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EXISTENCE OF SOLUTIONS OF FUNCTIONAL STOCHASTIC INCLUSION

JERZY MOTYL

Faculty of Mathematics, Computer Science and Econometrics,
University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland

ABSTRACT. We prove the existence of local solutions of the delayed stochastic inclusion $dX(t) \in F(X_t)dt + G(X_t)dW(t),\ X_0 = \xi$, with upper separated set-valued functions F and G.

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1. INTRODUCTION

Stochastic ordinary differential inclusions were investigated in early 90's of the last century among others by N. U. Ahmed in [1], [2], J. P. Aubin and G. Da Prato in [4], G. Da Prato and H. Frankowska in [7], M. Kisielewicz in [10], [11] and the author in [20], [21]. In the papers [16] and [22] stochastic inclusions driven by semimartingales have been studied. All the papers mentioned above refer mainly to strong solutions of stochastic inclusions with Lipschitz continuous or dissipative set-valued operators. We refer the reader to the survey works [12] and [13] for results on this topic. From the other side, stochastic functional equations with delay were investigated by many authors during last decades (see e.g.: [14], [15], [19] and references therein). Stochastic functional inclusions with delay have been considered by P. Balasubramanian, S. K. Ntouyas and D. Vinayagam in [5], [6].

In this work we prove the existence of local strong solutions for delay stochastic inclusion with upper separated set-valued drift and diffusion terms. Let us mention that such set-valued functions introduced in [17] need not be continuous in any sense.

2. MAIN RESULT

Let $r > 0$ be given. By $C([-r, 0], R^d)$ we denote the Banach space of continuous R^d-valued functions defined on $[-r, 0]$ and endowed with the supremum norm $\|\cdot\|$. Let $W = (W(t))_{t\geq 0}$ be an R^m-valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the standard Brownian filtration $(\mathcal{F}_t^W)_{t\geq 0}$. Let $\xi : \Omega \to$

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$C([-r,0],R^d)$ be an $(\mathcal{F}_t^W)$-independent $C([-r,0],R^d)$-valued random variable. By $\mathcal{F}_t$ we denote a σ-algebra $\mathcal{F}_t^W \vee \sigma(\xi)$.

For a stochastic proces X defined at least on $[t-r,t]$ we denote $X_t(s) = X(t+s)$, $s \in [-r,0]$, $t \geq 0$.

Let T be a predictable $(\mathcal{F}_t)$-stopping time and let $X : (\Omega \times [-r,0]) \cup [0,T) \to R^d$. A pair (X,T) is called a local $(\mathcal{F}_t)$-semimartingale up to time T starting from ξ if $X_0 = \xi$ P-almost surely and for any sequence of predictable stopping times $T_n \nearrow T$, the process $X(t \wedge T_n)$ is an $(\mathcal{F}_t)$-adapted semimartingale.

Let $ClConv(R^d)$ denote the family of all closed, convex and nonempty subsets of R^d.

Definition 2.1. Let $F : C([-r,0],R^d) \to ClConv(R^d)$ and $G : C([-r,0],R^d) \to ClConv(R^{d\times m})$ be given set-valued functions. Consider the stochastic functional inclusion (SFI):

$$dX(t) \in F(X_t)dt + G(X_t)dW(t),$$

$$X_0 = \xi.$$

A local $(\mathcal{F}_t)$-semimartingale (X,T) up to a predictable stopping time T is called a local strong solution of inclusion (SFI) if $X_0 = \xi$ and for any stopping time $T_n < T$ and $0 \leq s \leq t < \infty$

$$X(t \wedge T_n) - X(s \wedge T_n) \in \int_{s\wedge T_n}^{t\wedge T_n} F(X_u)du + \int_{s\wedge T_n}^{t\wedge T_n} G(X_u)dW(u).$$

Set-valued Lebesgue and Itô integrals are meant in the sense of Aumann. For detailed definitions and properies of such integrals see e.g.: [12].

The pair (X,T) is called a maximal local solution if

$$P(\{\exists \text{ compact } K \subset C([-r,0],R^d)\ \exists\ t_i \nearrow T : X_{t_i} \in K\} \cap \{T < \infty\}) = 0.$$

It means that (X_t) leaves any compact set $K \subset C([-r,0],R^d)$ for $t \to T$, P-almost surely on $\{T < \infty\}$.

Assume $(Y, \preceq)$ is an order complete Banach lattice with an order generated by a positive cone K^+ (i.e.: $x \preceq y$ iff $y - x \in K^+$).

We adjoin to Y the greatest element $+\infty$ together with the lowest element $-\infty$ and extend the vector space operations in a natural way. Let $\bar{Y} = Y \cup \{\pm\infty\}$.

Let Z be a Banach space. For a set-valued function $F : Z \to ClConv(Y)$ we define functions $V, W : Z \to \bar{Y}$ by formulas

$$V(x) = \sup\{a : a \in F(x)\} \text{ and } W(x) = inf\{b : b \in F(x)\}.$$

Let $\Pi_{F(x)}(a)$ denote the metric projection of a point $a \in Y$ onto the set $F(x)$. We define

$$\bar{V}(x) := \begin{cases} \Pi_{F(x)}(V(x)) & \text{for} \quad x \in DomV \\ +\infty & \text{for} \quad x \notin DomV \end{cases}$$

$$\bar{W}(x) := \begin{cases} \Pi_{F(x)}(W(x)) & \text{for} \quad x \in DomW \\ -\infty & \text{for} \quad x \notin DomW \end{cases}$$

where $Domf = \{x \in Z : \ f(x) \neq \pm\infty\}$.

Definition 2.2. A set-valued function $F : Z \to ClConv(Y)$ is upper separated if each point $(x, \bar{W}(x) - \epsilon)$ can be separated from the set $Epi\bar{V} = \{(x, a) \in Z \times Y : \ \bar{V}(x) \preceq a\}$ in the following sense:

for every $x \in Z$ and each $\epsilon \in K^+ \backslash \{0\}$ there exist $A \in \mathcal{L}(Z, Y)$, $a \in R^1$ and $\delta \in K^+ \backslash \{0\}$ such that for every $y \in Dom\bar{V}$ and each $b \in K^+$ the condition

$$A(x) - A(y) + a(\bar{W}(x) - \bar{V}(y) - \epsilon - b) - \delta \in K^+$$

holds where $\mathcal{L}(Z, Y)$ denotes the space of all linear and norm-continuous operators from Z to Y.

In an equivalent form the above condition means

$$A(x) + a(\bar{W}(x) - \epsilon) \succeq A(y) + a(\bar{V}(y) + b) + \delta.$$

Example 2.3. Let $Z = C([a, b], R^d)$ and let Y be an arbitrary order-complete Banach lattice with a positive cone K^+. Let $\mathrm{Var}(x)$ denote a total Jordan variation of the function x on the interval $[a, b]$. Let $z \in K^+ \backslash \{0\}$ be arbitrary fixed. We define a set-valued function $F : C([a, b], R^d) \to ClConv(Y)$ by the formula:

$$F(x) = \begin{cases} [\,0, z\,] & \text{for } x \text{ such that } \ \mathrm{Var}(x) < \infty \\ [\,-z, 0\,] & \text{for } x \text{ such that } \ \mathrm{Var}(x) = \infty \end{cases}$$

where $[\ 0, z\ ]$ and $[\ -z, 0\ ]$ are order intervals in Y. Observe that $V(y)$ is equal to z for finite variation functions $x \in C([a, b], R^d)$ and takes on the value 0 otherwise. Similarly $W(x)$ takes on 0 or $-z$ as its values. Therefore, taking $A \equiv 0$, $a = -1$, $\delta = \epsilon$ in Definition 2.2, and noting that $\bar{W}(x) = W(x)$, $\bar{V}(y) = V(y)$ we obtain the inequality

$$\forall_{x,y \in C([a,b],R^d)} \forall_{b \in K^+} \ \ V(y) + b \succeq W(x),$$

which is clearly fulfiled because of

$$V(y) + b \succeq 0 \succeq \max\{0, -z\} \succeq W(x).$$

This means that F is upper separated. Moreover, the above defined F is neither upper nor lower semicontinuous in any point $x \in C([a, b], R^d)$ because families of finite variation functions as well as infinite variation functions are dense subsets of $C([a, b], R^d)$.

Upper separated set-valued functions need not satisfy any type of Lipschitz nor monotone-dissipative conditions. For other examples of upper separated set-valued functions see [18].

A set $A \subset Y$ is called order bounded if it is contained in some order interval $[a, b] = \{y \in Y : a \preceq y \preceq b\}$. A set A is order convex if for each $x, y \in A$ the order interval $[x, y] \subset A$.

A set-valued function $F : Z \to 2^Y$ is majorized in the neighbourhood of x_0 if there exists an open neighbourhood U_{x_0} and $y \in Y$ such that for each $x \in U_{x_0}$ and every $a \in F(x)$ the inequality $a \preceq y$ holds.

Definition 2.4. A function $f : Z \to Y$ is locally Lipschitz if and only if for every $z \in Z$ there exist an open neighbourhood U_z and a constant $L_z > 0$ such that

$$\|f(x) - f(y\| \leq L_z \|x - y\| \text{ for every } x, y \in U_z.$$

Let us remark, that for an infinite dimensional space Z, e.g.: $Z = C([-r, 0], R^d)$, the above property is essentially weaker than the inequality $\|f(x)-f(y)\| \leq L_n\|x-y\|$ for every $x, y \in Z$ with $\|x\| < n, \|y\| < n$, or $\|f(x) - f(y)\| \leq L_{n,\epsilon}\|x - y\|$ with $\|x\| < n, \|y\| < n$, $\|x - y\| \leq \epsilon$, and called also "a local Lipschitz property" by many authors investigating existence of solutions of stochastic delay equations (see e.g.: [3], [9], [15], [19]).

The following result from [17] will be useful in the sequel:

Theorem 2.5. *Let $F : Z \to ClConv(Y)$ takes on order bounded and order convex values. Assume that there exists $x_1 \in Z$ such that F is majorized in a neighborhood of x_1. If F is upper separated then there exists a locally Lipschitz and order convex function f such that $f(x) \in F(x)$ for each $x \in Z$.*

Remark 2.6. Let us note, that if a set-valued function F admits an order-convex selection satisfying $\bar{W}(x) \preceq f(x) \preceq \bar{V}(x)$, then F should be upper separated (see: [17]). Therefore, the "upper separating property" gives the necessary and sufficient conditions for the existence of order-convex selections.

Consider R^d with the Euclidean norm $|\cdot|$ (resp.: $R^{d\times m}$ with the norm $|M| = \sqrt{tr(MM^*)}$) and the canonical order defined by the positive cone $K^+ := \{a \in R^d : a_i \geq 0, i = 1, 2, ..., d\}$ (resp.: $a_{i,j} \geq 0, i = 1, 2, ..., d,\ j = 1, 2, ..., m$). Then $(R^d, \preceq)$ (resp.: $(R^{d\times m}, \preceq)$ is an order complete Banach lattice.

Now, we are ready to prove the main result of the paper.

Theorem 2.7. *Let $F : C([-r, 0], R^d) \to ClConv(R^d)$ and $G : C([-r, 0], R^d) \to ClConv(R^{d\times m})$ be upper separated set-valued functions with order bounded and order convex values. Assume that F is majorized in the neighbourhood of some point*

$x_1 \in C([-r,0],R^d)$ and G is majorized in the neighbourhood of some point $x_2 \in C([-r,0],R^d)$. Then the inclusion (SFI) admits a maximal local strong solution (X,T).

Proof. Since F and G satisfy assumptions of Theorem 2.5 with $Z = C([-r,0],R^d)$, $Y = (R^d, \preceq)$ (resp.: $Y = (R^{d\times m}, \preceq)$, then there exist selections $f : C([-r,0],R^d) \to R^d$ of F and $g : C([-r,0],R^d) \to R^{d\times m}$ of G being locally Lipschitz in the sense of Definition 2.4. Observe first that f and g are Lipschitz continuous on every compact set $K \subset C([-r,0],R^d)$ with Lipschitz constants L_K and M_K depending only on the set K. Indeed, let K be an arbitrary fixed compact subset of $C([-r,0],R^d)$. For every $z \in C([-r,0],R^d)$ let $B(z;\epsilon_z)$ denote an open ball centered in z with radius ϵ_z on which f is Lipschitz with a constant L_z. A family $\{B(z;\epsilon_z)\}_{z\in C([-r,0],R^d)}$ is an open covering of $C([-r,0],R^d)$. From this covering we take the finite subcovering $\mathcal{A} = \{B(z_i;\epsilon_{z_i})\}_{i=1,2,...n}$ of a compact set K. There exists $\delta > 0$ such that $\{B(z;\delta)\}_{z\in K}$ covers K and each $B(z;\delta)$ is contained in some $B(z_i;\epsilon_{z_i})$ (see e.g.: [8] Th. 4.3.20). Let $x,y \in K$ be such that $\|x-y\| < \delta$. Then there exists some i, $i = 1,2,...n$, such that $x,y \in B(y;\delta) \subset B(z_i;\epsilon_{z_i})$.

Let

$$N = \sup\{|f(x)| : x \in K\} = \max_{1\le i\le n} \sup\{|f(x)| : x \in B(z_i;\epsilon_{z_i})\}.$$

For every $x \in B(z_i;\epsilon_{z_i})$ we have

$$|f(x)| \le |f(x) - f(z_i)| + |f(z_i)| \le L_{z_i}\|x - z_i\| + |f(z_i)| \le L_{z_i}\epsilon_{z_i} + |f(z_i)|.$$

Therefore,

$$N \le \max_{1\le i\le n}\{L_{z_i}\epsilon_{z_i} + |f(z_i)|\} < \infty.$$

Let $L_K = \max_{1\le i\le n}\{L_{z_i}; 2N/\delta\}$ and let $x,y \in K$ be arbitrary chosen. Two cases can occur:

(a) $\|x-y\| < \delta$

(b) $\|x-y\| \ge \delta$.

In the case (a) there exists some i, $i = 1,2,...n$, such that $x,y \in B(z_i;\epsilon_{z_i})$. Then

$$|f(x) - f(y)| \le L_{z_i}\|x-y\| \le L_K\|x-y\|.$$

In the case (b) we have

$$|f(x) - f(y)| \le 2N = 2N\delta/\delta \le L_K\delta \le L_K\|x-y\|,$$

and therefore, f is Lipschitz on K with a Lipschitz constant L_K. The same holds for g with some Lipschitz constant M_K.

By the above Lipschitz property we get

$$2\langle f(x) - f(y); x(0) - y(0)\rangle \le 2|f(x) - f(y)| \cdot |x(0) - y(0)|$$

$\leq 2L_K\|x-y\|\cdot|x(0)-y(0)|\leq 2L_K\|x-y\|^2$

for all $x,y\in K$, where $\langle\,\cdot,\ \cdot\rangle$ denots the inner product in R^d.

Therefore,

$$2\langle f(x)-f(y);x(0)-y(0)\rangle+|g(x)-g(y)|^2\leq(2L_K+M_K^2)\|x-y\|^2.$$

Now we are in position to use the following result of M.-K von Renesse and M. Scheutzow from [23]:

Assume that for each compact set $K\subset C([-r,0],R^d)$ there exists a number N_K such that for all $x,y\in K$

$$2\langle f(x)-f(y);x(0)-y(0)\rangle+|g(x)-g(y)|^2\leq N_K\|x-y\|^2.$$

Then the stochastic equation

$$dX(t)=f(X_t)dt+g(X_t)dW(t),\ X_0=\xi$$

admits a unique maximal local strong solution (X,T) up to a predictable stopping time T.

It means that $X_0=\xi$ and for any stopping time $T_n<T$ and $0\leq t<\infty$

$$X(t\wedge T_n)=X(0)+\int_0^{t\wedge T_n}f(X_u)du+\int_0^{t\wedge T_n}g(X_u)dW(u)\ \text{ P-a.s.}$$

Therefore,

$$X(t\wedge T_n)-X(s\wedge T_n)=\int_{s\wedge T_n}^{t\wedge T_n}f(X_u)du+\int_{s\wedge T_n}^{t\wedge T_n}g(X_u)dW(u)$$
$$\in\int_{s\wedge T_n}^{t\wedge T_n}F(X_u)du+\int_{s\wedge T_n}^{t\wedge T_n}G(X_u)dW(u),$$

because f and g are selections of F and G respectively. This proves the Theorem. □

Remark 2.8. Assume additionally that F and G satisfy the following global growth conditions with convex functions on their right sides:

there exists $c>0$ such that for every $x\in C([-r,0],R^d)$

$|F(x)|=\sup\{|a|:\ a\in F(x)\}\leq(c(1+\|x\|^2))^{1/2};\ \ |G(x)|\leq(c(1+\|x\|^2))^{1/2}.$

Then the solution (X,T) from Theorem 2.7 exists globally, i.e.: $T=+\infty$ P-almost surely.

Indeed, it suffices to observe that selections f and g used in the proof of Theorem 2.7 satisfy

$$2\langle f(x);x(0)\rangle+|g(x)|^2\leq 2(c(1+\|x\|^2))^{1/2}\cdot\|x\|+c(1+\|x\|^2)$$
$$\leq 2(c^{1/2}+1)(1+\|x\|^2)=\rho(\|x\|^2),$$

where the convex function $\rho(u)=2(c^{1/2}+1)(1+u)$ is non-decreasing with $\int_0^\infty 1/\rho(u)du=+\infty$.

Now, the remark follows by Theorem 2.3 of [23].

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