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ON MULTIVALUED STOCHASTIC INTEGRAL EQUATIONS DRIVEN BY A WIENER PROCESS IN THE PLANE

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ABSTRACT. In this paper we define a multivalued stochastic integral in the plane. Then a multivalued stochastic equation in the plane as an abstract stochastic equation in a hyperspace of subsets of the space of square integrable random vectors is investigated. We prove existence and uniqueness of solutions as well as some additional properties. Similar results to a fuzzy valued stochastic integral equation in the plane are stated as applications.

Keywords. Random Field, Set-valued Stochastic Integral Equation, Fuzzy Stochastic Integral Equation

AMS (MOS) Subject Classification. 60H20, 60G60, 26E25, 34A07

1. INTRODUCTION

The study on deterministic multivalued differential equations was started in [7] and it has been later extensively developed, among others, in connection with problems of control theory, differential inclusions and fuzzy differential equations (see [5], [8]–[10], [18], [23]–[26], [46] and references therein). Multivalued differential equations exhibit a natural extension of the theory of single-valued differential equations and one of their main advantages is that they can be used as a useful tool for studying properties of solutions to differential inclusions. In stochastic case, although there exists a wide literature where attempts have been made to investigate stochastic differential (or integral) inclusions (see e.g., [1]–[4], [19]–[22], [34], [35], [37]–[39] and references therein), the problem of existence and properties of solutions to stochastic multivalued or fuzzy-valued equations seems to be still insufficiently developed until now. The study concerning with such equations can be found in [28]–[33], [40] and [42]. In this paper we consider multivalued stochastic integral equations driven by

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Wiener process in the plane. The main tool used in this paper is a notion of set-valued stochastic integral in the plane. This integral is a subset of the space of square integrable random vectors. Such an integral is a set-valued counterpart of a single-valued two parameter stochastic integral developed earlier in literature (see e.g., [6], [12], [45], [49] and references therein). Consequently, multivalued stochastic equations are understood as abstract equations in the space of nonempty, closed, bounded and convex subsets of the space of square integrable random vectors. Hence evolutions of their solutions vary in an infinite dimensional space. We present the existence and uniqueness of solutions for such equations as well as their typical properties. We give also some applications to fuzzy-valued stochastic integral equations in the plane. The work presented here extends results obtained both for one-parameter multivalued (also fuzzy-valued) deterministic and stochastic differential equations as well as for single-valued stochastic differential equations in the plane studied earlier in [50]–[53].

2. PRELIMINARIES

Let $I\times J=[0,S]\times[0,T]$ denote the parameter set together with partial ordering:

$$(s,t)\preceq(s',t') \text{ if and only if } s\le s' \text{ and } t\le t'.$$

We will also write

$$(s,t)\prec(s',t') \text{ if and only if } s< s' \text{ and } t< t'.$$

Throughout the paper we shall deal with a complete filtered probability space $(\Omega,\mathbb{F},\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J},P)$, where $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ is a family of sub-σ-fields of $\mathbb{F}$ such that $\mathbb{F}_{s,t}\subset\mathbb{F}_{s',t'}$, if $(s,t)\preceq(s',t')$. We will assume that $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ satisfies the following additional conditions:

(i) $\mathbb{F}_{0,0}$ contains all P-null sets,
(ii) $\mathbb{F}_{s,t}=\bigcap_{(s,t)\prec(u,v)}\mathbb{F}_{u,v}$ for every $(s,t)\in[0,S)\times[0,T)$,
(iii) for every $(s,t)\in I\times J$, the σ-algebras $\mathbb{F}_{s,T}$ and $\mathbb{F}_{S,t}$ are conditionally independent relative to $\mathbb{F}_{s,t}$.

A stochastic process (or random field) $X:I\times J\times\Omega\to\mathbb{R}^d$ is said to be $\{\mathbb{F}_{s,t}\}$-adapted, if $X(s,t,\cdot):\Omega\to\mathbb{R}^d$ is an $\mathbb{F}_{s,t}$-measurable random vector for every fixed $(s,t)\in I\times J$. Let $\{B_{s,t}\}_{(s,t)\in I\times J}$ be a two-parameter real valued $\{\mathbb{F}_{s,t}\}$-Wiener process. In particular, it is a two-parameter, continuous Gaussian process such that $\mathbb{E}B_{s,t}=0$ and $\mathbb{E}(B_{s,t}B_{s',t'})=\min\{s,s'\}\cdot\min\{t,t'\}$ for every $s,s'\in I$ and $t,t'\in J$ (see [6]). Let $\mathcal{N}$ denote the σ-algebra of nonanticipating sets (elements) in $I\times J\times\Omega$, i.e.,

$$\mathcal{N}:=\{A\in\mathcal{B}(I\times J)\otimes\mathbb{F}: A^{s,t}\in\mathbb{F}_{s,t}\ \ \forall(s,t)\in I\times J\},$$

where $A^{s,t}=\{\omega\in\Omega:(s,t,\omega)\in A\}$. A stochastic process $X:I\times J\times\Omega\to\mathbb{R}^d$ is nonanticipating if it is an $\mathcal{N}$-measurable mapping. It is easy to see that a stochastic process X is nonanticipating if and only if it is $\mathcal{B}(I \times J) \otimes \mathbb{F}$-measurable and $\{\mathbb{F}_{s,t}\}$-adapted. Denote by λ the Lebesgue measure on the σ-algebra $\mathcal{B}(I \times J)$ of Borel sets in $I \times J$. For the sake of convenience we shall use the notations: $L^2_{\mathcal{N}}(\lambda \times P) := L^2(I \times J \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d)$, $L^2_{s,t} := L^2(\Omega, \mathbb{F}_{s,t}, P; \mathbb{R}^d)$ and $L^2 := L^2(\Omega, \mathbb{F}, P; \mathbb{R}^d)$. For $X \in L^2_{\mathcal{N}}(\lambda \times P)$ one can define both the stochastic Lebesgue integral process $\left(\int_0^s \int_0^t X_{u,v}\lambda(du, dv)\right)_{(s,t)\in I\times J}$ and Itô's type integral process $\left(\int_0^s \int_0^t X_{u,v} dB_{u,v}\right)_{(s,t)\in I\times J}$ (see [6] for details). Both of them are continuous and $\{\mathbb{F}_{s,t}\}$-adapted random fields. Moreover, the integral process $\left(\int_0^s \int_0^t X_{u,v} dB_{u,v}\right)_{(s,t)\in I\times J}$ is a continuous square integrable martingale with respect to the filtration $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ and it satisfies Itô's isometry:

$$\mathbb{E}\left\|\int_{s'}^{s}\int_{t'}^{t} X_{u,v} dB_{u,v}\right\|^2_{\mathbb{R}^d} = \mathbb{E}\int_{s'}^{s}\int_{t'}^{t} ||X_{u,v}||^2_{\mathbb{R}^d}\lambda(du, dv)$$

for all $(s,t), (s',t') \in I \times J$ with $(s',t') \preceq (s,t)$. In view of Doob's maximal inequality for two-parameter martingales we have (c.f. [6]):

$$\mathbb{E}\left(\sup_{(s,t)\in I\times J}\left\|\int_0^s\int_0^t X_{u,v} dB_{u,v}\right\|^2_{\mathbb{R}^d}\right) \leq 16\mathbb{E}\int_0^S\int_0^T ||X_{u,v}||^2_{\mathbb{R}^d}\lambda(du, dv).$$

Let $\mathfrak{X}$ be a separable Banach space. By a $\mathcal{K}^b(\mathfrak{X})$ we denote the family of all nonempty closed and bounded subsets of $\mathfrak{X}$ while by $\mathcal{K}^b_c(\mathfrak{X})$ we mean those of elements from $\mathcal{K}^b(\mathfrak{X})$ that are also convex subsets of $\mathfrak{X}$. The Hausdorff metric $H_{\mathfrak{X}}$ in $\mathcal{K}^b(\mathfrak{X})$ is defined by:

$$H_{\mathfrak{X}}(A, B) := \max\{\sup_{a\in A} \operatorname{dist}_{\mathfrak{X}}(a, B), \sup_{b\in B} \operatorname{dist}_{\mathfrak{X}}(b, A)\}$$

where $\operatorname{dist}_{\mathfrak{X}}(a, B) := \inf_{b\in B}||a - b||_{\mathfrak{X}}$ and $||\cdot||_{\mathfrak{X}}$ is a norm in $\mathfrak{X}$. Moreover $(\mathcal{K}^b(\mathfrak{X}), H_{\mathfrak{X}})$ is a complete metric space and $\mathcal{K}^b_c(\mathfrak{X})$ is a closed subspace of this space. Let us assume that $A, B, C, D \in \mathcal{K}^b_c(\mathfrak{X})$. Then it holds (see [15]):

$$H_{\mathfrak{X}}(A + B, C + D) \leq H_{\mathfrak{X}}(A, C) + H_{\mathfrak{X}}(B, D)$$

and

$$H_{\mathfrak{X}}(A + B, C + B) = H_{\mathfrak{X}}(A, C)$$

where $A+B$ denotes the Minkowski sum of A and B. For $A \in \mathcal{K}^b(\mathfrak{X})$ we set $|||A||| := H_{\mathfrak{X}}(A, \{0\}) = \sup_{a\in A}||a||_{\mathfrak{X}}$. If $A, B \in \mathcal{K}^b_c(\mathfrak{X})$ then $A \ominus B$ denotes the Hukuhara difference (if it exists) between the sets A and B, i.e., the set $C \in \mathcal{K}^b_c(\mathfrak{X})$ such that $A = B + C$.

Let $(U, \mathcal{U})$ be a measurable space and let $\mathcal{M}$ denote a set of $\mathcal{U}$-measurable mappings $f : U \to \mathfrak{X}$. Recall, the set $\mathcal{M}$ is said to be decomposable if for every $f_1, f_2 \in \mathcal{M}$ and every $A \in \mathcal{U}$ it holds: $f_1\mathbf{1}_A + f_2\mathbf{1}_{U\setminus A} \in \mathcal{M}$.

Let $F : I \times J \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a given set-valued mapping. It is called a two-parameter nonanticipating set-valued process if it is $\mathcal{N}$-measurable in the sense of set-valued analysis (c.f. [15]). It is called $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded set-valued stochastic process if

$$|||F||| \in L^2(I \times J \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}_+).$$

For such a mapping, by Kuratowski and Ryll-Nardzewski Measurable Selection Theorem (c.f. [15]) the set of its nonanticipating and square integrable selections

$$\mathcal{S}^2_{\mathcal{N}}(F, \lambda \times P) := \{f \in L^2_{\mathcal{N}}(\lambda \times P) : f \in F, \lambda \times P\text{-a.e.}\}$$

is nonempty.

Let $F, G : I \times J \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be set-valued and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded nonanticipating processes. We can define the following set-valued counterparts of single-valued stochastic integrals in the plane.

Definition 2.1. By a two-parameter trajectory stochastic Lebesgue integral of F and by a two-parameter set-valued trajectory Itô's integral of G, we mean the following sets contained in $L^2_{S,T}$

$$\int_0^S \int_0^T F_{u,v} \lambda(du, dv) := \left\{ \int_0^S \int_0^T f_{u,v} \lambda(du, dv) : f \in \mathcal{S}^2_{\mathcal{N}}(F, \lambda \times P) \right\}$$

and

$$\int_0^S \int_0^T G_{u,v} dB_{u,v} := \left\{ \int_0^S \int_0^T g_{u,v} dB_{u,v} : g \in \mathcal{S}^2_{\mathcal{N}}(G, \lambda \times P) \right\}$$

respectively. Similarly, we define:

$$\int_{s'}^s \int_{t'}^t F_{u,v} \lambda(du, dv) := \int_0^S \int_0^T \mathbf{1}_{[s',s] \times [t',t]} (u, v) \, F_{u,v} \lambda(du, dv)$$

and

$$\int_{s'}^s \int_{t'}^t G_{u,v} dB_{u,v} := \int_0^S \int_0^T \mathbf{1}_{[s',s] \times [t',t]} (u, v) \, G_{u,v} dB_{u,v}$$

for every $(s, t), (s', t') \in I \times J$ with $(s', t') \preceq (s, t)$.

We shall present some properties of set-valued trajectory Lebesgue and set-valued trajectory Itô's integrals in the plane.

Theorem 2.2. *Let $F : I \times J \times \Omega \to \mathcal{K}^b_c(\mathbb{R}^d)$ be a set-valued and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded nonanticipating process. Then*
a) $\mathcal{S}^2_{\mathcal{N}}(F, \lambda \times P)$ is a nonempty, closed, bounded, convex, decomposable and weakly compact subset of $L^2_{\mathcal{N}}(\lambda \times P)$.

b) both integrals $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ *and* $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ *are nonempty, closed, bounded, convex and weakly compact subsets of* $L^2_{s,t}$ *for every* $(s,t),(s',t') \in I\times J$ *with* $(s',t') \preceq (s,t)$.

Proof. a) As it was mentioned earlier, nonemptiness of $\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ is assured by Kuratowski and Ryll-Nardzewski Measurable Selection Theorem. Other properties of this set follow immediately from the assumptions imposed on F. The decomposability of this set follows by Theorem 3.1 in [14].
b) The convexity of the sets $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ and $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ follows from the same property of the set $S^2_{\mathcal{N}}(F,\lambda\times P)$. The boundedness of the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ in the space $L^2_{s,t}$ is a consequence of the following inequality

$$\mathbb{E}\left\|\int_{s'}^{s}\int_{t'}^{t} f_{u,v}\lambda(du,dv)\right\|^2_{\mathbb{R}^d} \le (s-s')(t-t')\int_{[s',s]\times[t',t]\times\Omega} ||f||^2_{\mathbb{R}^d} d\lambda\times dP, \tag{2.1}$$

for every $f\in S^2_{\mathcal{N}}(F,\lambda\times P)$. So, by reflexivity of the space $L^2_{s,t}$ the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ is sequentially relatively weakly compact (c.f. [11]). In order to show its norm closedness in the space $L^2_{s,t}$ let us take an arbitrary sequence $\{u_n\}\subset \int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ such that $u_n \to u$ in $L^2_{s,t}$ strongly. Then there exists a sequence $\{f^{(n)}\}\subset S^2_{\mathcal{N}}(F,\lambda\times P)$ such that $u_n = \int_{s'}^{s}\int_{t'}^{t} f^{(n)}_{u,v}\lambda(du,dv)$ for every $n\ge 1$. Since the set $S^2_{\mathcal{N}}(F,\lambda\times P)$ is weakly compact, there exists a subsequence $\{f^{(n_k)}\}$ of $\{f^{(n)}\}$ such that $f^{(n_k)} \rightharpoonup f$ in $L^2_{\mathcal{N}}(\lambda\times P)$, where $\rightharpoonup$ denotes weak convergence. In view of part a) the set $S^2_{\mathcal{N}}(F,\lambda\times P)$ is weakly compact. Thus we conclude $f\in S^2_{\mathcal{N}}(F,\lambda\times P)$. For fixed $(s',s),(t',t)\in I\times J$, where $(s',s)\preceq(t',t)$, let us consider a linear operator $T^{s,t}_{s',t'}: L^2_{\mathcal{N}}(\lambda\times P)\to L^2_{s,t}$ defined by $T^{s,t}_{s',t'}(g) := \int_{s'}^{s}\int_{t'}^{t} g_{u,v}\lambda(du,dv)$. In view of (2.1), it follows that the operator $T^{s,t}_{s',t'}$ is norm-to-norm continuous. Hence, by Theorem 15 in chap. 5.3 [11], it is equivalent to its continuity with respect to weak topologies. Therefore $u_{n_k} = T^{s,t}_{s',t'}(f^{(n_k)}) \rightharpoonup T^{s,t}_{s',t'}(f)$ in $L^2_{s,t}$. Thus we have $u = T^{s,t}_{s',t'}(f)$, which proves the closedness of the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ in the norm topology of the space $L^2_{s,t}$. Consequently, it proves its weak compactness as well.

In a similar way one can proceed with the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$. Indeed, let $u\in \int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$. Then there exists a selection $f \in \mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ such that $u =$

$\int_{s'}^{s}\int_{t'}^{t} f_{u,v}dB_{u,v}$. Thus by Doob's maximal inequality and Itô's isometry, we obtain

$$\mathbb{E}\left(||u||^2_{\mathbb{R}^d}\right) \le \mathbb{E}\left(\sup_{(s,t)\in I\times J}\left\|\int_0^s\int_0^t f_{u,v}dB_{u,v}\right\|^2_{\mathbb{R}^d}\right)$$

$$\le 16\mathbb{E}\int_0^S\int_0^T ||f_{u,v}||^2_{\mathbb{R}^d}\,\lambda(du,dv) \le 16\int_{I\times J\times\Omega} |||F|||^2 d\lambda\times dP < \infty.$$

Thus we conclude that $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ is a bounded subset of the space $L^2_{s,t}$ for (s',s), $(t',t)\in I\times J$, where $(s',s)\preceq(t',t)$. Let us take an arbitrary sequence $\{u_n\}\subset \int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ converging strongly to some element $u\in L^2_{s,t}$. Then for every $n\ge 1$ there exists $f^{(n)}\in\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ such that $u_n=\int_{s'}^{s}\int_{t'}^{t} f^n_{u,v}dB_{u,v}$. We shall show that $u=\int_{s'}^{s}\int_{t'}^{t} f_{u,v}dB_{u,v}$ for some $f\in\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$. By Itô's isometry we have:

$$\|u_n-u_m\|^2_{L^2} = \left\|\int_{s'}^{s}\int_{t'}^{t} f^{(n)}_{u,v}dB_{u,v} - \int_{s'}^{s}\int_{t'}^{t} f^{(m)}_{u,v}dB_{u,v}\right\|^2_{L^2}$$

$$= \int_{[s',s]\times[t',t]\times\Omega} \left\|f^{(n)}-f^{(m)}\right\|^2_{\mathbb{R}^d} d\lambda\times dP.$$

Consequently, $\{f^{(n)}\}$ is a Cauchy sequence in $L^2_{\mathcal{N}}(\lambda\times P)$. Hence it is convergent to some element $f\in L^2_{\mathcal{N}}(\lambda\times P)$. Since the set $\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ is closed, we conclude that $u=\int_{s'}^{s}\int_{t'}^{t} f_{u,v}dB_{u,v}$. This proves the closedness of the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ in the space $L^2_{s,t}$. In a similar way as in the case of the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ one can prove the weak compactness of the set $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$. □

Remark 2.3. Similarly as in the one-parameter case ([36]), the set-valued trajectory integral $\int_0^S\int_0^T F_{u,v}dB_{u,v}$ need not be a decomposable subset of the space $L^2(\Omega,\mathbb{F}_{S,T},P;\mathbb{R}^d)$ in general. Indeed, let $I\times J=[0,1]^2$ and let us take an one-dimensional two-parameter Wiener process $\{B_{s,t}\}_{(s,t)\in I\times J}$ on a given filtered probability space $(\Omega,\mathbb{F},\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J},P)$. Let $F: I\times J\times\Omega\to\mathcal{K}^b_c(\mathbb{R}^1)$ be a constant set $F=[0,1]$. Hence F is nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded multifunction (in fact it is bounded). Then, in particular, the random variables $u=0$ and $v=B_{1,1}$ belong to the set $\int_0^1\int_0^1 F_{u,v}dB_{u,v}$. If this set were decomposable in the space $L^2(\Omega,\mathbb{F}_{1,1},P;\mathbb{R}^1)$, then for every $A\in\mathbb{F}_{1,1}$

there would exists a selection $f \in \mathcal{S}^2_{\mathcal{N}}(F, \lambda \times P)$ such that $\mathbf{1}_A B_{1,1} = \int_0^1 \int_0^1 f_{u,v} dB_{u,v}$. Thus $\mathbb{E}(\mathbf{1}_A B_{1,1}) = 0$ for every $A \in \mathbb{F}_{1,1}$. But taking $A_\alpha = \{B_{1,1} \geq \alpha\}$ for $\alpha > 0$ we have $\mathbb{E}(\mathbf{1}_{A_\alpha} B_{1,1}) \geq \alpha P(A_\alpha) > 0$ which leads to contradiction. In fact, in a similar way as in one-parameter case (c.f. [17]) one can show that the set $\int_0^S \int_0^T F_{u,v} dB_{u,v}$ is decomposable if and only if the set-valued mapping F is single-valued.

Theorem 2.4. *Let $F^{(n)} : I \times J \times \Omega \to \mathcal{K}^b(\mathbb{R}^d)$ be a set-valued, nonanticipating process for $n \in \mathbb{N}$ and let $F^{(1)}$ be $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded. Let us also assume $F^{(1)} \supset F^{(2)} \supset \ldots \supset F$ $\lambda \times P$-a.e., where $F := \bigcap_{n=1}^{\infty} F^{(n)}$ $\lambda \times P$-a.e. Then*

$$\int_{s'}^{s} \int_{t'}^{t} F_{u,v} \lambda(du, dv) = \bigcap_{n=1}^{\infty} \int_{s'}^{s} \int_{t'}^{t} F^{(n)}_{u,v} \lambda(du, dv)$$

and

$$\int_{s'}^{s} \int_{t'}^{t} F_{u,v} dB_{u,v} = \bigcap_{n=1}^{\infty} \int_{s'}^{s} \int_{t'}^{t} F^{(n)}_{u,v} dB_{u,v}$$

for every $(s,t), (s',t') \in I \times J$ such that $(s',t') \preceq (s,t)$.

Proof. Firstly, by virtue of Theorem 3.3 in [16] a set-valued mapping F is nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded. Since

$S^2_{\mathcal{N}}(F^{(1)}, \lambda \times P) \supset S^2_{\mathcal{N}}(F^{(2)}, \lambda \times P) \supset \ldots \supset S^2_{\mathcal{N}}(F, \lambda \times P)$, we get

$$\bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P) \supset S^2_{\mathcal{N}}(F, \lambda \times P).$$

Suppose that there exist $f \in \left(\bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P) \right) \setminus S^2_{\mathcal{N}}(F, \lambda \times P)$. Then for every $n \in \mathbb{N}$ it holds $f \in F^{(n)}$ $\lambda \times P$-a.e. and therefore $f \in F$ $\lambda \times P$-a.e. Since $f \in L^2_{\mathcal{N}}(\lambda \times P)$, we have $f \in S^2_{\mathcal{N}}(G, \lambda \times P)$. This fact leads to contradiction. Therefore $S^2_{\mathcal{N}}(F, \lambda \times P) = \bigcap_{n=1}^{\infty} S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P)$. Thus it is easy to see that for every fixed (s,t), $(s',t') \in I \times J$, $(s',t') \preceq (s,t)$ we have

$$\left\{ \int_{s'}^{s} \int_{t'}^{t} f_{u,v} dB_{u,v} : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\}$$

$$\subset \bigcap_{n=1}^{\infty} \left\{ \int_{s'}^{s} \int_{t'}^{t} f_{u,v} dB_{u,v} : f \in S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P) \right\}$$

and also

$$\left\{ \int_{s'}^{s} \int_{t'}^{t} f_{u,v} \lambda(du, dv) : f \in S^2_{\mathcal{N}}(F, \lambda \times P) \right\}$$

$$\subset \bigcap_{n=1}^{\infty} \left\{ \int_{s'}^{s} \int_{t'}^{t} f_{u,v} \lambda(du, dv) : f \in S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P) \right\}.$$

Now, let $\eta \in \bigcap_{n=1}^{\infty} \int_{s'}^{s} \int_{t'}^{t} F^{(n)}_{u,v} dB_{u,v}$. Then for every $n \geq 1$ there exists $f^{(n)} \in S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P)$ such that $\eta = \int_{s'}^{s} \int_{t'}^{t} f^{(n)}_{u,v} dB_{u,v}$. Since the family of sets $\{S^2_{\mathcal{N}}(F^{(n)}, \lambda \times P) : n \geq 1\}$ decreases it follows that $\left\{f^{(n)}\right\} \subset S^2_{\mathcal{N}}(F^{(1)}, \lambda \times P)$. By Theorem 2.2 a) there exists a subsequence $\left\{f^{(n_k)}\right\}$ of $\left\{f^{(n)}\right\}$ such that $\left\{f^{(n_k)}\right\}$ converges weakly to some $f \in L^2_{\mathcal{N}}(\lambda \times P)$. Since the mapping $L^2_{\mathcal{N}}(\lambda \times P) \ni g \mapsto \int_{s'}^{s} \int_{t'}^{t} g_{u,v} dB_{u,v} \in L^2_{s,t}$ is weakly continuous, we get $\eta = \int_{s'}^{s} \int_{t'}^{t} f_{u,v} dB_{u,v}$. Let us note that $\{f^{(n_k)} : k \geq k_0\} \subset S^2_{\mathcal{N}}(F^{(n_{k_0})}, \lambda \times P)$ for every $k_0 \geq 1$. Thus $f \in S^2_{\mathcal{N}}(F^{(n_{k_0})}, \lambda \times P)$ for every $k_0 \geq 1$ as well and finally we conclude that $f \in S^2_{\mathcal{N}}(F, \lambda \times P)$. Hence $\eta \in \int_{s'}^{s} \int_{t'}^{t} F_{u,v} dB_{u,v}$. This proves the equality

$$\int_{s'}^{s} \int_{t'}^{t} F_{u,v} dB_{u,v} = \bigcap_{n=1}^{\infty} \int_{s'}^{s} \int_{t'}^{t} F^{(n)}_{u,v} dB_{u,v}.$$

In a similar way one can show that

$$\int_{s'}^{s} \int_{t'}^{t} F_{u,v} \lambda(du, dv) = \bigcap_{n=1}^{\infty} \int_{s'}^{s} \int_{t'}^{t} F^{(n)}_{u,v} \lambda(du, dv).$$

□

Theorem 2.5. *Let $F, G : I \times J \times \Omega \to K^b(\mathbb{R}^d)$ be the set-valued, nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded stochastic processes. Then*

$$H^2_{L^2}\left(\int_{s'}^{s} \int_{t'}^{t} F_{u,v} \lambda(du, dv), \int_{s'}^{s} \int_{t'}^{t} G_{u,v} \lambda(du, dv) \right) \leq$$

$$\leq (s - s')(t - t') \int_{[s',s] \times [t',t] \times \Omega} H^2_{\mathbb{R}^d}(F, G)\, d\lambda \times dP$$

and

$$H^2_{L^2}\left(\int_{s'}^{s} \int_{t'}^{t} F_{u,v} dB_{u,v}, \int_{s'}^{s} \int_{t'}^{t} G_{u,v} dB_{u,v} \right) \leq$$

$$\leq \int_{[s',s] \times [t',t] \times \Omega} H^2_{\mathbb{R}^d}(F, G)\, d\lambda \times dP.$$

for every $(s, t), (s', t') \in I \times J$, $(s', t') \preceq (s, t)$.

Proof. Let $(s',t'),(s,t) \in I\times J$ and $(s',t') \preceq (s,t)$ be fixed. For every $f \in S^2_{\mathcal{N}}(F,\lambda\times P)$ we have:

$$\begin{aligned}
&\operatorname{dist}^2_{L^2}\left(\int_{s'}^{s}\int_{t'}^{t} f_{u,v}\lambda(du,dv), \int_{s'}^{s}\int_{t'}^{t} G_{u,v}\lambda(du,dv)\right)\\
&= \inf_{g\in S^2_{\mathcal{N}}(G,\lambda\times P)} E\left\|\int_{s'}^{s}\int_{t'}^{t}(f_{u,v}-g_{u,v})\lambda(du,dv)\right\|^2_{\mathbb{R}^d}\\
&\le (s-s')(t-t')\inf_{g\in S^2_{\mathcal{N}}(G,\lambda\times P)}\int_{[s',s]\times[t',t]\times\Omega}||f-g||^2_{\mathbb{R}^d}d\lambda\times dP.
\end{aligned}$$

Let us take a function $\varphi : I\times J\times\Omega\times\mathbb{R}^d \to \mathbb{R}^+$ defined by $\varphi(u,v,\omega,x) := ||f_{u,v}(\omega) - x||^2_{\mathbb{R}^d}$. Then the mapping $\varphi(\cdot,\cdot,\cdot,x)$ is nonanticipating for every fixed $x\in\mathbb{R}^d$ while the mapping $\varphi(u,v,\omega,\cdot)$ is continuous for every fixed $(s,t,\omega)\in I\times J\times\Omega$. Hence, by Theorem 2.2 in [14] we get

$$\begin{aligned}
&\inf_{g\in S^2_{\mathcal{N}}(G,\lambda\times P)}\int_{[s',s]\times[t',t]\times\Omega}||f-g||^2_{\mathbb{R}^d}d\lambda\times dP\\
&= \int_{[s',s]\times[t',t]\times\Omega}\inf_{x\in G_{u,v}(\omega)}\varphi(u,v,\omega,x)\lambda(du,dv)\times P(d\omega)\\
&\le \int_{[s',s]\times[t',t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)d\lambda\times dP.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sup_{f\in S^2_{\mathcal{N}}(F,\lambda\times P)} dist^2_{L^2}\left(\int_{s'}^{s}\int_{t'}^{t} f_{u,v}\lambda(du,dv), \int_{s'}^{s}\int_{t'}^{t} G_{u,v}\lambda(du,dv)\right)\\
&\le (s-s')(t-t')\int_{[s',s]\times[t',t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)d\lambda\times dP.
\end{aligned}$$

As a consequence we obtain first part of the theorem. Using similar argumentation and Itô's isometry second part of the theorem can be proved. □

By the above theorem the following result can be stated.

Corollary 2.6. *Let $F : I\times J\times\Omega \to K^b_c(\mathbb{R}^d)$ be a set valued, nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded stochastic processes. Then both*

$$I\times J \ni (s,t) \mapsto \int_0^s\int_0^t F_{u,v}\lambda(du,dv) \in K^b_c(L^2)$$

and

$$I \times J \ni (s,t) \mapsto \int_0^s \int_0^t F_{u,v} dB_{u,v} \in K_c^b(L^2)$$

are continuous set-valued mappings with respect to the metric H_{L^2}.

We end this section with the following Wendroff's type inequality needed in the sequel (c.f. [43]) .

Lemma 2.7. *Let* $u(s,t)$, $w(s,t)$ *and* $a(s,t)$ *be non-negative continuous real-valued functions defined for* $(s,t) \in I \times J$. *Let the function* $w(s,t)$ *be non-decreasing in each variable. If*

$$u(s,t) \leq w(s,t) + \int_0^s \int_0^t a(u,v)u(u,v)\lambda(du,dv) \text{ for every } (s,t) \in I \times J,$$

then

$$u(s,t) \leq w(s,t) \exp\left(\int_0^s \int_0^t a(u,v)u(u,v)\lambda(du,dv) \right) \text{ for every } (s,t) \in I \times J.$$

3. MULTIVALUED STOCHASTIC INTEGRAL EQUATION

From now on we assume that $\mathbb{F}$ is separable with respect to probability P. In this part we consider the following set-valued stochastic integral equation in the plane:

$$(3.1)\quad \begin{aligned} X(s,t) + \xi_{0,0} = \xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X(u,v)))\lambda(du,dv) \\ + \int_0^s \int_0^t G(u,v,X(u,v)))dB_{u,v} \text{ for } (s,t) \in I \times J, \end{aligned}$$

where F, $G : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$ are given set-valued mappings and the integrals above are defined as in preceding section. Here, $\xi : I \times J \to K_c^b(L^2)$ is a given set-valued mapping. Thus the equation (3.1) is thought as an abstract relation in the hyperspace of nonempty, bounded, closed and convex subsets of the space L^2. Note also that for F, G and ξ being single-valued maps, multivalued equation (3.1) reduces to single-valued one considered in [50]-[53].

Definition 3.1. By a solution to equation (3.1) we mean an H_{L^2}-continuous mapping $X : I \times J \to K_c^b(L^2)$ such that (3.1) is satisfied.

Below we formulate main assumptions imposed on set-valued mappings F, G and ξ:

(A1) for every $U \in K_c^b(L^2)$ the mappings

$$F(\cdot,\cdot,\cdot,U),\ G(\cdot,\cdot,\cdot,U) : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$$

are nonanticipating set-valued two-parameter stochastic processes.

(A2) there exists a constant $K > 0$ such that

$$H_{\mathbb{R}^d}(F(s,t,\omega,A),F(s,t,\omega,B)) + H_{\mathbb{R}^d}(G(s,t,\omega,A),G(s,t,\omega,B)) \le KH_{L^2}(A,B),$$

for every $(s,t) \in I \times J$, every $A, B \in K_c^b(L^2)$, and P-a.e.

(A3) there exists a constant $C > 0$ such that

$$H_{\mathbb{R}^d}(F(s,t,\omega,A),\{\theta\}) + H_{\mathbb{R}^d}(G(s,t,\omega,A),\{\theta\}) \le C(1 + H_{L^2}(A,\{\Theta\})),$$

for every $(s,t) \in I \times J$, every $A \in K_c^b(L^2)$, and P-a.e.

(A4) the mapping $\xi : I \times J \to K_c^b(L^2)$ is assumed to be continuous with respect to the Hausdorff metric H_{L^2} and such that the Hukuhara difference $(\xi_{s,0} + \xi_{0,t}) \ominus \xi_{0,0}$ exists for every $(s,t) \in I \times J$, and

$$\int\limits_{I\times J} H_{L^2}^2((\xi_{s,0} + \xi_{0,t}) \ominus \xi_{0,0}, \{\Theta\})\lambda(ds,dt) < \infty.$$

The symbols θ and Θ denote the zero elements in $\mathbb{R}^d$ and L^2, respectively. Consider the space $\mathcal{C} = C(I \times J, K_c^b(L^2))$ with the metric

$$\varrho(X,Y) = \sup_{(s,t)\in I\times J} H_{L^2}(X(s,t),Y(s,t)).$$

Then $(\mathcal{C},\varrho)$ is a complete metric space (cf. Proposition 1.6.19 in [9]).

Theorem 3.2. *Let F, $G : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$ and $\xi : I \times J \to K_c^b(L^2)$ satisfy conditions* (A1)–(A4). *Then equation* (3.1) *has a unique solution.*

Proof. By assumptions imposed on the initial mapping ξ we can define a sequence $\{X_n\}, X_n : I \times J \to K_c^b(L^2), n = 0,1,2,...$ of successive approximations as follows:

$$X_0(s,t) + \xi_{0,0} = \xi_{s,0} + \xi_{0,t}$$

and for $n = 1,2,...$

$$X_n(s,t) + \xi_{0,0} = \xi_{s,0} + \xi_{0,t} + \int_0^s\int_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv) + $$

$$+ \int_0^s\int_0^t G(u,v,X_{n-1}(u,v))dB_{u,v}$$

for every $(s,t) \in I \times J$. By Theorem 2.5 and the assumption (A3) one has:

$$H_{L^2}^2\left(\int_0^s\int_0^t F(u,v,X_0(u,v))\lambda(du,dv),\{\Theta\}\right)$$

$$\le st \int\limits_{[0,s]\times[0,t]\times\Omega} H_{\mathbb{R}^d}^2(F(u,v,X_0(u,v)),\{\theta\})d\lambda \times dP$$

$$\leq st \int\limits_{[0,s]\times[0,t]} 2C^2(1+H^2_{L^2}(X_0(u,v),\{\Theta\}))\lambda(du,dv).$$

Let $\eta = 2C^2 \int\limits_{I\times J} (1+H^2_{L^2}(X_0(u,v),\{\Theta\}))\lambda(du,dv)$. Then we have

$$H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t F(u,v,X_0(u,v))\lambda(du,dv),\{\Theta\}\right) \leq st\eta.$$

In a similar way we obtain the following estimation

$$\begin{aligned} &H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t G(u,v,X_0(u,v))dB_{u,v},\{\Theta\}\right) \\ &\leq \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(u,v,X_0(u,v)),\{\theta\})d\lambda\times dP \\ &\leq \int\limits_{[0,s]\times[0,t]} 2C^2(1+H^2_{L^2}(X_0(u,v),\{\Theta\})\lambda(du,dv) \leq \eta. \end{aligned}$$

Hence

$$\begin{aligned} &H^2_{L^2}(X_1(t,s),X_0(t,s)) \\ &= H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t F(u,v,X_0(u,v))\lambda(du,dv) + \int\limits_0^s\int\limits_0^t G(u,v,X_0(u,v))dB_{u,v},\{\Theta\}\right) \\ &\leq 2H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t F(u,v,X_0(u,v))\lambda(du,dv),\{\Theta\}\right) \\ &+2H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t G(u,v,X_0(u,v))dB_{u,v},\{\Theta\}\right) \\ &\leq 2\eta(1+st) \leq 2\eta(1+ST). \end{aligned}$$

Similarly, by the assumption (A2) and Theorem 2.5, we obtain the following chain of estimates

$$\begin{aligned} &H^2_{L^2}\left(X_n(t,s),X_{n-1}(t,s)\right) \\ &\leq 2H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv),\right. \\ &\left.\int\limits_0^s\int\limits_0^t F(u,v,X_{n-2}(u,v))\lambda(du,dv)\right) \\ &+2H^2_{L^2}\left(\int\limits_0^s\int\limits_0^t G(u,v,X_{n-1}(u,v))dB_{u,v},\int\limits_0^s\int\limits_0^t G(u,v,X_{n-2}(u,v))dB_{u,v}\right) \end{aligned}$$

$$\leq 2ST \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}\left(F(u,v,X_{n-1}(u,v)),F(u,v,X_{n-2}(u,v))\right)d\lambda\times dP$$

$$+2 \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}\left(G(u,v,X_{n-1}(u,v)),G(u,v,X_{n-2}(u,v))\right)d\lambda\times dP$$

$$\leq 2K^2(ST+1) \int\limits_{[0,s]\times[0,t]} H^2_{L^2}\left(X_{n-1}(u,v),X_{n-2}(u,v)\right)\lambda(du,dv).$$

In particular, we infer that

$$\begin{aligned} & H^2_{L^2}\left(X_2(s,t),X_1(s,t)\right) \\ \leq\ & 2K^2(ST+1) \int\limits_{[0,s]\times[0,t]} H^2_{L^2}\left(X_1(u,v),X_0(u,v)\right)\lambda(du,dv) \\ \leq\ & 2K^2(ST+1) \int\limits_{[0,s]\times[0,t]} 2\eta(ST+1)\lambda(du,dv) \\ =\ & 2K^2(ST+1)^2 2\eta st. \end{aligned}$$

Therefore, by mathematical induction

$$H^2_{L^2}\left(X_n(s,t),X_{n-1}(s,t)\right) \leq (2K^2)^{n-1}(ST+1)^n 2\eta \frac{s^{n-1}}{(n-1)!}\frac{t^{n-1}}{(n-1)!}$$

$$\leq (2K^2)^{n-1}(ST+1)^n 2\eta \frac{S^{n-1}}{(n-1)!}\frac{T^{n-1}}{(n-1)!}.$$

Hence

$$\varrho^2\left(X_n,X_{n-1}\right) \leq (2K^2)^{n-1}(ST+1)^n 2\eta \frac{S^{n-1}}{(n-1)!}\frac{T^{n-1}}{(n-1)!}.$$

Consequently, for $m<n$ we have:

$$\begin{aligned} \varrho(X_n,X_m) &\leq \sum_{k=m+1}^{n} \sqrt{(2K^2)^{k-1}(ST+1)^k 2\eta \frac{S^{k-1}}{(k-1)!}\frac{T^{k-1}}{(k-1)!}} \\ &\leq \alpha \sum_{k=m+1}^{n} \frac{\beta^{k-1}}{(k-1)!}, \end{aligned}$$

where

$$\alpha = \sqrt{2\eta\big((\max\{S,T\})^2+1\big)}$$

and

$$\beta = \sqrt{2K^2\big((\max\{S,T\})^2+1\big)}\max\{S,T\}.$$

Since the last sum converges to zero as $m,n\to\infty$, it follows that $\{X_n\}$ is a Cauchy sequence in $\mathcal{C}$. Hence, there exists $X\in\mathcal{C}$ such that $\varrho(X_n,X)\to 0$ as

$n \to \infty$. Now we shall show that X is a solution to equation (3.1). Indeed, let us fix $(s,t) \in I \times J$. Then we have

$$H_{L^2}^2 \Bigg(X(s,t) + \xi_{0,0}, \xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X(u,v))\,\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X(u,v))\,dB_{u,v} \Bigg)$$

$$\leq 3H_{L^2}^2 \left(X_n(s,t) + \xi_{0,0}, X(s,t) + \xi_{0,0} \right)$$

$$+3H_{L^2}^2 \Bigg(X_n(s,t) + \xi_{0,0}\ ,$$

$$\xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X_{n-1}(u,v))dB_{u,v} \Bigg)$$

$$+3H_{L^2}^2 \Bigg(\xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X_{n-1}(u,v))dB_{u,v}\ ,$$

$$\xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v} \Bigg).$$

By the definition of X_n the second term above equals zero. On the other hand, again by Theorem 2.5 and (A2), the last term can be estimated as follows:

$$3H_{L^2}^2 \Bigg(\xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X_{n-1}(u,v))dB_{u,v}\ ,$$

$$\xi_{s,0} + \xi_{0,t} + \int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v} \Bigg)$$

$$\leq 6H_{L^2}^2 \Bigg(\int_0^s \int_0^t F(u,v,X_{n-1}(u,v))\lambda(du,dv),$$

$$\int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv)\Bigg)$$

$$+6H_{L^2}^2\left(\int_0^s \int_0^t G(u,v,X_{n-1}(u,v))dB_{u,v}, \int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v}\right)$$

$$\leq 6st \int_{[0,s]\times[0,t]\times\Omega} H_{\mathbb{R}^d}^2\left(F(u,v,X_{n-1}(u,v)), F(u,v,X(u,v))\right) d\lambda \times dP$$

$$+6 \int_{[0,s]\times[0,t]\times\Omega} H_{\mathbb{R}^d}^2\left(G(u,v,X_{n-1}(u,v)), G(u,v,X(u,v))\right) d\lambda \times dP$$

$$\leq 6K^2ST \int_{I\times J} \sup_{(u,v)\in I\times J} H_{L^2}^2(X_{n-1}(u,v), X(u,v))\lambda(du,dv)$$

$$+6K^2 \int_{I\times J} \sup_{(u,v)\in I\times J} H_{L^2}^2(X_{n-1}(u,v), X(u,v))\lambda(du,dv)$$

$$\leq (6K^2S^2T^2 + 6K^2ST)\varrho(X_n, X).$$

Consequently, letting $n \to \infty$ we get

$$H_{L^2}\Bigg(X(t,s) + \xi_{0,0}, \xi_{s,0} + \xi_{0,t}$$

$$+\int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v}\Bigg) = 0.$$

Finally, we show the uniqueness. Let us assume X and Y are solutions to equation (3.1). Then in view of Theorem 2.5 and condition (A2):

$$H_{L^2}^2(X(t,s), Y(t,s))$$

$$= H_{L^2}^2\Bigg(\int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v}\ ,$$

$$\int_0^s \int_0^t F(u,v,Y(u,v))\lambda(du,dv) + \int_0^s \int_0^t G(u,v,Y(u,v))dB_{u,v}\Bigg)$$

$$\leq 2H_{L^2}^2\left(\int_0^s \int_0^t F(u,v,X(u,v))\lambda(du,dv), \int_0^s \int_0^t F(u,v,Y(u,v))\lambda(du,dv)\right)$$

$$+2H_{L^2}^2\left(\int_0^s \int_0^t G(u,v,X(u,v))dB_{u,v}, \int_0^s \int_0^t G(u,v,Y(u,v))dB_{u,v}\right)$$

$$\leq 2st \int_{[0,s]\times[0,t]\times\Omega} H_{\mathbb{R}^d}^2(F(u,v,X(u,v)), F(u,v,Y(u,v)))d\lambda \times dP$$

$$+2 \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(u,v,X(u,v)),G(u,v,Y(u,v)))d\lambda\times dP$$

$$\leq 2K^2(ST+1) \int\limits_{[0,s]\times[0,t]} H^2_{L^2}(X(u,v),Y(u,v))\lambda(du,dv).$$

By Lemma 2.7 we conclude

$$H^2_{L^2}(X(s,t),Y(s,t))=0$$

for $(s,t)\in I\times J$ which completes the proof.

□

We proceed in a similar way in order to obtain the following estimate.

Theorem 3.3. *Under assumptions of Theorem* 3.2 *the solution* X *to equation* (3.1) *satisfies:*

$$H^2_{L^2}(X(s,t),\{\Theta\})$$

$$\leq [3 \sup_{(s,t)\in I\times J} H^2_{L^2}(\xi_{s,0}+\xi_{0,t},\xi_{0,0})+6C^2st(st+1)]\exp\{6C^2st(st+1)\},$$

for every $(s,t)\in I\times J$.

As a consequence of the next result we have a continuous dependence of the solution to (3.1) on initial mapping. To this end, let us consider the following equations

$$\begin{aligned} X(s,t)+\xi_{0,0} &= \xi_{s,0}+\xi_{0,t} \\ &+\int_0^s\int_0^t F(u,v,X(u,v))\lambda(du,dv)+\int_0^s\int_0^t G(u,v,X(u,v))dB_{u,v}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} Y(s,t)+\widetilde{\xi}_{0,0} &= \widetilde{\xi}_{s,0}+\widetilde{\xi}_{0,t} \\ &+\int_0^s\int_0^t F(u,v,Y(u,v))\lambda(du,dv)+\int_0^s\int_0^t G(u,v,Y(u,v))dB_{u,v.} \end{aligned} \tag{3.3}$$

Then the following result holds true.

Theorem 3.4. *Let* ξ, $\widetilde{\xi}$, F, G *satisfy conditions* (A1)–(A4). *Let* $\widetilde{\xi}_{0,0}=\xi_{0,0}$. *Let* X *and* Y *denote unique solutions of equation* (3.2) *and* (3.3), *respectively. Then*

$$H^2_{L^2}(X(s,t),Y(s,t))$$

$$\leq 4[\sup_{s\in I} H^2_{L^2}(\xi_{s,0},\widetilde{\xi}_{s,0})+\sup_{t\in J} H^2_{L^2}(\xi_{0,t},\widetilde{\xi}_{0,t})]\exp\{4K^2st(st+1)\}$$

for every $(s,t)\in I\times J$.

Proof. Indeed, by assumptions (A1)–(A4) and Theorem 2.5 we have the following estimations:

$$H^2_{L^2}(X(s,t),Y(s,t)) \le 4H^2_{L^2}(\xi_{s,0}\ ,\ \widetilde{\xi}_{s,0}) + 4H^2_{L^2}(\xi_{0,t}\ ,\ \widetilde{\xi}_{0,t})$$

$$+4H^2_{L^2}\left(\int_0^s\int_0^t F(u,v,X(u,v))\lambda(du,dv)\ ,\ \int_0^s\int_0^t F(u,v,Y(u,v))\lambda(du,dv)\right)$$

$$+4H^2_{L^2}\left(\int_0^s\int_0^t G(u,v,X(u,v))dB_{u,v}\ ,\ \int_0^s\int_0^t G(u,v,Y(u,v))dB_{u,v}\right)$$

$$\le 4H^2_{L^2}(\xi_{s,0}\ ,\ \widetilde{\xi}_{s,0}) + 4H^2_{L^2}(\xi_{0,t}\ ,\ \widetilde{\xi}_{0,t})$$

$$+4st\int_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}(F(u,v,X(u,v)\ ,\ F(u,v,Y(u,v)))d\lambda\times dP$$

$$+4\int_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}(G(u,v,X(u,v)\ ,\ G(u,v,Y(u,v)))d\lambda\times dP$$

$$\le 4\sup_{s\in I} H^2_{L^2}(\xi_{s,0}\ ,\ \widetilde{\xi}_{s,0}) + 4\sup_{t\in J} H^2_{L^2}(\xi_{0,t}\ ,\ \widetilde{\xi}_{0,t})$$

$$+4(st+1)K^2\int_{[0,s]\times[0,t]} H^2_{L^2}(X(u,v)\ ,\ Y(u,v))\lambda(du,dv).$$

Hence, by Lemma 2.7, the assertion follows. □

Finally, we consider problem of stability of solutions to the system of set-valued stochastic integral equations. Let us consider the equations

$$X(s,t) + \xi_{0,0} = \xi_{s,0} + \xi_{0,t} + \int_0^s\int_0^t F(u,v,X(u,v))\lambda(du,dv)$$

$$+\int_0^s\int_0^t G(u,v,X(u,v))dB_{u,v}$$

and

$$X_n(s,t) + \xi^n_{0,0} = \xi^n_{s,0} + \xi^n_{0,t} + \int_0^s\int_0^t F_n(u,v,X_n(u,v))\lambda(du,dv)$$

$$+\int_0^s\int_0^t G_n(u,v,X_n(u,v))dB_{u,v}$$

and their solutions X and X_n, respectively for $n = 1, 2, ...$

Theorem 3.5. *Let ξ, ξ^n, F, G, F_n and G_n satisfy assumptions* (A1)–(A4) *(with the same constants K and C). Suppose that $\xi^n_{0,0} = \xi_{0,0}$ for $n \geq 1$. Moreover, let us assume that*

$$\max\left\{\sup_{s\in I} H_{L^2}\left(\xi^n_{s,0}, \xi_{s,0}\right), \sup_{t\in J} H_{L^2}\left(\xi^n_{0,t}, \xi_{0,t}\right)\right\} \to 0 \text{ as } n \to \infty$$

and

$$\max\left\{H_{\mathbb{R}^d}\left(F_n\left(s,t,\omega,A\right), F\left(s,t,\omega,A\right)\right),\right.$$
$$\left.H_{\mathbb{R}^d}\left(G_n\left(s,t,\omega,A\right), G\left(s,t,\omega,A\right)\right)\right\} \to 0 \text{ as } n \to \infty$$

for every $(s,t,A) \in I \times J \times \mathcal{K}^b_c(L^2)$ and P-a.e. Then

$$\varrho\left(X_n, X\right) \to 0 \text{ as } n \to \infty.$$

Proof. Let $(s,t) \in I \times J$ be fixed. Using Theorem 2.5 one can show the inequalities

$$H^2_{L^2}\left(X_n(s,t), X(s,t)\right) = H^2_{L^2}\left(X_n(s,t) + \xi^n_{0,0}, X(s,t) + \xi_{0,0}\right)$$
$$\leq 4\left(H^2_{L^2}(\xi^n_{s,0}, \xi_{s,0}) + H^2_{L^2}(\xi^n_{0,t}, \xi_{0,t})\right)$$
$$+4st \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}\left(F(u,v,X(u,v)), F_n(u,v,X_n(u,v))\right) d\lambda \times dP$$
$$+4 \int\limits_{[0,s]\times[0,t]\times\Omega} H^2_{\mathbb{R}^d}\left(G(u,v,X(u,v)), G_n(u,v,X_n(u,v))\right) d\lambda \times dP$$
$$\leq 4\left(\sup_{s\in I} H^2_{L^2}(\xi^n_{s,0}, \xi_{s,0}) + \sup_{t\in J} H^2_{L^2}(\xi^n_{0,t}, \xi_{0,t})\right) + 8stA_n$$
$$+8B_n + 8K^2(ST+1) \int\limits_{[0,s]\times[0,t]} H^2_{L^2}\left(X_n(u,v), X(u,v)\right) \lambda(du,dv),$$

where

$$A_n := \int\limits_{[0,S]\times[0,T]\times\Omega} H^2_{\mathbb{R}^d}(F(u,v,X(u,v)), F_n(u,v,X(u,v))) d\lambda \times dP,$$

$$B_n := \int\limits_{[0,S]\times[0,T]\times\Omega} H^2_{\mathbb{R}^d}(G(u,v,X(u,v)), G_n(u,v,X(u,v))) d\lambda \times dP.$$

Thus, again by Lemma 2.7 we obtain

$$H^2_{L^2}\left(X_n(s,t), X(s,t)\right)$$
$$\leq \left(4\sup_{s\in I} H^2_{L^2}(\xi^n_{s,0}, \xi_{s,0}) + 4\sup_{t\in J} H^2_{L^2}(\xi^n_{0,t}, \xi_{0,t}) + 8(stA_n + B_n)\right) e^{8K^2(ST+1)st}.$$

Using Lebesgue's dominated convergence theorem we have that $A_n \to 0$ and $B_n \to 0$ as $n \to \infty$. This together with assumptions imposed on the sequence $\{\xi^n\}$ ends the proof. ∎

Remark 3.6. Note that under assumptions (A1)–(A4) equation (3.1) can be rewritten as

$$X(s,t) = (\xi_{s,0} + \xi_{0,t}) \ominus \xi_{0,0} + \int_0^s \int_0^t F(u,v,X(u,v)))\lambda(du,dv)$$

$$+ \int_0^s \int_0^t G(u,v,X(u,v)))dB_{u,v} \text{ for } (s,t) \in I \times J.$$

Then one can replace the equation above by a more general relation

$$X(s,t) = C(s,t) + \int_0^s \int_0^t F(u,v,X(u,v)))\lambda(du,dv)$$

$$+ \int_0^s \int_0^t G(u,v,X(u,v)))dB_{u,v} \text{ for } (s,t) \in I \times J,$$

where initial value mapping $C : I \times J \to K_c^b(L^2)$ can be chosen as a continuous set-valued function such that

$$\int_{I \times J} H_{L^2}^2(C(s,t), \{\Theta\})\lambda(ds,dt) < \infty.$$

In this case the same conclusion as in Theorem 3.2 still holds, and consequently analogs of the rest of results can also be proved.

Remark 3.7. Similar results as in Sections 2 and 3 can be obtained in the case of n-parameter random fields for $n > 2$.

4. APPLICATIONS TO FUZZY STOCHASTIC INTEGRAL EQUATIONS IN THE PLANE

In this section we show an applicability of the results of previous sections to fuzzy stochastic integral equations driven by Wiener process in the plane. We shall begin with some basic notions concerning fuzzy sets and their main properties (see [10] for details). Let $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$ be a given separable Banach space. By a fuzzy set u of the space $\mathfrak{X}$ we mean a mapping $u : \mathfrak{X} \to [0,1]$. The space of all fuzzy sets of $\mathfrak{X}$ will be denoted by the symbol $\mathcal{F}(\mathfrak{X})$. For $\alpha \in (0,1]$ let $[u]^\alpha := \{x \in \mathfrak{X} : u(x) \geq \alpha\}$ and $[u]^0 := \mathrm{cl}_{\mathfrak{X}}\{x \in \mathfrak{X} : u(x) > 0\}$ where $\mathrm{cl}_{\mathfrak{X}}$ denotes the closure in $(\mathfrak{X}, ||\cdot||_{\mathfrak{X}})$. In the sequel we deal with fuzzy sets belonging to the following family

$$\mathcal{F}_c^b(\mathfrak{X}) = \{u \in \mathcal{F}(\mathfrak{X}) : [u]^\alpha \in \mathcal{K}_c^b(\mathfrak{X}) \text{ for every } \alpha \in [0,1]\}.$$

Let $u, v \in \mathcal{F}_c^b(\mathfrak{X})$. Then the addition $u + v$ is defined level-set-wise, i.e.,

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha \text{ for } \alpha \in [0,1].$$

Moreover, if there exists $w \in \mathcal{F}_c^b(\mathfrak{X})$ such that $u = v + w$ then w is called a difference of u and v and we denote it by $u \ominus v$. We shall use a metric $D_{\mathfrak{X}}$ in $\mathcal{F}_c^b(\mathfrak{X})$ described as follows

$$D_{\mathfrak{X}}(u,v) := \sup_{\alpha\in[0,1]} H_{\mathfrak{X}}([u]^\alpha, [v]^\alpha) \text{ for } u, v \in \mathcal{F}_c^b(\mathfrak{X}).$$

By properties of Hausdorff metric $H_{\mathfrak{X}}$ one has

$$D_{\mathfrak{X}}(u_1 + u_2, v_1 + v_2) \le D_{\mathfrak{X}}(u_1, v_1) + D_{\mathfrak{X}}(u_2, v_2)$$

and

$$D_{\mathfrak{X}}(u_1 + v, u_2 + v) = D_{\mathfrak{X}}(u_1, u_2)$$

for $u_1, u_2, v_1, v_2, v \in \mathcal{F}_c^b(\mathfrak{X})$. It is known (c.f. [44]) that $\left(\mathcal{F}_c^b(\mathfrak{X}), D_{\mathfrak{X}}\right)$ is a complete metric space. For the aims of this section we consider two cases: $\mathfrak{X} = \mathbb{R}^d$ or $\mathfrak{X} = L^2$. By a fuzzy random variable we mean a function $u\colon \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$ such that $[u(\cdot)]^\alpha\colon \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ is an $\mathbb{F}$-measurable set-valued mapping for every $\alpha \in (0,1]$.

A fuzzy set-valued mapping $f\colon I \times J \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$ is called a two-parameter fuzzy-valued stochastic process (or fuzzy-valued random field) if $f(s,t,\cdot)\colon \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$ is a fuzzy random variable for every $(s,t) \in I \times J$.

A fuzzy-valued random field $f\colon I \times J \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$ is said to be nonanticipating if set-valued mapping $[f]^\alpha\colon I \times J \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ is $\mathcal{N}$-measurable for every $\alpha \in (0,1]$. A fuzzy-valued random field f is called $\{\mathbb{F}_{s,t}\}$-adapted, if $f(s,t,\cdot)$ is an $\mathbb{F}_{s,t}$-measurable fuzzy random variable for every $(s,t) \in I \times J$. It is called $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded, if $\||[f]^0|\| \in L^2_{\mathcal{N}}(\lambda \times P)$. For α-levels ($\alpha \in [0,1]$) of such a fuzzy-valued random field f one can consider set-valued trajectory stochastic integrals: $\int_0^S\int_0^T [f_{u,v}]^\alpha \lambda(du,dv)$ and $\int_0^S\int_0^T [f_{u,v}]^\alpha dB_{u,v}$ defined in Section 2. Then by Theorem 2.4 and Theorem of Negoita and Ralescu (cf. [41]) for every $(s,t) \in I \times J$ there exist fuzzy sets in $\mathcal{F}_c^b(L^2)$ denoted by

$$(\mathcal{F})\int_0^s\int_0^t f_{u,v}\lambda(du,dv)) \quad\text{and}\quad (\mathcal{F})\int_0^s\int_0^t f_{u,v} dB_{u,v}$$

such that for every $\alpha \in [0,1]$

$$\left[(\mathcal{F})\int_0^s\int_0^t f_{u,v}\lambda(du,dv))\right]^\alpha = \int_0^s\int_0^t [f_{u,v}]^\alpha \lambda(du,dv),$$

$$\left[(\mathcal{F})\int_0^s\int_0^t f_{u,v} dB_{u,v}\right]^\alpha = \int_0^s\int_0^t [f_{u,v}]^\alpha dB_{u,v}.$$

Definition 4.1. For every $(s,t) \in I \times J$, the fuzzy sets

$$(\mathcal{F})\int_0^s\int_0^t f_{u,v}\lambda(du,dv) \text{ and } (\mathcal{F})\int_0^s\int_0^t f_{u,v}dB_{u,v}$$

are called the fuzzy stochastic Lebesgue trajectory integral and fuzzy Itô trajectory integral (respectively) of nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded fuzzy-valued random field $f\colon I \times J \times \Omega \to \mathcal{F}^b_c(\mathbb{R}^d)$.

Remark 4.2. Note that the sum of these integrals exists in the sense of addition of fuzzy sets and

$$(\mathcal{F})\int_0^s\int_0^t f_{u,v}\lambda(du,dv) + (\mathcal{F})\int_0^s\int_0^t g_{u,v}dB_{u,v} \in \mathcal{F}^b_c(L^2)$$

for $f,g\colon I\times J\times\Omega \to \mathcal{F}^b_c(\mathbb{R}^d)$ being nonanticipating and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded fuzzy-valued random fields.

As a consequence of Theorem 2.5 and Corollary 2.6 we have the following results.

Corollary 4.3. *Let $f,g : I \times J \times \Omega \to \mathcal{F}^b_c(\mathbb{R}^d)$ be nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded fuzzy-valued random fields. Then*

$$D^2_{L^2}\left((\mathcal{F})\int_{s'}^s\int_{t'}^t f_{u,v}\lambda(du,dv), (\mathcal{F})\int_{s'}^s\int_{t'}^t g_{u,v}\lambda(du,dv)\right)$$
$$\le (s-s')(t-t')\int_{[s',s]\times[t',t]\times\Omega} D^2_{\mathbb{R}^d}(f,g)d\lambda\times dP$$

and

$$D^2_{L^2}\left((\mathcal{F})\int_{s'}^s\int_{t'}^t f_{u,v}dB_{u,v}, (\mathcal{F})\int_{s'}^s\int_{t'}^t g_{u,v}dB_{u,v}\right)$$
$$\le \int_{[s',s]\times[t',t]\times\Omega} D^2_{\mathbb{R}^d}(f,g)d\lambda\times dP$$

for every $(s,t),(s',t') \in I \times J$ such that $(s',t') \preceq (s,t)$.

Corollary 4.4. *Let $f\colon I \times J \times \Omega \to \mathcal{F}^b_c(\mathbb{R}^d)$ be a nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded fuzzy-valued random field. Then the mappings*

$$I \times J \ni (s,t) \mapsto (\mathcal{F})\int_0^s\int_0^t f_{u,v}dB_{u,v} \in \mathcal{F}^b_c(L^2)$$

and

$$I \times J \ni (s,t) \mapsto (\mathcal{F})\int_0^s\int_0^t f_{u,v}\lambda(du,dv) \in \mathcal{F}^b_c(L^2)$$

are continuous with respect to the metric D_{L^2}.

Let $f, g\colon I \times J \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ and let $\gamma : I \times J \to \mathcal{F}_c^b(L^2)$ be given. By a fuzzy stochastic integral equation in the plane we mean the following relation in the metric space $\big(\mathcal{F}_c^b(L^2), D_{L^2}\big)$:

$$\begin{aligned} x_{s,t} + \gamma_{0,0} = \gamma_{s,0} + \gamma_{0,t} + (\mathcal{F}) \int_0^s \int_0^t f(u, v, x_{u,v}) \lambda(du, dv) \\ +(\mathcal{F}) \int_0^s \int_0^t g(u, v, x_{u,v}) dB_{u,v} \end{aligned} \tag{4.1}$$

for $(s, t) \in I \times J$.

Definition 4.5. By a solution to (4.1) we mean a D_{L^2}-continuous mapping $x\colon I \times J \to \mathcal{F}_c^b(L^2)$ that satisfies (4.1). A solution $x\colon I \times J \to \mathcal{F}_c^b(L^2)$ to (4.1) is unique if

$$x_{s,t} = y_{s,t} \text{ for every } (s, t) \in I \times J$$

where $y\colon I \times J \to \mathcal{F}_c^b(L^2)$ is any solution of (4.1).

Assume that $f, g\colon I \times J \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ and $\gamma : I \times J \to \mathcal{F}_c^b(L^2)$ satisfy:

(a1) for every $u \in \mathcal{F}_c^b(L^2)$ the mappings

$$f(\cdot, \cdot, \cdot, u), g(\cdot, \cdot, \cdot, u)\colon I \times J \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d)$$

are the nonanticipating fuzzy-valued random fields,

(a2) there exists a constant $K > 0$ such that

$$D_{\mathbb{R}^d}(f(s, t, \omega, u), f(s, t, \omega, v)) + D_{\mathbb{R}^d}(g(s, t, \omega, u), g(s, t, \omega, v))$$

$$\leq K D_{L^2}(u, v),$$

for every $(s, t) \in I \times J$, every $u, v \in \mathcal{F}_c^b(L^2)$, and P-a.e.,

(a3) there exists a constant $C > 0$ such that

$$D_{\mathbb{R}^d}(f(s, t, \omega, u), \hat{\theta}) + D_{\mathbb{R}^d}(g(s, t, \omega, u), \hat{\theta})$$

$$\leq C(1 + D_{L^2}(u, \hat{\Theta})),$$

for every $(s, t) \in I \times J$, every $u \in \mathcal{F}_c^b(L^2)$, and P-a.e.,

(a4) the mapping $\gamma : I \times J \to \mathcal{F}_c^b(L^2)$ is continuous with respect to the metric D_{L^2} and such that the difference $(\gamma_{s,0} + \gamma_{0,t}) \ominus \gamma_{0,0}$ exists for every $(s, t) \in I \times J$, and

$$\int_{I \times J} D_{L^2}^2((\gamma_{s,0} + \gamma_{0,t}) \ominus \gamma_{0,0}, \hat{\Theta}) \lambda(ds, dt) < \infty.$$

Here the symbols $\hat{\theta}, \hat{\Theta}$ denote the fuzzy counterparts of zero elements θ and Θ in $\mathbb{R}^d$ and L^2, respectively, i.e., $[\hat{\theta}]^\alpha = \{\theta\}$ and $[\hat{\Theta}]^\alpha = \{\Theta\}$ for every $\alpha \in [0, 1]$. Using the properties of fuzzy trajectory stochastic integrals in the plane and proceeding similarly as in Section 3 the following result can be proved.

Theorem 4.6. *Let* $f, g\colon I \times J \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ *and* $\gamma : I \times J \to \mathcal{F}_c^b(L^2)$ *satisfy conditions* (a1)–(a4). *Then equation* (5) *has a unique solution* x *such that*

$$D^2_{L^2}(x_{s,t}, \hat{\Theta})$$
$$\leq [3 \sup_{(s,t)\in I\times J} D^2_{L^2}(\gamma_{s,0} + \gamma_{0,t}, \gamma_{0,0}) + 6C^2 st(st+1)] \exp\{6C^2 st(st+1)\}$$

for every $(s,t) \in I \times J$.

Similarly as in the case of set-valued stochastic equations in the plane let us consider two fuzzy stochastic integral equations:

$$\begin{aligned} x_{s,t} + \gamma_{0,0} = \gamma_{s,0} + \gamma_{0,t} + (\mathcal{F}) \int_0^s \int_0^t f(u,v,x_{u,v})\lambda(du,dv) \\ +(\mathcal{F}) \int_0^s \int_0^t g(u,v,x_{u,v}) dB_{u,v} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} y_{s,t} + \sigma_{0,0} = \sigma_{s,0} + \sigma_{0,t} + (\mathcal{F}) \int_0^s \int_0^t f(u,v,y_{u,v})\lambda(du,dv) \\ +(\mathcal{F}) \int_0^s \int_0^t g(u,v,y_{u,v}) dB_{u,v} \end{aligned} \tag{4.3}$$

for $(s,t) \in I \times J$. Then the following fuzzy counterpart of Theorem 3.4 holds.

Theorem 4.7. *Assume that the mappings* $\gamma, \sigma : I \times J \to \mathcal{F}_c^b(L^2)$ *and fuzzy-valued random fields* $f, g\colon I \times J \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ *satisfy the conditions* (a1)–(a4). *Let us also assume that* $\gamma_{0,0} = \sigma_{0,0}$. *Let* x *and* y *be solutions to equations* (4.2) *and* (4.3) *respectively. Then*

$$D^2_{L^2}(x_{s,t}, y_{s,t}) \leq 4[\sup_{s\in I} D^2_{L^2}(\gamma_{s,0}, \sigma_{s,0}) + \sup_{t\in J} D^2_{L^2}(\gamma_{0,t}, \sigma_{0,t})] \exp\{4K^2 st(st+1)\}$$

for every $(s,t) \in I \times J$.

Finally, let us consider the system of fuzzy stochastic integral equations:

$$x_{s,t} + \gamma_{0,0} = \gamma_{s,0} + \gamma_{0,t} + (\mathcal{F}) \int_0^s \int_0^t f(u,v,x_{u,v})\lambda(du,dv)$$

$$+(\mathcal{F}) \int_0^s \int_0^t g(u,v,x_{u,v}) dB_{u,v}$$

and for $n \geq 1$

$$x^n_{s,t} + \gamma^n_{0,0} = \gamma^n_{s,0} + \gamma^n_{0,t} + (\mathcal{F}) \int_0^s \int_0^t f_n(u,v,x^n_{u,v})\lambda(du,dv)$$

$$+(\mathcal{F}) \int_0^s \int_0^t g_n(u,v,x^n_{u,v}) dB_{u,v}$$

for $(s,t) \in I \times J$, with their solutions x, x^n for $n \geq 1$.

Applying similar methods as those used in the proof of Theorem 3.5 the following assertion holds.

Theorem 4.8. *Let* $f, g, f_n, g_n \colon I \times J \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d)$ *satisfy conditions* (a1)–(a3) *with the same constants* K, C. *Let also the mappings* $\gamma, \gamma^n : I \times J \to \mathcal{F}_c^b(L^2)$ *satisfy condition* (a4) *and* $\gamma_{0,0}^n = \gamma_{0,0}$ *for every* $n \in \mathbb{N}$. *We assume additionally that*

$$\max\left\{\sup_{s\in I} D_{L^2}\left(\gamma_{s,0}^n, \gamma_{s,0}\right), \sup_{t\in J} D_{L^2}\left(\gamma_{0,t}^n, \gamma_{0,t}\right)\right\} \to 0 \text{ as } n \to \infty$$

and

$$\max\left\{D_{\mathbb{R}^d}\left(f_n\left(s,t,\omega,u\right), f\left(s,t,\omega,u\right)\right),\right.$$
$$\left. D_{\mathbb{R}^d}\left(g_n\left(s,t,\omega,u\right), g\left(s,t,\omega,u\right)\right)\right\} \to 0 \text{ as } n \to \infty,$$

for every $(s,t,u) \in I \times J \times \mathcal{F}_c^b(L^2)$ *and* P-*a.e. Then*

$$\sup_{(s,t)\in I\times J} D_{L^2}\left(x_{s,t}^n, x_{s,t}\right) \to 0, \quad \text{as } n \to \infty.$$

REFERENCES

[1] N. U. Ahmed, Nonlinear stochastic differential inclusions on Banach spaces, *Stoch. Anal. Appl.*, 12:1–10, 1994.

[2] N. U. Ahmed, Impulsive perturbation of C_0 semigroups and stochastic evolution inclusions, *Discuss. Math. Differ. Incl. Control Optim.*, 22:125–149, 2002.

[3] N. U. Ahmed, Optimal relaxed controls for nonlinear infinite dimensional stochastic differential inclusions, *Optimal Control of Differential Equations*, M. Dekker Lect. Notes, 160:1–19, 1994.

[4] N. U. Ahmed, Differential inclusions operator valued measures and optimal control, *Dynam. Systems Appl.*, 16:13–36, 2007.

[5] Z. Artstein, A calculus for set-valued maps and set-valued evolusion equations, *Set-Valued Anal.*, 3:216–261, 1995.

[6] R. Cairoli and J. B. Walsh, Stochastic integrals in the plane. *Acta Math.*, 134:111–183, 1975.

[7] F. S. De Blasi and F. Iervolino, Equazioni differenziali con soluzioni a valore compatto convesso, *Boll. Unione Math Ital.*, 4:194–501, 1969.

[8] F. S. De Blasi, Banach-Saks-Mazur nad Kakutani-Ky-Fan theorems in spaces of multifunctions and applications to set differential inclusions, *Dynam. Systems Appl.*, 16:73–89, 2007.

[9] Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis and Its Applications, Part I*, Kluwer Acad. Publ., Boston, 2003.

[10] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994.

[11] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Int. Publ., INC, New York, 1958.

[12] A. A. Gushchin, On the general theory of random fields on the plane. (in Russian) *Uspekhi Mat. Nauk*, 37:53–74, 1982.

[13] G. N. Galanis, T. Gnana Bhaskar, V. Lakshmikantham and P. K. Palamides, Set valued functions in Frechet spaces: continuity, Hukuhara differentiability and applications to set differential equations, *Nonlinear Anal. TMA*, 61:559–575, 2005.

[14] F. Hiai and H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivariate Anal.*, 7:149–182, 1977.

[15] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Vol.1, Theory*, Kluwer Acad. Publ., Boston, 1997.

[16] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Acad. Publ., Dordrecht, 1991.

[17] M. Kisielewicz, Some properties of set-valued stochastic integrals, *J. Math. Anal. Appl.*, 388:984–995, 2012.

[18] M. Kisielewicz, Method of averaging for differential equations with compact convex valued solutions, *Rend. Math.*, 6:397–408, 1976.

[19] M. Kisielewicz, Backward stochastic differential inclusions, *Dynam. Systems Appl.*, 16:121–140, 2007.

[20] M. Kisielewicz, Weak compactness of weak solutions to backward stochastic differential inclusions, *Dynam. Systems Appl.*, 17:351–370, 2008.

[21] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part I. Existence and regularity properties, *Dynam. Systems Appl.*, 12:405–431, 2003.

[22] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part II. Viability and semimartingale issue, *Dynam. Systems Appl.*, 12:433–466, 2003.

[23] V. Lakshmikantham, S. Lela and A. Vatsala, Interconnection between set and fuzzy differential equations, *Nonlinear Anal. TMA*, 54:351–360, 2003.

[24] V. Lakshmikantham and R.N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis Publishers, London, 2003.

[25] V. Lakshmikantham and A.A. Tolstonogov, Existence and interrelation between set and fuzzy differential equations, *Nonlinear Anal. TMA*, 55:255–268, 2003.

[26] V. Lakshmikantham, T. Gnana Bhaskar and J. Vasundhara Devi, *Theory of Set Differential Equations in a Metric Space*, Cambridge Scientific Publ., 2006.

[27] Z. Liang and M. Zheng, Estimates on moments of the solutions to stochastic differential equations with respect to martingales in the plane, *Stochastic Process. Appl.*, 62:263–276, 1996.

[28] M. T. Malinowski, On random fuzzy differential equations, *Fuzzy Sets and Systems*, 160:3152–3165, 2009.

[29] M. T. Malinowski, Existence theorems for solutions to random fuzzy differential equations, *Nonlinear Anal. TMA*, 73:1515–1532, 2010.

[30] M. T. Malinowski, Strong solutions to stochastic fuzzy differential equations of Itô type, *Math. Comput. Modelling*, 55:918–928, 2012.

[31] M. T. Malinowski, Random fuzzy differential equations under generalized Lipschitz condition, *Nonlinear Anal. RWA*, 13:860–881, 2012.

[32] M. T. Malinowski and M. Michta, Fuzzy stochastic integral equations. *Dynam. Systems Appl.*, 19:473–493, 2010.

[33] M. T. Malinowski and M. Michta, Stochastic fuzzy differential equations with an application, *Kybernetika*, 41:123–143, 2011.

[34] M. Michta, Optimal solutions to stochastic differential inclusions, *Applicationes Math.*, 29:387–398, 2002.

[35] M. Michta, On weak solutions to stochastic differential inclusions driven by semimartingales, *Stoch. Anal. Appl.*, 22:1341–1361, 2004.

[36] M. Michta, On set-valued stochastic integrals and fuzzy stochastic equations, *Fuzzy Sets and Systems*, 177:1–19, 2011.

[37] M. Michta and J. Motyl, Weak solutions to a stochastic inclusion with multivalued integrator, *Dynam. Systems Appl.*, 14:323–334, 2005.

[38] M. Michta and J. Motyl, Second order stochastic inclusions and equations, *Dynam. Systems Appl.*, 15:301–315, 2006.

[39] M. Michta and J. Motyl, Set valued Stratonovich integral and Stratonovich type stochastic inclusion, *Dynam. Systems Appl.*, 16:141–154, 2007.

[40] I. Mitoma, Y. Okazaki and J. Zhang, Set-valued stochastic differential equations in M-type 2 Banach space, *Comm. Stoch. Anal.*, 4:215–237, 2010.

[41] C.V. Negoita and D.A. Ralescu, *Applications of Fuzzy Sets to System Analysis*, Wiley, New York, 1975.

[42] Y. Ogura, On stochastic differential equations with fuzzy set coefficients. In: *Soft Methods for Handling Variability and Imprecision*, pp. 263–270, Springer, Berlin, 2008.

[43] B.G. Pachpatte, *Inequalities for differential and integral equations.* Mathematics in Science and Engineering, 197, Academic Press, San Diego, 1998.

[44] D. Qiu and L. Shu, Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings, *Inform. Sci.*,178:3595–3604, 2008.

[45] E.N. Strukov, A stochastic integral for Gaussian processes in the plane, *J. Math. Sci.*, 3:2486–2491, 1988.

[46] A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer, Dordrecht, 2000.

[47] C. Tudor, Stochastic integrals equations in the plane, *Rev. Roumaine Math. Pures Appl.*, 26:507–538, 1981.

[48] J. Turo, Nonlinear stochastic functional intergal equations in the plane, *J. Appl. Math. Stochastic Anal.*, 4:371–379, 1995.

[49] E. Wong and M. Zakai, Martingales and stochastic integrals for processes with a multidimensional parameter, *Z. Wahrsch. Verw. Gebiete*, 29:109–122, 1974.

[50] J. Yeh, Existence of strong solutions for stochastic differential equations in the plane. *Pacific J. Math.*, 97:217–247, 1981.

[51] J. Yeh, Existence of weak solutions to stochastic differential equations in the plane with continuous coefficients, *Trans. Amer. Math. Soc.*, 290:345–361, 1985.

[52] J. Yeh, Uniqueness of strong solutions to stochastic differential equations in the plane with deterministic boundary process, *Pacific J. Math.*, 128:391–400, 1987.

[53] X. Zhang and J. Zhu, Non-Lipschitz stochastic differential equations driven by multi-parameter Brownian motions, *Stoch. Dyn.*, 6:329–340, 2006.