(x,t) - w^4(x,t)|^2 + |u'(x,t) - w^5(x,t)|^2 \right) dx + \\
&\quad + \int_{\Gamma_C} \left(|\beta(x,t) - w^6(x,t)|^2 + |\beta'(x,t) - w^7(x,t)|^2 \right) d\Gamma\, dt, \\
F_4(\phi, y) &= \int_0^T \int_\Gamma |u(x,t) - w^8(x,t)|^2 + |u'(x,t) - w^9(x,t)|^2\, d\Gamma\, dt,
\end{aligned}$$

where $\phi = (f, u^0, u^1, \beta^0)$, $y = (u, u', \beta)$, $0 < t_1 < t_2 < \ldots < t_r \leq T$ are points of measurements and $w_i^1 \in V$, $w_i^2 \in V$, $w_i^3 \in L^2(\Gamma_C)$, $w^4, w^5 \in \hat{\mathcal{H}}$, $w^6, w^7 \in L^2(0,T;L^2(\Gamma_C))$, $w^8, w^9 \in L^2(0,T;L^2(\Gamma;\mathbb{R}^d))$ are fixed targets,

$$F_5(\phi, y) = \int_0^T \int_\Omega \rho(x,t) \|\sigma^D(u)(x,t) - \sigma_g(x,t)\|^2_{\mathcal{S}_d}\, dx\, dt,$$

where $\sigma^D = \sigma - \frac{1}{d}(tr\sigma)I$ is the stress deviator, $tr\sigma$ is the trace of σ and I is the identity matrix, ρ is a smooth weight function and σ_g is a given target.

REFERENCES

[1] J. P. Aubin and H. Frankowska, *Set-Valued Analysis,* Birkhäuser, Boston-Basel-Berlin (1990).

[2] E. J. Balder, Necessary and sufficient conditions for L_1-strong-weak lower semicontinuity of integral functionals, *Nonlinear Analysis: Theory, Methods & Applications*, 11 (1987), 1399-1404.

[3] K. Bartosz, Hemivariational inequality approach to the dynamic viscoelastic sliding contact problem with wear, *Nonlinear Analysis: Theory, Methods & Applications*, 65 (2006), 546–566.

[4] K. Bartosz, *Hemivariational Inequalities Modeling Dynamic Contact Problems in Mechanics*, PhD Thesis, Jagiellonian University, Kraków (2007).

[5] K. Bartosz, Hemivariational Inequalities Modeling Dynamic Contact Problems with Adhesion, *Nonlinear Analysis: Theory, Methods & Applications*, 71 (2009), 1747–1762.

[6] L. Carbone and R. de Arcangelis, *Unbounded Functionals in the Calculus of Variations*, Chapman & Hall/CRC (2001).

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, Interscience, New York (1983).

[8] Z. Denkowski and S. Migórski, Optimal shape design problems for a class of systems described by hemivariational inequalities, *Journal of Global Optimization*, 12 (1998), 37–59.

[9] Z. Denkowski, S. Migórski and A. Ochal, Optimal control for a class of mechanical thermoviscoelastic frictional contact problems, *Control and Cybernetics*, 36 (2007), 611–632.

[10] G. Duvaut and J. L. Lions, *Les Inéquations en Mécanique et en Physique*, Dunod, Paris (1972).

[11] M. Frémond, Adhérence des solides, *Journal de Mécanique Théorique et Appliquée*, 6 (1987), 383–407.

[12] M. Frémond, *Non-Smooth Thermomechanics*, Springer, Berlin (2002).

[13] L. Gasiński, Evolution hemivariational inequality with hysteresis and optimal control problem, *Lecture Notes in Pure and Applied Mathematics*, 240 (2005), 157–168.

[14] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, American Mathematical Society, International Press (2002).

[15] M. Miettinen, *Approximation of Hemivariational Inequalities and Optimal Control Problems*, PhD Thesis, University of Jyväskylä, Finland (1993).

[16] P. Kalita, Decay of energy for second-order boundary hemivariational inequalities with coercive damping, *Nonlinear Analysis: Theory, Methods & Applications*, 74 (2011), 1164–1181.

[17] P. Kalita, Regularity and Rothe method error estimates for parabolic hemivariational inequality, to appear in *Journal of Mathematical Analysis and Applications*, doi:10.1016/j.jmaa.2011.12.007.

[18] S. Migórski, Optimal control for a class of hyperbolic hemivariational inequalities, *Proceedings of the International Conference "Control and Applications"*, Cancun, Mexico, M. H. Hamza Ed., Acta Press, 75–80 (2002).

[19] S. Migórski, Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction, *Applicable Analysis*, 84 (2005), 669–699.

[20] S. Migórski, Boundary hemivariational inequalities of hyperbolic type and applications, *Journal of Global Optimization*, 31 2005), 505–533.

[21] S. Migórski and A. Ochal, Optimal control of parabolic hemivariational inequalities, *Journal of Global Optimization*, 17 (2000), 285–300.

[22] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications,* Marcel Dekker, Inc., New York, Basel, Hong Kong (1995).

[23] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic anf Elasto-plastic Bodies: An Introduction*, Elsevier, Amsterdam, Oxford, New York (1981).

[24] A. Ochal, *Optimal Control Problems for Evolution Hemivariational Inequalities of Second Order*, PhD Thesis, Jagiellonian University, Kraków (2001).

[25] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*, Birkhäuser, Basel (1985).

[26] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin (1993).

[27] P. D. Panagiotopoulos and J. Haslinger, Optimal control and identification of structures involving multivalued nonmonotonicities. Existence and approximation results, *European Journal of Mechanics. A. Solids*, 11 (1992), 425–445 .

[28] M. Raous, L. Cangémi and M. Cocu, A consistent model coupling adhesion, friction and unilateral contact, *Computer Methods in Applied Mechanics and Engineering*, 177 (1999), 383–399.

[29] J. Rojek and J. J. Telega, Numerical simulation of bone-implant systems using a more realistic model of the contact interfaces with adhesion, *Journal of Theoretical and Applied Mechanics*, 37 (1999), 659–686.

[30] J. Rojek, J. J. Telega and S. Stupkiewicz, Contact problems with friction adhesion and wear in orthopaedic biomechanics II: Numerical implementation and application to implanted knee joints, *Journal of Theoretical and Applied Mechanics*, 39 (2001), 679–706.

[31] M. Selmani and M. Sofonea, Viscoelastic frictionless contact problems with adhesion, *Journal of Inequalities and Applications*, 2006 Article ID 36130 (2006), 22 pages.

[32] M. Shillor, M. Sofonea and J. J. Telega, *Models and Analysis of Quasistatic Contact*, Springer, Berlin (2004).

[33] M. Sofonea, W. Han and M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics, vol. 276, Chapman & Hall/CRC, Boca Raton (2006).

[34] N. Strömberg, L. Johansson and A. Klarbring, Derivation and analysis of a generalized standard model for contact, friction and wear, *International Journal of Solids and Structures*, 33 (1996), 1817–1836.