Dynamic Systems and Applications 21 (2012) 339-350

EXISTENCE OF ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS TO NONLINEAR DELAY INTEGRAL EQUATIONS

JIAN-QIONG ZHAO, YONG-KUI CHANG, AND G. M. N'GUÉRÉKATA

Department of Mathematics, Lanzhou Jiaotong University
Lanzhou 730070, China
zhao_jianqiong@163.com
Department of Mathematics, Lanzhou Jiaotong University
Lanzhou 730070, China
lzchangyk@163.com Corresponding Address.
Department of Mathematics, Morgan State University,1700 E. Cold Spring Lane,
Baltimore, M.D. 21251, USA
nguerekata@aol.com Corresponding Address.

ABSTRACT. In this paper, we first establish a new fixed point theorem for mixed monotone operators in a cone, and then apply it to prove the existence of asymptotically almost automorphic solutions to a nonlinear delay integral equation.

AMS (MOS) Subject Classification. 34K14, 60H10, 35B15, 34F05

1. INTRODUCTION

The existence of periodic, almost periodic and pseudo almost periodic solutions for integral equations is an interesting topic, see such as [1, 2, 4] and the references therein. The concept of almost automorphy introduced by Bochner [3], which is an important generalization of the classical almost periodicity, has received lots of attention recently, see the books by N'Guérékata [12, 13]. The concept of asymptotically almost automorphic functions treated here was introduced in the literature in 1981 by N'Guérékata [14]. See also [8, 10, 11] for recent developments and applications to abstract differential equations.

The starting point of this paper is the works in papers [6, 7, 9, 15]. Specifically, Fink and Gatica in [9] considered the existence of a positive almost periodic solution for the following delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \tag{1.1}$$

Received November 15, 2011 1056-2176 $15.00 ©Dynamic Publishers, Inc.

which is a model for the spread of a disease (for more detail [2, 5]). Torrejón [15] studied positive almost periodic solution to the following equation:

$$x(t) = \int_{t-\tau(t)}^{t} f(s, x(s))ds. \tag{1.2}$$

In [6], Ding et al studied positive almost automorphic solutions for the following equation:

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} \sum_{i=1}^{n} f_i(s, x(s))g_i(s, x(s))ds. \tag{1.3}$$

And recently, Ding et al [7] considered positive almost automorphic solutions and asymptotically almost automorphic solutions for Eq (1.2).

Motivated by the above-mentioned works, in this paper we investigate the existence of asymptotically almost automorphic solutions to the following more general equations:

$$x(t) = \gamma x(t-\tau(t)) + (1-\gamma)\int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, x(s))g_i(s, x(s))ds, \tag{1.4}$$

where $f_i(t, \cdot)$, $g_i(t, \cdot)$, $i = 1, \ldots, n$ are nonincreasing, $\gamma \in (0, 1)$. To the best of our knowledge, there are no results available in the literature on asymptotically almost automorphic solutions for Eq (1.4). In this work, we first establish a new fixed point theorem for mixed monotone operators in a cone, and then apply it to prove the existence of asymptotically almost automorphic solutions to Eq (1.4).

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we present our main result and their proofs.

2. PRELIMINARIES

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, and by $\mathbb{X}$ a real Banach space with the norm $\|\cdot\|$, by Ω a subset of $\mathbb{R}$. First, let us recall some definitions, notations and basic results which are main from [12, 13].

Definition 2.1 ([13] (Bochner))**.** A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost automorphic if for every sequence of real numbers (s_m) there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R})$.

Definition 2.2 ([13])**.** A continuous function $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is called almost automorphic in t uniformly for x in compact subset of Ω if for every compact subset K of Ω and every real sequence (s_m), there exist a subsequence (s_n) such that

$$g(t, x) := \lim_{n\to\infty} f(t + s_n, x)$$

for each $t \in \mathbb{R}, x \in K$, and

$$\lim_{n\to\infty} g(t - s_n, x) = f(t, x)$$

for each $t \in \mathbb{R}, x \in K$. Denote $AA(\mathbb{R} \times \Omega)$ the set of all such functions.

Denote by $C_0(\mathbb{R}^+)$ the space of all continuous functions $h : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{t\to+\infty} h(t) = 0$, and by $C_0(\mathbb{R}^+ \times \Omega)$ the space of all continuous functions $h : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ such that $\lim_{t\to+\infty} h(t, x) = 0$ uniformly for x in any compact subset of Ω.

Definition 2.3 ([13])**.** A continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ ($\mathbb{R}^+ \times \Omega \to \mathbb{R}$) is called asymptotically almost automorphic (asymptotically almost automorphic in t uniformly for x in compact subsets of Ω) if it admits a decomposition

$$f = g + h,\ t \in \mathbb{R}^+,$$

where $g \in AA(\mathbb{R})$ ($AA(\mathbb{R} \times \Omega)$ and $h \in C_0(\mathbb{R}^+)$ ($C_0(\mathbb{R}^+ \times \Omega)$). Denote by $AAA(\mathbb{R}^+)$ ($AAA(\mathbb{R}^+ \times \Omega)$) the set of all such functions.

Lemma 2.4. *Assume that $f, f_1, f_2 \in AAA(\mathbb{R}^+)$, then the following hold true:*
(a) $f_1 + f_2, c \cdot f \in AAA(\mathbb{R}^+)$(c is a scalar), $\sup_{t\in R^+} \|f_i(t)\| < \infty, i = 1, 2$;
(b) If $f \in AAA(\mathbb{R}^+)$ and μ is a scalar function in $AAA(\mathbb{R}^+)$, then $\mu \cdot f \in AAA(\mathbb{R}^+)$;
(c) The decomposition of asymptotically almost automorphic function is unique;
(d) $AAA(\mathbb{R}^+)$ is a Banach space with the norm

$$\|f\| = \sup_{t\in\mathbb{R}} \|g(t)\| + \sup_{t\in\mathbb{R}^+} \|h(t)\|,$$

where $f = g + h$ with $g \in AA(\mathbb{R})$ and $h \in C_0(\mathbb{R}^+)$.

Proof. (a,c,d) are proved in [7, Lemma 2.9]. (b) is easy to prove. □

Definition 2.5 ([6])**.** Let $\mathbb{X}$ be a real Banach space, a closed convex set A in $\mathbb{X}$ is called a convex cone if the following conditions are satisfied:
(i) if $x \in A$, then $\lambda x \in A$ for any $\lambda > 0$;
(ii) if $x \in A$ and $-x \in A$, then $x = 0$.
The cone A induces a partial ordering $\le$ in $\mathbb{X}$ through

$$x \le y \ \text{ if and only if } \ y - x \in A.$$

The cone A is called normal if there exists a constant $k > 0$ such that

$$0 \le x \le y \ \text{ if and only if } \ \|x\| \le k\|y\|,$$

where $\|\cdot\|$ is the norm on $\mathbb{X}$. We denote by A^0 the interior set of A. The cone A is called a solid cone if $A^0 \neq \emptyset$.

Definition 2.6 ([6]). Let $\mathbb{X}$ be a real Banach space and $E \subset \mathbb{X}$. An operator $T : E \times E \to \mathbb{X}$ is called a mixed monotone operator if $T(x,y)$ is nondecreasing in x and nonincreasing in y, i.e. $x_i, y_i \in E$ $(i = 1, 2)$, $x_1 \leq x_2$ and $y_1 \geq y_2$ implies that $T(x_1, y_1) \leq T(x_2, y_2)$. An element $x^* \in E$ is called a fixed point of T if $T(x^*, x^*) = x^*$.

3. MAIN RESULT

In the proof of our main result, we will need the following fixed point theorem which is slight different from [6] by Ding.

Theorem 3.1. *Let A be a normal and solid cone in a real Banach space $\mathbb{X}$, suppose that the operator $T = B + D^* : A^0 \times A^0 \to A^0$ satisfies:*
(c1) $B : A^0 \times A^0 \to A^0$ is a mixed monotone operator and there exist a constant $\alpha \in (0,1)$ and a function $\phi : (0,1) \times A^0 \times A^0 \to (0, +\infty)$ such that for each $x, y \in A^0$, we have

$$B(\alpha x, \alpha^{-1} y) \geq \phi(\alpha, x, y) B(x,y), \quad \inf_{x,y \in A^0} \phi(\alpha, x, y) > \alpha;$$

(c2) There exists $z \in A^0$ such that $T(z,z) \geq z$;
(c3) $D^*(x,y) := D(x)$ for any $(x,y) \in \mathbb{X} \times \mathbb{X}$, where $D : \mathbb{X} \to \mathbb{X}$ is a positive linear operator satisfying $D(A^0) \subset A^0 \cup \{\theta\}$.

Then T has a unique fixed point x^* in A^0. Moreover, if we construct the iterative sequences $z_n = T(z_{n-1}, z_{n-1})$ for any initial $z_0 \in A^0$, we have $\|z_n - x^*\| \to 0, n \to +\infty$.

Proof. Because B is a mixed monotone operator and $T = B + D^*$, so T is also mixed monotone operator, then exists a constant $\varepsilon > 0$, $[x_0, y_0] \subset A^0$, such that

$$\varepsilon T(x,y) \leq \varepsilon T(y_0, x_0) \leq B(x_0, y_0) \leq B(x,y), \ \forall x, y \in [x_0, y_0].$$

Then we deduce

$$\begin{aligned} T(\alpha x, \alpha^{-1} y) &= B(\alpha x, \alpha^{-1} y) + D(\alpha x) = B(\alpha x, \alpha^{-1} y) + \alpha D(x) \\ &\geq \phi(\alpha, x, y) B(x,y) + \alpha(T(x,y) - B(x,y)) \\ &\geq [(\phi(\alpha, x, y) - \alpha)\varepsilon + \alpha] T(x,y) \\ &\geq \psi(\alpha, x, y) T(x,y), \end{aligned} \tag{3.1}$$

for all $x, y \in [x_0, y_0]$ and $\alpha \in (0,1)$, where

$$\psi(\alpha, x, y) := [(\phi(\alpha, x, y) - \alpha)\varepsilon + \alpha] > \alpha.$$

Moreover, by (c1), we have

$$\inf_{x,y \in A^0} \psi(\alpha, x, y) > \alpha, \ \text{ for all } \alpha \in (0,1).$$

On the other hand, choose $\alpha \in (0,1)$ and $N \in \mathbb{N}$, such that

$$T(z,z) \le \alpha^{-1} z, \quad \psi^N(\alpha, z, z) \ge \alpha^{N-1}$$

Let

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}),\ n \in \mathbb{N}$$
$$x_0 = \alpha^N z, \quad y_0 = \alpha^{-N} z,\ N \in \mathbb{N};$$

then $x_1 = T(x_0, y_0),\ y_1 = T(y_0, x_0)$.
We have

$$\begin{aligned} x_1 = T(\alpha^N z, \alpha^{-N} z) &\ge \psi(\alpha, \alpha^{N-1} z, \alpha^{-(N-1)} z) T(\alpha^{(N-1)} z, \alpha^{-(N-1)} z) \\ &\ge \alpha T(\alpha^{N-1} z, \alpha^{-(N-1)} z) \ge \cdots \ge \alpha^N T(z,z) \\ &\ge \alpha^N z = x_0, \end{aligned}$$

and

$$\begin{aligned} y_1 = T(\alpha^{-N} z, \alpha^N z) &\le \frac{1}{\psi(\alpha, \alpha^{-N} z, \alpha^N z)} T(\alpha^{-(N-1)} z, \alpha^{N-1} z) \\ &\le \frac{1}{\psi^N(\alpha, z, z)} T(z,z) \\ &\le \alpha^{(1-N)} \cdot \alpha^{-1} z \\ &= \alpha^{-N} z = y_0. \end{aligned}$$

Since T is a mixed monotone operator, we deduce

$$x_0 \le x_1 = T(x_0, y_0) \le x_2 \le \cdots \le x_n = T(x_{n-1}, y_{n-1}) \le \cdots,$$

$$y_0 \ge y_1 = T(y_0, x_0) \ge y_2 \ge \cdots \ge y_n = T(y_{n-1}, x_{n-1}) \ge \cdots,$$

So we have

$$x_0 \le x_1 \le x_2 \le \cdots \le x_n \le \cdots \le y_n \le \cdots \le y_0.$$

Define

(3.2) $$\alpha_n := \sup\{\beta > 0 : x_n \ge \beta y_n\},$$

for each $n \in \mathbb{N}$, then $x_n \ge \alpha_n y_n$ and

(3.3) $$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots \le 1.$$

We claim that

$$\alpha_\infty := \lim_{n\to\infty} \alpha_n = 1.$$

If this is not true, then $0 < \alpha_\infty < 1$, take $\xi \in (0,1)$, we consider the following two cases:
(a) $\alpha_\infty = \xi$.
There exists $P \in \mathbb{N}$ such that $\alpha_n = \xi$, for all $n > P$. Now, for $n > P$, from above

definition, we can know $x_n \geq \xi_n y_n,\ y_n \leq \xi^{-1} x_n$, for $x_n, y_n \in [x_0, y_0]$, and through (3.1), we can obtain

$$x_{n+1} = T(x_n, y_n) \geq T(\xi y_n, \xi^{-1} x_n) \geq \psi(\xi, x_n, y_n) T(y_n, x_n) = \psi(\xi, x_n, y_n) y_{n+1},$$

by the form of (3.2), we have $\alpha_{n+1} \geq \psi(\xi, x_n, y_n) > \xi$, this is a contradiction.
(b) $\alpha_\infty < \xi$.
Since $\xi y_n, \xi^{-1} x_n \in [\xi x_0, \xi^{-1} y_0]$, by the form of (3.1) and (3.2),

$$\begin{aligned} x_{n+1} = T(x_n, y_n) &\geq T(\alpha_n y_n, \alpha_n^{-1} x_n) = T(\frac{\alpha_n}{\xi} \cdot \xi y_n, (\frac{\alpha_n}{\xi})^{-1} \cdot \xi^{-1} x_n) \\ &\geq \psi(\frac{\alpha_n}{\xi}, \xi y_n, \xi^{-1} x_n) T(\xi y_n, \xi^{-1} x_n) \\ &\geq \frac{\alpha_n}{\xi} T(\xi y_n, \xi^{-1} x_n) \\ &\geq \frac{\alpha_n}{\xi} \psi(\xi, y_n, x_n) T(y_n, x_n), \end{aligned}$$

which implies that

$$\alpha_{n+1} \geq \alpha_n \frac{\psi(\xi, y_n, x_n)}{\xi}.$$

Let $n \to \infty$, then

$$\alpha_\infty \geq \alpha_\infty \frac{\psi(\xi, y_n, x_n)}{\xi} \geq \alpha_\infty \frac{\inf_{x,y \in [x_0, y_0]} \psi(\xi, x, y)}{\xi} > \alpha_\infty,$$

this is a contradiction, so we have $\alpha_\infty = 1$.
Combining the form of (3.2) and (3.3), we can obtain

$$0 \leq x_{n+p} - x_n \leq y_n - x_n \leq y_n - \alpha_n y_n \leq (1 - \alpha_n) y_0,$$

and

$$0 \leq y_n - y_{n+p} \leq y_n - x_n \leq y_n - \alpha_n y_n \leq (1 - \alpha_n) y_0,$$

for each $n, p \in \mathbb{N}$. Because A is a normal cone, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, therefore there exists $u^*, v^* \in [x_0, y_0]$ such that

$$\lim_{n \to \infty} x_n = u^*, \quad \lim_{n \to \infty} y_n = v^*,$$

and

$$0 \leq v^* - u^* \leq y_n - x_n \leq y_n - \alpha_n y_n \leq (1 - \alpha_n) y_0,$$

so $0 \leq v^* - u^* \leq 0$, which means that $v^* = u^*$. Let $x^* = u^* = v^*$, then

$$T(x^*, x^*) = T(u^*, v^*) \geq T(x_n, y_n) = x_{n+1},$$

$$T(x^*, x^*) = T(u^*, v^*) \leq T(y_n, x_n) = y_{n+1},$$

which means that $u^* \leq T(x^*, x^*) \leq v^*$. Hence, $T(x^*, x^*) = x^*$, that is x^* is a fixed point of T.

It remains to show that x^* is the unique fixed point of T. Suppose $y^* \in [x_0, y_0]$ is another fixed point of T, it follows from the definition of x_n and y_n, that $x_n \leq y^* \leq y_n$.

Let $r := \sup\{k > 0 : k^{-1}y^* \geq x^* \geq ky^*\}$, then $r^{-1}y^* \geq x^* \geq ry^*$ and $0 < r \leq 1$. If $0 < r < 1$, then

$$w := \min\{\psi(r, v^*, v^*), \psi(r, r^{-1}v^*, rv^*)\} > r.$$

Noticing that

$$x^* = T(x^*, x^*) \geq T(ry^*, r^{-1}y^*) \geq \psi(r, y^*, y^*)y^* \geq wy^* > ry^*,$$

and

$$x^* = T(x^*, x^*) \leq T(r^{-1}y^*, ry^*) \leq [\psi(r, r^{-1}y^*, ry^*)y^*]^{-1}y^* \leq wy^* < r^{-1}y^*,$$

so we have $ry^* < wy^* \leq x^* \leq wy^* < r^{-1}y^*$, because r is defined supremum, this is a contradiction. Therefore $r = 1$. So $y* \geq x^* \geq y^*$, that is $x^* = y^*$. Thus x^* is a unique fixed point of T. Moreover, for any initial $z_0 \in A^0$ and the iterative sequences $z_n = T(z_{n-1}, z_{n-1})$, we can choose $n \in \mathbb{N}$, such that $x_n \leq z_n \leq y_n$. Therefore, $\|z_n - x^*\| \to 0$, when $n \to \infty$.
The proof is now complete. □

Now we are in position to investigate the existence of asymptotically almost automorphic solutions to Eq. (1.4). For the sake of convenience, we list all the hypotheses to be used in this section as follows:
(H1) $f_i,\ g_i \in AAA(\mathbb{R}^+ \times \mathbb{R}^+)$, $i = 1, 2, \ldots, n$, are nonnegative functions and $\tau \in AAA(\mathbb{R}^+)$ is a positive function, moreover, $t \geq \tau(t)$ for all $t \in \mathbb{R}^+$.
(H2) For every $t \in \mathbb{R}^+$, $f_i(t, \cdot)$ are nondecreasing and $g_i(t, \cdot)$ are nonincreasing in $\mathbb{R}^+$, $i = 1, 2, \ldots, n$.
(H3) For each $x \in \mathbb{R}^+$ and each $i \in \{1, 2, \ldots, n\}$, $\{f_i(t, \cdot)\}_{t\in\mathbb{R}^+}$ and $\{g_i(t, \cdot)\}_{t\in\mathbb{R}^+}$ are equi-continuous in x.
(H4) There exist positive functions $\varphi_i,\ \psi_i$ defined on $(0, 1) \times (0, +\infty)$ such that

$$f_i(t, \alpha x) \geq \varphi_i(\alpha, x)f_i(t, x), \quad g_i(t, \alpha^{-1}y) \geq \psi_i(\alpha, y)g_i(t, y),$$

$\varphi_i(\alpha, x) > \alpha$ for all $x, y > 0$, $\alpha \in (0, 1)$, $t \in \mathbb{R}^+$ and $i \in \{1, 2, \ldots, n\}$; moreover, for any $0 < a < b < +\infty$, $\inf_{x,y\in[a,b]} \varphi_i(\alpha, x)\psi_i(\alpha, y) > \alpha$, $\alpha \in (0, 1), i = 1, 2, \ldots, n$.
(H5) For any $d > 0$, there exists a constant c with $0 < c \leq d$ such that

$$\inf_{t\in\mathbb{R}^+} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, c)g_i(s, d)ds \geq c.$$

Lemma 3.2 ([7, Lemma 3.9])**.** *If $f \in AAA(\mathbb{R}^+)$, $\{f(t, \cdot)\}_{t\in\mathbb{R}^+}$ are equi-continuous everywhere on $\mathbb{R}^+$, $x \in AAA(\mathbb{R}^+)$ and $x(t) \geq 0$ for every $t \in \mathbb{R}^+$, then $f(\cdot, x(\cdot)) \in AAA(\mathbb{R}^+)$.*

Lemma 3.3 ([7, Lemma 3.10])**.** *Let $f \in AAA(\mathbb{R}^+), \tau \in AAA(\mathbb{R}^+)$ and $t \geq \tau(t)$ for all $t \in \mathbb{R}^+$, then*

$$F(t) = \int_{t-\tau(t)}^{t} f(s)ds \in AAA(\mathbb{R}^+).$$

Theorem 3.4. *Assume the conditions (H1)–(H5) are satisfied, then Eq* (1.4) *has a unique asymptotically almost automorphic solution* x^**. Moreover, for any initial* $x_0 \in AAA(\mathbb{R}^+)$ *with positive infinimum and the iterative sequences*

$$x_k(t) = \gamma x_{k-1}(t - \tau(t)) + (1 - \gamma) \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, x_{k-1}(s)) g_i(s, x_{k-1}(s)) ds,$$

we have

$$\lim_{k\to+\infty} \|x_k - x^*\| = 0.$$

Proof. Let A be a cone and define in the Banach space $AAA(\mathbb{R}^+)$ by

$$A = \{x \in AAA(\mathbb{R}^+) : x(t) \geq 0, \forall t \in \mathbb{R}^+\}$$

It can be verified that A is a normal and solid cone in $AAA(\mathbb{R}^+)$, and

$$A^0 = \{x \in AAA(\mathbb{R}^+) : \exists \varepsilon > 0, \text{ such that } x(t) > \varepsilon, \forall t \in \mathbb{R}^+\},$$

then A^0 is the interior set of A.

For $x, y \in A^0$ and $t \in \mathbb{R}^+$, we define the operators

$$B(x, y)(t) = (1 - \gamma) \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, x(s)) g_i(s, y(s)) ds,$$

$$D(x)(t) = \gamma x(t - \tau(t)).$$

Then Eq (1.4) is equivalent to the equation $x = T(x, x)$ with $T = D + B$. We will verify all the assumptions of Theorem 3.1.

Let $x, y \in A^0$, it follows from (H1), (H3) and lemma 3.2 that

$$f_i(\cdot, x(\cdot)), g_i(\cdot, y(\cdot)) \in AAA(\mathbb{R}^+),\ i = 1, 2, \ldots, n.$$

Combining this with Lemma 2.4 (a), we deduce

$$\sum_{i=1}^{n} f_i(s, x(s)) g_i(s, x(s)) \in AAA(\mathbb{R}^+).$$

By Lemma 3.3, $B(x, y) \in AAA(\mathbb{R}^+ \times \mathbb{R}^+)$. Also there exist $\varepsilon, M > 0$, such that $x(t) \geq \varepsilon$ and $y(t) \leq M$ for all $t \in \mathbb{R}^+$. So by (H5) there exists $a > 0$, such that

$$\inf_{t\in\mathbb{R}^+} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, a) g_i(s, M) ds \geq a.$$

If $\varepsilon \le a$, by (H4) and (H5), we have

$$\begin{aligned}
B(x,y)(t) &= (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,x(s))g_i(s,y(s))ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,\varepsilon)g_i(s,M)ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} \varphi_i(\frac{\varepsilon}{a},a)f_i(s,a)g_i(s,M)ds \\
&\ge (1-\gamma)\frac{\varepsilon}{a}\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,a)g_i(s,M)ds \\
&\ge (1-\gamma)\varepsilon > 0.
\end{aligned}$$

If $\varepsilon > a$, by (H5) that

$$\begin{aligned}
B(x,y)(t) &= (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,x(s))g_i(s,y(s))ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,\varepsilon)g_i(s,M)ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,a)g_i(s,M)ds \\
&\ge (1-\gamma)a > 0.
\end{aligned}$$

Thus $B(x,y) \in A^0$, this is, B is from $A^0 \times A^0 \to A^0$. On the other hand, it follows easily from (H2) that B is a mixed monotone operator.

Suppose $x, y \in A^0$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}
B(\alpha x,\alpha^{-1}y)(t) &= (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} f_i(s,\alpha x(s))g_i(s,\alpha^{-1}y(s))ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} \varphi_i(\alpha,x(s))\psi_i(\alpha,y(s))f_i(s,x(s))g_i(s,y(s))ds \\
&\ge (1-\gamma)\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} \phi_i(\alpha,x,y)f_i(s,x(s))g_i(s,y(s))ds \\
&\ge \phi(\alpha,x,y)B(x,y)(t),
\end{aligned}$$

where

$$\phi_i(\alpha,x,y) = \varphi_i(\alpha, \inf_{s\in\mathbb{R}^+} x(s))\psi_i(\alpha, \inf_{s\in\mathbb{R}^+} y(s)),\ i = 1,2,\ldots,n.$$

and

$$\phi(\alpha,x,y) = \min_{i=1,2,\ldots,n} \phi_i(\alpha,x,y),$$

which means that $B(\alpha x, \alpha^{-1}y)(t) \geq \phi(\alpha, x, y)B(x, y)(t)$, for each $x, y \in A^0$, $\alpha \in (0, 1)$. By (H4) that,

$$\begin{aligned} \inf_{x,y\in A^0} \phi(\alpha, x, y) &= \min_{i=1,2,\dots,n} \inf_{x,y\in A^0} \phi_i(\alpha, x, y) \\ &= \min_{i=1,2,\dots,n} \inf \varphi_i(\alpha, \inf_{s\in\mathbb{R}^+} x(s))\psi_i(\alpha, \inf_{s\in\mathbb{R}^+} y(s)) > \alpha, \end{aligned}$$

for each $\alpha \in (0, 1)$. Thus, the assumption (c1) in Theorem 3.1 is satisfied.

Next, let us check the assumption (c2) of Theorem 3.1. By (H5), for any $d > 0$, there exists a constant c, with $0 < c \leq d$ such that

$$\inf_{t\in\mathbb{R}^+} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, c)g_i(s, d)ds \geq c.$$

Therefore, we have

$$T(c, d)(t) = \gamma c + (1-\gamma) \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, c)g_i(s, d)ds \geq \gamma c + (1-\gamma)c = c.$$

Let $0 < z = c \leq d$, then

$$\begin{aligned} T(z, z)(t) &= \gamma z + (1-\gamma) \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, z)g_i(s, z)ds \\ &\geq \gamma z + (1-\gamma) \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} f_i(s, z)g_i(s, d)ds \\ &\geq \gamma z + (1-\gamma)z = z. \end{aligned}$$

So we can claim that there exists a constant $z > 0$, such that $T(z, z) \geq z$. Then the assumption (c2) of Theorem 3.1 are justified.

Finally, let us check the assumption (c3) of Theorem 3.1. For $t \in \mathbb{R}^+, x(t) \in A^0$ and D is a positive linear operator, we have $D(x)(t) = \gamma x(t - \tau(t)) \in A^0$ for $\gamma \in \mathbb{R}^+, t \geq \tau(t)$, that is $D(A^0) \subset A^0 \cup \{\theta\}$. Thus Theorem 3.1 yields that T has a unique fixed point x^* in A^0, which is just the unique asymptotically almost automorphic solution with a positive infinimum to Eq (1.4). Moreover, applying Theorem 3.1, we get that the iterative sequences $x_k = T(x_{k-1}, x_{k-1}), k = 1, 2, \dots,$ satisfy

$$\lim_{k\to+\infty} \|x_k - x^*\| = 0.$$

This completes the proof. □

Acknowledgements: The second author's work was supported by NNSF of China (10901075), Program for New Century Excellent Talents in University (NCET-10-0022), the Key Project of Chinese Ministry of Education (210226), and NSF of Gansu Province of China (1107RJZA091).

REFERENCES

[1] E. Ait Dads, K. Ezzinbi, Almost periodic solution for some neutral nonlinear integral equation, Nonlinea Anal.TMA 28(1997) 1479–1489.

[2] E. Ait Dads, K. Ezzinbi, Existence of positive pseudo-almost-periodic solution for some neutral nonlinear infinite delay integral equations arising in epidemic problems, Nonlinea Anal 41 (2000) 1–13.

[3] S. Bochner, A new approach to almost-periodicity, Proc. Natl. Acad. Sci. USA 48 (1962) 2039–2043.

[4] T. A. Burton, Testuo Furumochi, Existence theorem and periodic solutions of neutral integral equation,Nonlinea Anal. 43 (2001) 527–546.

[5] K. L. Cooke, J. L. Kaplan, A perodicity threshold theorem for epidemics and population growth, Math.Biosci.31 (1976) 87–104.

[6] H. S. Ding, J. Liang, G. M. N'Guérékata , T.J.Xiao, Existence of positive almost automorphic solutions to neutral nonlinear integral equations, Nonlinear Anal 69 (2008) 1188–1199.

[7] H. S. Ding, T. J. Xiao, J. Liang, Existence of positive almost automorphic solutions to nonlinear delay integral equations, Nonlinear Anal.70 (2009) 2216–2231.

[8] H.S. Ding, T. J. Xiao, J. Liang, Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions, J. Math. Anal. Appl. 338 (2008) 141–151.

[9] A. M. Fink, J. A. Gatica, Positive almost periodic solutions of some delay integral equation, J. Differential equations, 83 (1990) 166–178.

[10] J. Liang, J. Zhang, T. J. Xiao, Composition of pseudo almost automorphic and asymptotically almost automorphic functions, J. Math. Anal. Appl. 340 (2008) 1493–1499.

[11] G. M. Mophou, G. M. N'Guérékata , On some classes of almost automorphic functions and applications to fractional differential equations, Comput. Math. Appl. 59 (2010) 1310–1317.

[12] G. M. N'Guérékata , Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces, Kluwer Academic Plenum Publishers, New York, London, Moscow, 2001.

[13] G. M. N'Guérékata , Topics in Almost Automorphy, Springer, New York, Boston, Dordrecht, London Moscow, 2005.

[14] G. M. N'Guérékata , Sue les solutions presqu'automorphes d'équations différentielles abstraites, Annales des Sciences Mathématqiues du Québec, 51 (1981) 69–79.

[15] R. Torrejón, Positive almost periodic solutions of a state-dependent delay nonlinear integral equation, Nonlinear Anal. 20 (1993) 1383–1416.