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EXISTENCE OF SOLUTIONS FOR A CLASS OF HISTORY-DEPENDENT EVOLUTION HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. We deal with a class of abstract second order evolution inclusions involving a nonlinear history-dependent operator. For this class we prove an existence and uniqueness result. The proof is based on arguments of evolution inclusions with monotone operators and the Banach fixed point theorem. We apply this result to prove the solvability of a class of second order hemivariational inequalities with nonlinear memory term and, under an additional assumption, its unique solvability.

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1. INTRODUCTION

In this paper we study a class of evolution inclusions of second order in Banach spaces. Our goal is to provide a new result of the existence and uniqueness of solutions. The main feature of the inclusion under consideration is the presence of a memory term (sometimes called a history-dependent operator) which allow for the applications to hemivariational inequalities. The novelty consists in the fact that the memory term in the inclusion is nonlinear.

The study is motivated by mechanical problems with multivalued boundary contact conditions which can be described by the Clarke generalized gradients of nonconvex and nonsmooth energy functionals. These mechanical problems can be formulated as hemivariational inequalities. They were introduced by Panagiotopoulos in the early eighties (cf. Panagiotopoulos [16, 17]) as generalizations of variational inequalities. For motivation and mathematical results on hemivariational inequalities we refer to Panagiotopoulos [16, 17], Naniewicz and Panagiotopoulos [15] and Migorski et al. [14]. We remark that today the theory of hemivariational inequalities plays an important role in the analysis of nonlinear boundary value problems arising in mechanics, physics and engineering sciences. For this reason the mathematical literature in this field is extensive, see for instance [2, 3, 15, 17, 9, 14] and the references therein. A part of the progress in hemivariational inequalities was motivated by new models involving nonconvex energy functions arising in contact mechanics.

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The present paper represents a continuation and an extension of [11] where an abstract evolution inclusion of second order involving a linear Volterra-type integral term was considered. The extension and introduction of a nonlinear memory term is essential since in the applications to problems of mechanics one can study nonlinear constitutive laws which can be introduced to model the nonlinear viscoelastic materials.

2. PRELIMINARIES

In this paper we use standard notation for the Lebesque and Sobolev spaces of functions defined on a time interval $[0,T]$, $T>0$ with values in a Banach space E with a norm $\|\cdot\|_E$. The dual space to E is denoted by E^* and $\langle\cdot,\cdot\rangle_{E^*\times E}$ is the duality pairing of E and E^*. For a set $U\subset E$ we define $\|U\|_E=\sup\{\|u\|_E \mid u\in U\}$. The notation $\mathcal{L}(E,F)$ stands for the space of linear bounded operators defined on the Banach space E with values in the Banach space F.

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary Γ and let Γ_C be a measurable part of Γ such that $\Gamma_C\subseteq\Gamma$. Let V be a closed subspace of $H^1(\Omega;\mathbb{R}^d)$, $Z=H^\delta(\Omega;\mathbb{R}^d)$ where $\delta\in(1/2,1)$ and let $H=L^2(\Omega;\mathbb{R}^d)$. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing of V and V^*, by $\|\cdot\|$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$ the norms on the spaces V, H and V^*, respectively. It is well known that $V\subset Z\subset H\subset Z^*\subset V^*$ continuously and $V\subset Z$ compactly. We introduce the trace operator $\gamma\colon Z\to L^2(\Gamma;\mathbb{R}^d)$ and its adjoint $\gamma^*\colon L^2(\Gamma;\mathbb{R}^d)\to Z^*$. We also consider the spaces

$$\mathcal{V}=L^2(0,T;V),\quad \mathcal{Z}=L^2(0,T;Z),\quad \mathcal{W}=\{v\in\mathcal{V}\mid v'\in\mathcal{V}^*\},$$

where v' denotes the time derivative in the sense of vector-valued distributions. Endowed with the norm $\|v\|_{\mathcal{W}}=\|v\|_{\mathcal{V}}+\|v'\|_{\mathcal{V}^*}$, the space $\mathcal{W}$ becomes a separable, reflexive Banach space. We have $\mathcal{W}\subset\mathcal{V}\subset\mathcal{Z}\subset\widehat{\mathcal{H}}\subset\mathcal{Z}^*\subset\mathcal{V}^*$, where $\widehat{\mathcal{H}}=L^2(0,T;H)$, $\mathcal{Z}^*=L^2(0,T;Z^*)$ and $\mathcal{V}^*=L^2(0,T;V^*)$. Finally, for $t\in[0,T]$, we denote by $C(0,t;E)$ the space of continuous functions from $[0,t]$ to E, with the norm $\|v\|_{C(0,t;E)}=\max_{s\in[0,t]}\|v(s)\|_E$. It is well known (cf. e.g. [21, 3]) that the space $\mathcal{W}$ is embedded continuously in $C(0,T;H)$, i.e. every element of $\mathcal{W}$, after a possible modification on a set of measure zero, has a unique continuous representative in $C(0,T;H)$.

Let $h\colon E\to\mathbb{R}$ be a locally Lipschitz function. Then the generalized directional derivative of h at $x\in E$ in the direction $v\in E$, denoted by $h^0(x;v)$, is defined by

$$h^0(x;v)=\limsup_{y\to x,\ \lambda\downarrow 0}\frac{h(y+\lambda v)-h(y)}{\lambda}$$

and the generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x; v) \geq \langle \zeta, v \rangle_{E^* \times E} \text{ for all } v \in E\}.$$

A locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$. For properties of the generalized directional derivative and the generalized gradient we refer to [1, 2, 14].

The following properties related to the generalized directional derivative and the generalized gradient can be found in Theorem 2.3.10 of [1].

Proposition 2.1. *Let X and Y be Banach spaces, $A \in \mathcal{L}(Y, X)$ and let $f\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then*

(i) $(f \circ A)^0(x; z) \leq f^0(Ax; Az)$ *for* $x, z \in Y$,
(ii) $\partial(f \circ A)(x) \subseteq A^* \partial f(Ax)$ *for* $x \in Y$,

where $A^* \in \mathcal{L}(X^*, Y^*)$ denotes the adjoint operator to A. If in addition either f or $-f$ is regular, then (i) and (ii) are replaced by the corresponding equalities.

3. EVOLUTION INCLUSION

In this section we state and prove a result on the existence and uniqueness of the solution to an abstract second order evolution inclusion.

We consider the following evolution inclusion of second order of the following form

$$\tag{3.1} \begin{cases} \text{find } u \in \mathcal{V} \text{ with } u' \in \mathcal{W} \text{ such that} \\ u''(t) + A(t, u'(t)) + Bu(t) + \mathcal{S}u(t) + \gamma^* \, \partial J(t, \gamma u'(t)) \ni f(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0, T), \\ u(0) = u_0, \;\; u'(0) = v_0. \end{cases}$$

In the study of problem (3.1) we need the following definition.

Definition 3.1. A function $u \in \mathcal{V}$ is called a solution of (3.1) if and only if $u' \in \mathcal{W}$ and there exists $\zeta \in \mathcal{Z}^*$ such that

$$\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) + \mathcal{S}u(t) + \zeta(t) = f(t) \quad \text{a.e. } t \in (0, T), \\ \zeta(t) \in \gamma^* \left(\partial J(t, \gamma u'(t))\right) \quad \text{a.e. } t \in (0, T), \\ u(0) = u_0, \;\; u'(0) = v_0. \end{cases}$$

We consider the following hypotheses on the data.

<u>$H(A)$</u>: $\quad A\colon (0, T) \times V \to V^*$ is such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$;
(ii) $A(t,\cdot)$ is strongly monotone i.e. $\langle A(t,u) - A(t,v), u-v\rangle \ge m_1\|u-v\|^2$ for all u, $v \in V$, a.e. $t \in (0,T)$ with $m_1 > 0$;
(iii) $||A(t,v)||_{V^*} \le a(t) + b||v||$ for all $v \in V$, a.e. $t \in (0,T)$ with $a \in L^2(0,T)$, $a \ge 0$, $b > 0$;
(iv) $\langle A(t,v), v\rangle \ge \alpha||v||^2$ for all $v \in V$, a.e. $t \in (0,T)$ with $\alpha > 0$.

<u>$H(B)$</u> : $B\colon V \to V^*$ is a bounded, linear, monotone and symmetric operator, i.e. $B \in \mathcal{L}(V, V^*)$, $\langle Bv, v\rangle \ge 0$ for all $v \in V$, $\langle Bv, w\rangle = \langle Bw, v\rangle$ for all v, $w \in V$.

<u>$H(\mathcal{S})$</u> : $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ is such that

$$\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_{V^*} \le L_{\mathcal{S}} \int_0^t \|u_1(s) - u_2(s)\|_V \, ds$$

for all u_1, $u_2 \in \mathcal{V}$, a.e. $t \in (0,T)$ with $L_{\mathcal{S}} > 0$.

<u>$H(J)$</u> : $J\colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ is a functional such that

(i) $J(\cdot, v)$ is measurable for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$ and $J(\cdot, 0) \in L^1(0,T)$;
(ii) $J(t,\cdot)$ is locally Lipschitz for a.e. $t \in (0,T)$;
(iii) $\|\partial J(t,v)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le c_0\left(1 + \|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with $c_0 > 0$;
(iv) $J^0(t,v;-v) \le d_0\left(1 + \|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with $d_0 \ge 0$;
(v) $(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \ge -m_2\|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$ for all $z_i \in \partial J(t,w_i)$, $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i = 1,2$, a.e. $t \in (0,T)$ with $m_2 \ge 0$.

<u>(H_0)</u> : $f \in \mathcal{V}^*$, $u_0 \in V$, $v_0 \in H$.

<u>(H_1)</u> : $m_1 > m_2\,\|\gamma\|^2$, where $\|\gamma\| = \|\gamma\|_{\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^d))}$.

We note that the condition stated in $H(\mathcal{S})$ is satisfied for the operator $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ defined by

$$(\mathcal{S}v)(t) = R\Big(t, \int_0^t v(s)\,ds + v_0\Big) \quad \text{for all } v \in \mathcal{V}, \text{ a.e. } t \in (0,T), \tag{3.2}$$

where $R\colon (0,T) \times V \to V^*$ is such that $R(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$, $R(t,\cdot)$ is a Lipschitz continuous operator for a.e. $t \in (0,T)$ and $v_0 \in V$. Moreover, $H(\mathcal{S})$ holds for the Volterra operator $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ given by

$$(\mathcal{S}v)(t) = \int_0^t C(t-s)\,v(s)\,ds \quad \text{for all } v \in \mathcal{V}, \text{ a.e. } t \in (0,T), \tag{3.3}$$

where $C \in L^\infty(0,T;\mathcal{L}(V,V^*))$. Since for the operators (3.2) and (3.3), the current value $(\mathcal{S}v)(t)$ at $t \in (0,T)$ depends on the history of the values of v at the moments $s \in (0,t)$, we refer to the operators of the form (3.2) or (3.3) as history–dependent operators. In what follows, we extend this definition to all the operators $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ which satisfy condition $H(\mathcal{S})$.

Furthermore, let us observe that if the functional J is such that $J(t,\cdot)$ is convex for a.e. $t \in (0,T)$, then its Clarke subdifferential coincides with the subdifferential in the sense of convex analysis, in this case the hypotheses $H(J)(\mathrm{v})$ holds with $m_2 = 0$, and, in consequence, (H_1) is trivially satisfied.

The result on the problem (3.1) is given by the following existence and uniqueness theorem.

Theorem 3.2. *Under hypotheses $H(A)$, $H(B)$, $H(\mathcal{S})$, $H(J)$, (H_0) and (H_1), the problem* (3.1) *admits a unique solution.*

Proof. The proof consists of two steps. First, given $\eta \in \mathcal{V}^*$, we consider the following evolution inclusion of second order without memory term

$$(3.4) \qquad \begin{cases} \text{find } u_\eta \in \mathcal{V} \text{ with } u_\eta' \in \mathcal{W} \text{ such that} \\ u_\eta''(t) + A(t, u_\eta'(t)) + Bu_\eta(t) + \gamma^* \, \partial J(t, \gamma u_\eta'(t)) \ni f(t) - \eta(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,T) \\ u_\eta(0) = u_0, \ u_\eta'(0) = v_0. \end{cases}$$

Applying Proposition 15 in [8], it follows that the problem (3.4) has a unique solution. Furthermore, by Proposition 9 of [8], the unique solution $u_\eta \in \mathcal{V}$ satisfies the estimate

$$(3.5) \qquad \|u_\eta\|_{C(0,T;V)} + \|u_\eta'\|_{\mathcal{W}} \le \bar{c}\,(1 + \|u_0\| + \|u_1\|_H + \|f\|_{\mathcal{V}^*} + \|\eta\|_{\mathcal{V}^*})$$

with a positive constant $\bar{c}$.

In the second step we use a fixed point theorem. To this end, we consider the operator $\Lambda\colon \mathcal{V}^* \to \mathcal{V}^*$ defined by

$$(3.6) \qquad (\Lambda\eta)(t) = (\mathcal{S}u_\eta)(t) \ \text{ for } \ \eta \in \mathcal{V}^*, \ t \in (0,T),$$

where $u_\eta \in \mathcal{V}$ is the unique solution to (3.4). We check that the operator Λ is well defined and it has a unique fixed point. Indeed, for $\eta \in \mathcal{V}^*$, by using $H(\mathcal{S})$, we have

$$\begin{aligned} \|(\mathcal{S}u_\eta)(t)\|_{V^*} &\le \|(\mathcal{S}u_\eta)(t) - (\mathcal{S}0)(t)\|_{V^*} + \|(\mathcal{S}0)(t)\|_{V^*} \le \\ &\le L_{\mathcal{S}} \int_0^t \|u_\eta(s)\|_V \, ds + \|(\mathcal{S}0)(t)\|_{V^*} \le \\ &\le L_{\mathcal{S}} \sqrt{T} \|u_\eta\|_{L^2(0,t;V)} + \|(\mathcal{S}0)(t)\|_{V^*} \end{aligned}$$

for a.e. $t \in (0,T)$. Hence

$$\begin{aligned} \|\Lambda\eta\|_{\mathcal{V}^*}^2 &= \int_0^T \|(\Lambda\eta)(t)\|_{V^*}^2 \, dt = \int_0^T \|(\mathcal{S}u_\eta)(t)\|_{V^*}^2 \, dt \le \\ &\le 2 \int_0^T \Big(L_{\mathcal{S}}^2 T \|u_\eta\|_{L^2(0,t;V)}^2 + \|(\mathcal{S}0)(t)\|_{V^*}^2 \Big) \, dt \le \\ &\le c_1 (1 + \|u_\eta\|_{\mathcal{V}}^2) \end{aligned}$$

with $c_1 > 0$. It is clear from (3.5) that the operator Λ takes vales in $\mathcal{V}^*$.

In order to prove that the operator Λ has a unique fixed point, let $\eta_1,\ \eta_2 \in \mathcal{V}^*$, and let $u_1 = u_{\eta_1}$ and $u_2 = u_{\eta_2}$ be the corresponding solutions to (3.4) such that $u_i \in \mathcal{V}$ and $u_i' \in \mathcal{W}$ for $i = 1,\ 2$. Thus, we have

$$u_1''(t) + A(t, u_1'(t)) + Bu_1(t) + \zeta_1(t) = f(t) - \eta_1(t) \quad \text{a.e. } t \in (0,T), \tag{3.7}$$

$$u_2''(t) + A(t, u_2'(t)) + Bu_2(t) + \zeta_2(t) = f(t) - \eta_2(t) \quad \text{a.e. } t \in (0,T), \tag{3.8}$$

$$\zeta_1(t) \in \gamma^* \partial J(t, \gamma u_1'(t)), \quad \zeta_2(t) \in \gamma^* \partial J(t, \gamma u_2'(t)) \quad \text{a.e. } t \in (0,T), \tag{3.9}$$

$$u_1(0) = u_2(0) = u_0,\ u_1'(0) = u_2'(0) = v_0. \tag{3.10}$$

Subtracting (3.8) from (3.7), multiplying the result by $u_1'(t) - u_2'(t)$ and integrating by parts with the initial conditions (3.10) we obtain, for all $t \in [0,T]$

$$\frac{1}{2}\|u_1'(t) - u_2'(t)\|_H^2 + \int_0^t \langle A(s, u_1'(s)) - A(s, u_2'(s)), u_1'(s) - u_2'(s)\rangle \, ds + \tag{3.11}$$

$$+ \int_0^t \langle Bu_1(s) - Bu_2(s), u_1'(s) - u_2'(s)\rangle \, ds + \int_0^t \langle \zeta_1(s) - \zeta_2(s), u_1'(s) - u_2'(s)\rangle \, ds =$$

$$= \int_0^t \langle \eta_2(s) - \eta_1(s), u_1'(s) - u_2'(s)\rangle \, ds.$$

Next, from (3.9) we infer that $\zeta_i(t) = \gamma^* z_i(t)$ with $z_i(t) \in \partial J(t, \gamma u_i'(t))$ for a.e. $t \in (0,T)$ and $i = 1,\ 2$. Therefore, by using $H(J)$(v), we have

$$\int_0^t \langle \zeta_1(s) - \zeta_2(s), u_1'(s) - u_2'(s)\rangle \, ds =$$

$$= \int_0^t (z_1(s) - z_2(s), \gamma u_1'(s) - \gamma u_2'(s))_{L^2(\Gamma_C;\mathbb{R}^d)} \, ds \geq$$

$$\geq -m_2 \int_0^t \|\gamma u_1'(s) - \gamma u_2'(s)\|^2_{L^2(\Gamma_C;\mathbb{R}^d)} \, ds \geq -m_2 \|\gamma\|^2 \int_0^t \|u_1'(s) - u_2'(s)\|^2 \, ds$$

for all $t \in [0,T]$. Employing in (3.11) the previous inequality, $H(A)$(ii) and the following relation

$$\int_0^t \langle Bu_1(s) - Bu_2(s), u_1'(s) - u_2'(s)\rangle \, ds =$$

$$= \frac{1}{2}\int_0^t \frac{d}{ds}\langle B(u_1(s) - u_2(s)), u_1(s) - u_2(s)\rangle \, ds =$$

$$= \frac{1}{2}\langle B(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle \geq 0$$

for all $t \in [0,T]$, we get

$$\frac{1}{2}\|u_1'(t) - u_2'(t)\|_H^2 + c\int_0^t \|u_1'(s) - u_2'(s)\|^2 \, ds \leq$$

$$\leq \int_0^t \|\eta_1(s) - \eta_1(s)\|_{V^*} \|u_1'(s) - u_2'(s)\| \, ds$$

for all $t \in [0,T]$ with $c = m_1 - m_2\|\gamma\|^2 > 0$. Hence

$$c\,\|u_1' - u_2'\|^2_{L^2(0,t;V)} \le \|\eta_1 - \eta_1\|_{L^2(0,t;V^*)}\,\|u_1' - u_2'\|_{L^2(0,t;V)}$$

for all $t \in [0,T]$, which implies that

$$\|u_1' - u_2'\|_{L^2(0,t;V)} \le \frac{1}{c}\|\eta_1 - \eta_1\|_{L^2(0,t;V^*)} \tag{3.12}$$

for all $t \in [0,T]$. Since u_1, $u_2 \in H^1(0,T;V)$ and V is reflexive, by Theorem 3.4.11 and Remark 3.4.9 of [3], we know that u_1, $u_2 \in AC^{1,2}(0,T;V)$ and, using (3.10), we have

$$u_i(t) = u_0 + \int_0^t u_i'(s)\,ds \quad \text{for} \quad i = 1,2.$$

Thus

$$\|u_1(t) - u_2(t)\| \le \int_0^t \|u_1'(s) - u_2'(s)\|\,ds \le \sqrt{T}\|u_1' - u_2'\|_{L^2(0,t;V)} \tag{3.13}$$

for all $t \in [0,T]$. From (3.12) and (3.13) we obtain

$$\|u_1(t) - u_2(t)\| \le \frac{\sqrt{T}}{c}\|\eta_1 - \eta_2\|_{L^2(0,t;V^*)} \quad \text{for all } t \in [0,T]. \tag{3.14}$$

On the other hand, from $H(\mathcal{S})$ and (3.13), we deduce

$$\begin{aligned}\|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)\|^2_{V^*} &\le L^2_{\mathcal{S}}\left(\int_0^t \|u_1(s) - u_2(s)\|_{V^*}\,ds\right)^2 \\ &\le \frac{L^2_{\mathcal{S}}T^2}{c^2}\,t\,\|\eta_1 - \eta_2\|_{L^2(0,t;V^*)}\end{aligned}$$

and

$$\begin{aligned}\|(\Lambda^2\eta_1)(t) - (\Lambda^2\eta_2)(t)\|^2_{V^*} &= \|\Lambda(\Lambda\eta_1)(t) - \Lambda(\Lambda\eta_2)(t)\|^2_{V^*} \\ &\le \frac{L^2_{\mathcal{S}}T^2}{c^2} t \int_0^t \|(\Lambda\eta_1)(s) - (\Lambda\eta_2)(s)\|^2_{V^*}\,ds \\ &\le \frac{L^4_{\mathcal{S}}T^5}{c^4}\frac{t^2}{2}\|\eta_1 - \eta_2\|^2_{\mathcal{V}^*}\end{aligned}$$

for all $t \in [0,T]$. Reiterating this inequality k times, we obtain

$$\begin{aligned}\|(\Lambda^k\eta_1)(t) - (\Lambda^k\eta_2)(t)\|^2_{V^*} &\le \frac{L^{2k}_{\mathcal{S}}T^{2k+1}}{c^{2k}}\frac{t^k}{k!}\|\eta_1 - \eta_2\|^2_{\mathcal{V}^*} \\ &\le \frac{L^{2k}_{\mathcal{S}}T^{3k+1}}{c^{2k}}\frac{1}{k!}\|\eta_1 - \eta_2\|^2_{\mathcal{V}^*}\end{aligned}$$

for all $t \in [0,T]$. Hence

$$\|\Lambda^k\eta_1 - \Lambda^k\eta_2\|_{\mathcal{V}^*} \le \frac{L^k_{\mathcal{S}}T^{\frac{3k}{2}+1}}{c^k}\frac{1}{\sqrt{k!}}\|\eta_1 - \eta_2\|^2_{\mathcal{V}^*}.$$

Therefore, we deduce that for k sufficiently large, the operator Λ^k is a contraction on $\mathcal{V}^*$. Hence, there exists a unique $\eta^* \in \mathcal{V}^*$ such that $\eta^* = \Lambda^k\eta^*$. It is clear that $\Lambda^k(\Lambda\eta^*) = \Lambda(\Lambda^k\eta^*) = \Lambda\eta^*$, so $\Lambda\eta^*$ is also a fixed point of Λ^k. By the uniqueness of fixed point of Λ^k, we have $\eta^* = \Lambda\eta^*$. Thus $\eta^* \in \mathcal{V}^*$ is the unique fixed point of the operator Λ. Finally, it is clear that u_{η^*} is a solution to (3.1), which concludes the existence part of the theorem.

The uniqueness part is a consequence of the fixed point of Λ. Namely, let $u \in \mathcal{V}^*$ with $u' \in \mathcal{W}$ be a solution to (3.1) and define the element $\eta \in \mathcal{V}^*$ by

$$\eta(t) = (\mathcal{S}u)(t) \quad \text{for all} \quad t \in [0, T].$$

It follows that u is the solution to the problem (3.4) and by the uniqueness of solutions to (3.4), we obtain $u = u_\eta$. This implies $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of Λ we obtain $\eta = \eta^*$, so $u = u_{\eta^*}$, which concludes the proof. □

We conclude this section with a remark that existence results on evolution inclusions of the form (3.1) with the multivalued Clarke subdifferential operator depending on both u and its derivative u' can be found in [10, 9, 14].

4. HEMIVARIATIONAL INEQUALITY

In this section we apply Theorem 3.2 in the study of a class of second order hemivariational inequalities. The problem we are interested in is formulated as follows

$$(4.1) \qquad \begin{cases} \text{find } u \in \mathcal{V} \text{ with } u' \in \mathcal{W} \text{ such that} \\ \langle u''(t) + A(t, u'(t)) + Bu(t) + \mathcal{S}u(t), v\rangle + \\ \qquad\qquad + \displaystyle\int_{\Gamma_C} j^0(x, t, u'(t); v)\, d\Gamma \geq \langle f(t), v\rangle \\ \qquad\qquad\qquad \text{for all } v \in V \text{ and a.e. } t \in (0, T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

In the study of (4.1) we need the following hypothesis.

<u>$H(j)$</u>: $\quad j\colon \Gamma_C \times (0, T) \times \mathbb{R}^d \to \mathbb{R}$ is such that

(i) $j(\cdot, \cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}$ and $j(\cdot, \cdot, 0) \in L^1(\Gamma_C \times (0, T))$;

(ii) $j(x, t, \cdot)$ is locally Lipschitz for a.e. $(x, t) \in \Gamma_C \times (0, T)$;

(iii) $|\partial j(x, t, \xi)| \leq \widetilde{c}\,(1 + \|\xi\|_{\mathbb{R}^d})$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma_C \times (0, T)$ with $\widetilde{c} > 0$;

(iv) $j^0(x, t, \xi; -\xi) \leq \widetilde{d}\,(1 + \|\xi\|_{\mathbb{R}^d})$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma_C \times (0, T)$ with $\widetilde{d} \geq 0$;

(v) $(\eta_1 - \eta_2, \xi_1 - \xi_2)_{\mathbb{R}^d} \geq -m_2\|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$ for all $\eta_i \in \partial j(x, t, \xi_i)$, $\xi_i \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma_C \times (0, T)$, $i = 1, 2$ with $m_2 \geq 0$,

where j^0 and ∂j denote the directional derivative and the Clarke generalized gradient of $j(x, t, \cdot)$, respectively.

We consider the functional $J\colon (0, T) \times L^2(\Gamma_C; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$(4.2) \qquad J(t, v) = \int_{\Gamma_C} j(x, t, v(x))\, d\Gamma \quad \text{for a.e. } t \in (0, T) \text{ and } v \in L^2(\Gamma_C; \mathbb{R}^d).$$

We have the following result on the properties of the functional (4.2).

Proposition 4.1. *Under the hypothesis* $H(j)$, *the functional* J *given by* (4.2) *satisfies* $H(J)$, *and for all* u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$, *we have*

$$J^0(t,u;v) \le \int_{\Gamma_C} j^0(x,t,u(x);v(x))\,d\Gamma, \tag{4.3}$$

where $J^0(t,u;v)$ *denotes the directional derivative of* $J(t,\cdot)$ *at a point* $u \in L^2(\Gamma_C;\mathbb{R}^d)$ *in the direction* $v \in L^2(\Gamma_C;\mathbb{R}^d)$.

Proof. The conditions $H(J)$(i)−−(iii) and (4.3) follow from $H(j)$(i)−−(iii) (analogously as in Lemma 3 in [8]). The sign condition $H(J)$(iv) is a consequence of (4.3) and $H(j)$(iv). For the proof of $H(J)$(v), consider $z_i \in \partial J(t,w_i)$ and $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, for $i = 1, 2$. By the formula

$$\partial J(t,v) \subset \int_{\Gamma_C} \partial j(x,t,v(x))\,d\Gamma \quad \text{a.e. } t \in (0,T) \text{ and } v \in L^2(\Gamma_C;\mathbb{R}^d)$$

(cf. Theorem 2.7.5 of [1]), we have $z_i(x) \in \partial j(x,t,w_i(x))$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, $i = 1, 2$. Therefore, using $H(j)$(v), we have

$$(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma_C;\mathbb{R}^d)} = \int_{\Gamma_C} (z_1(x) - z_2(x), w_1(x) - w_2(x))_{\mathbb{R}^d}\,d\Gamma \ge$$

$$\ge -m_2 \int_{\Gamma_C} \|w_1(x) - w_2(x)\|^2_{\mathbb{R}^d}\,d\Gamma = -m_2\|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}.$$

We conclude that assumption $H(J)$(v) is satisfied, which completes the proof of the proposition. □

Combining Theorem 3.2 and Proposition 4.1 to obtain the following existence result.

Corollary 4.2. *Under the hypotheses* $H(A)$, $H(B)$, $H(C)$, $H(j)$, (H_0) *and* (H_1), *the hemivariational inequality* (4.1) *has at least a solution.*

Proof. Let us denote by u the solution of the problem (3.1) with J given by (4.2). Note that the existence and uniqueness of this solution is guaranteed by Theorem 3.2 and Proposition 4.1. Therefore, by Definition 3.1 we have $u \in \mathcal{V}$, $u' \in \mathcal{W}$,

$$u''(t) + A(t,u'(t)) + Bu(t) + \mathcal{S}u(t) + \zeta(t) = f(t), \tag{4.4}$$

where $\zeta(t) = \gamma^* z(t) \in Z^*$ and $z(t) \in \partial J(t,\gamma u'(t))$ for a.e. $t \in (0,T)$. The latter is equivalent to

$$(z(t),w)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t,\gamma u'(t);w) \tag{4.5}$$

for all $w \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. We combine now (4.3)–(4.5) to obtain

$$\langle f(t) - u''(t) - A(t,u'(t)) - Bu(t) - \mathcal{S}u(t), v\rangle = \langle \zeta(t), v\rangle_{Z^*\times Z} =$$

$$= (z(t),\gamma v)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t,\gamma u'(t);\gamma v) \le \int_{\Gamma_C} j^0(x,t,u'(t);v)\,d\Gamma.$$

for all $v \in V$, a.e. $t \in (0,T)$. It follows from the last inequality that u is a solution to (4.1), which concludes the proof. □

We complete the result of Corollary 4.2 with the following uniqueness result.

Corollary 4.3. *Under the hypotheses of Corollary* 4.2*, if either* $j(x,t,\cdot)$ *or* $-j(x,t,\cdot)$ *is regular for a.e.* $(x,t) \in \Gamma_C \times (0,T)$*, then the hemivariational inequality* (4.1) *admits a unique solution.*

Proof. Let u be a solution to (4.1) obtained in Corollary 4.2. It is well known (cf. [1, 2]) that if either $j(x,t,\cdot)$ or $-j(x,t,\cdot)$ is regular for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then either $J(t,\cdot)$ or $-J(t,\cdot)$ is regular for a.e. $t \in (0,T)$, respectively, and (4.3) holds with equality. Therefore, using the equality in (4.3), we have

$$\langle u''(t) + A(t,u'(t)) + Bu(t) + \mathcal{S}u(t) - f(t), v\rangle + J^0(t,\gamma u'(t);\gamma v) \geq 0$$

for all $v \in V$ and a.e. $t \in (0,T)$. From Proposition 2.1(i), we deduce

$$\langle f(t) - u''(t) - A(t,u'(t)) - Bu(t) - \mathcal{S}u(t), v\rangle \leq (J \circ \gamma)^0(t,u'(t);v)$$

for all $v \in V$ and a.e. $t \in (0,T)$. Now, using the definition of the Clarke subdifferential, Proposition 2.1(ii) and the previous inequality, we have that

$$f(t) - u''(t) - A(t,u'(t)) - Bu(t) - \mathcal{S}u(t) \in \partial(J \circ \gamma)(t,u'(t)) = \gamma^* \partial J(t,\gamma u'(t))$$

for a.e. $t \in (0,T)$. This means that u is a solution to (3.1). The uniqueness of solution to (4.1) follows now from the uniqueness part in Theorem 3.2. This concludes the proof. □

We conclude this section with a remark on possible applications of Corollary 4.3 to problems of contact mechanics. Corollary 4.3 can be used to obtain a result on the unique solvability of a dynamic viscoelastic frictional contact problem with a nonlinear constitutive law involving a nonlinear memory term. An application of Corollary 4.3 to contact problem of viscoelasticity with a linear memory term can be found in Migorski et al. [11]. The weak formulation of these mechanical problems leads to a hemivariational inequality (4.1) for the displacement field. For recent results on the theory of hemivariational inequalities we refer to [17, 15, 8, 14, 10, 9]. Applications of evolution hemivariational inequalities to problems of contact mechanics can be found, for instance, in [12, 13] while the results on modeling and analysis of contact problem are contained in [5, 4, 6, 7, 18, 19, 20] and references therein.

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